Electronic Journal of Differential Equations, Vol. 2007(2007), No. 119, pp. 1–23.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

CENTRES AND LIMIT CYCLES FOR AN EXTENDED KUKLES SYSTEM

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Abstract. We present conditions for the origin to be a centre for a class of cubic systems. Some of the centre conditions are determined by finding complicated invariant functions. We also investigate the coexistence of fine foci and the simultaneous bifurcation of limit cycles from them.

1. Introduction

In this paper we establish some properties of the cubic differential system

$$\begin{aligned}\dot{x} &= P(x,y) = \lambda x + y + kxy,\\ \dot{y} &= Q(x,y) = -x + \lambda y + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3,\end{aligned} \tag{1.1}$$

where the a_i and k are real. We first became interested in this class of systems when considering transformations to generalised Liénard form [1]. It was also brought to our attention that a system used to model predator-prey interactions with intratrophic predation could be transformed so that it is an example of a system of type (1.1). We investigated this particular case of system (1.1) in [7]. In [6] we found conditions for the origin to be an isochronous centre for system (1.1).

When $\lambda = 0$ the origin is said to be a *fine focus*; then system (1.1) is derived from a second order scalar equation and it has an invariant line $kx = -1$. When $k = 0$, (1.1) is often referred to as the Kukles system; this system has been extensively studied, see [2], [12] and [14] for example.

Here we derive conditions for the origin to be a centre for system (1.1) and consider the simultaneous bifurcation of limit cycles from several fine foci. We shall see, for example, that at most two fine foci of (1.1) can coexist; when one fine focus is of order one, the other is of maximum order six and when one fine focus is of order two, the other is of maximum order two. We show that in the latter case a large amplitude limit cycle can surround the two fine foci and conjecture that this is also true in the former.

We obtain necessary conditions for a critical point to be a centre for (1.1) by calculating the *focal values*, which are polynomials in the coefficients k, a_i. There is a function V, analytic in a neighbourhood of the origin, such that its rate of change along orbits, $\dot{V}$, is of the form $\eta_2 r^2 + \eta_4 r^4 + \cdots$, where $r^2 = x^2 + y^2$. The η_{2j} are the

2000 *Mathematics Subject Classification.* 34C07, 37G15.
Key words and phrases. Nonlinear differential equations; invariant curves; limit cycles.

Submitted August 10, 2007. Published September 6, 2007.

focal values and the origin is a centre if, and only if, they are all zero. The relations $\eta_2 = \eta_4 = \cdots = \eta_{2j} = 0$ are used to eliminate some of the variables from η_{2j+2}. This reduced focal value η_{2j+2}, with strictly positive factors removed, is known as the *Liapunov quantity* $L(j)$. We note that $L(0) = \lambda$. The circumstances under which the calculated $L(j)$ are zero yield possible centre conditions. The origin is a fine focus of *order* j if $L(i) = 0$ for $i = 0, 1, \ldots, j-1$ and $L(j) \neq 0$; at most j small amplitude limit cycles can bifurcate from a fine focus of order j.

Various methods are used to prove the sufficiency of centre conditions; in this paper we require three of them. The simplest is that the origin is a centre if the system is symmetric in either axis, that is, it remains invariant under the transformation $(x, y, t) \mapsto (x, -y, -t)$ or $(x, y, t) \mapsto (-x, y, -t)$. Another technique which we employ involves a transformation of the system to Liénard form

$$\dot{x} = y, \quad \dot{y} = -f(x)y - g(x). \tag{1.2}$$

The relevant results are as follows; proofs can be found in [3].

Lemma 1.1. *Consider system* (1.2) *where* f, g *are analytic,* $g(0) = 0$, $xg(x) > 0$ *for* $x \neq 0$ *and* $g'(0) > 0$. *Let* $F(x) = \int_0^x f(\mu)d\mu$ *and* $G(x) = \int_0^x g(\mu)d\mu$.

(i) *The origin is a centre for system (1.2) if and only if there is an analytic function* Φ *with* $\Phi(0) = 0$ *such that* $G(x) = \Phi(F(x))$ *in a neighbourhood of* $x = 0$.

(ii) *The origin is a centre for system (1.2) if and only if there is a function* $z(x)$ *satisfying* $z(0) = 0$, $z'(0) < 0$ *such that* $F(z) = F(x)$ *and* $G(z) = G(x)$.

The third approach, and the one which is of particular interest to us here, is the possibility of finding an *integrating factor*. If the origin is a critical point of focus type then it is a centre if there is a function $D \neq 0$ such that

$$\frac{\partial}{\partial x}(DP) + \frac{\partial}{\partial y}(DQ) = 0 \tag{1.3}$$

in a neighbourhood of the origin. Such a function is called an integrating factor or *Dulac function*. The existence of the function D means the system is integrable and the origin is a centre.

We make a systematic search for an integrating factor of the form $D = \Pi_{i=1}^n C_i^{\alpha_i}$, where each C_i is an invariant algebraic function. In this context an invariant function is such that $\dot{C}_i = C_i L_i$, where L_i, known as the cofactor of C_i, is of degree one less than that of the system. We require

$$\frac{\partial}{\partial x}(DP) + \frac{\partial}{\partial y}(DQ) = D(P_x + Q_y + \alpha_1 L_1 + \cdots + \alpha_n L_n) = 0. \tag{1.4}$$

The α_i and the coefficients in the C_i, L_i are functions of the coefficients k, a_i. We note that the C_i, L_i, α_i may be complex; a real Dulac function is then constructed from these together with their conjugates. In any given situation there may well be no invariant functions and even though there is an upper bound for the possible degree of an invariant curve it is not known how to determine this bound. Darboux [5] showed that if $n \geq \frac{1}{2}m(m+1)+2$ invariant functions exist, where m is the degree of the system, then the n functions can be combined to form a first integral. In practice we find that fewer such functions are required. As will be apparent later, finding such functions is non-trivial. However it is a relatively straightforward matter to confirm that the functions found actually satisfy the relation (1.3).

These techniques for finding centre conditions are well established but the computational problems encountered are often formidable. We are constantly pushing the available software to its limits. The reduction of the focal values to obtain the Liapunov quantities is one area which causes difficulties and here we demonstrate the usefulness of our suite of programs INVAR [11] in the search for invariant functions. We are unable to complete the reduction the focal values for (1.1) to obtain the conditions that are necessary for the origin to be a centre, but we can find sufficient conditions by searching for invariant functions. We then try to determine whether or not we have a complete set of conditions for the origin to be a centre.

The necessary and sufficient conditions for the origin to a centre for the Kukles system are known; we summarise them in Theorem 1.2. We note that the condition given in [8, Theorem 3.3], with $a_2 = 0$, is covered by condition (v) of Theorem 1.2. In [9] it was conjectured that, when $a_7 \neq 0$, the origin is a centre for the Kukles system if and only if one of the conditions given in Theorems 2.1 or 2.2 therein is satisfied. This was verified in [10].

Theorem 1.2. *Let $\lambda = k = 0$. The origin is a centre for system* (1.1) *if and only if one of the following conditions holds:*

(i) $a_2 = a_5 = a_7 = 0$;
(ii) $a_1 = a_3 = a_5 = a_7 = 0$;
(iii) $a_4 = a_3(a_1+a_3)$, $a_5 = -a_2(a_1+a_3)$, $(a_1+2a_3)a_6+a_3^2(a_1+a_3) = 0$, $a_7 = 0$;
(iv) $a_5 + 3a_7 + a_2(a_1 + a_3) = 0$, $9a_6a_2^2 + 2a_2^4 + 27a_7\mu + 9\mu^2 = 0$, $a_4a_2^2 + a_5\mu = 0$, $(3a_7\mu + \mu^2 + a_6a_2^2)a_5 - 3a_7\mu^2 - a_6a_2^2\mu = 0$, *where* $\mu = 3a_7 + a_2a_3$;
(v) $a_5 + 3a_7 + a_2(a_1 + a_3) = 0$, $18a_4a_5 - 27a_4a_7 + 9a_5a_1^2 + 9a_5a_6 + 2a_5a_2^2 = 0$, $27a_4a_1 + 4a_5a_2 + 9a_1^3 + 2a_1a_2^2 = 0$, $18a_4^2 + 9a_4a_1^2 + 2a_4a_2^2 + 2a_5^2 = 0$, $18a_4a_2 + 9a_5a_1 + 9a_5a_3 + 9a_1^2a_2 - 27a_1a_7 + 9a_6a_2 + 2a_2^3 = 0$.

For a proof of the above theorem, see [2, 8, 9, 10].

This is one particular sub-class of system (1.1). In the next section we shall present conditions that are necessary and sufficient for the origin to be a centre for two other sub-classes of system (1.1); one with $a_7 = 0$ and one with $a_2 = 0$. Presenting the results in this way allows for a clearer description of the general case and gives us insight into the types of invariant functions we should seek for system (1.1) in general. In section 3 we derive sufficient conditions for the origin to be a centre for system (1.1) and in section 4 we investigate whether there are any other conditions. The coexistence of fine foci and the bifurcation of small amplitude limit cycles is considered in section 5, and in section 6 we investigate the possibility of the existence of large amplitude limit cycles.

2. Sub-classes $a_7 = 0$ and $a_2 = 0$

In this section we consider the sub-classes of system (1.1) with $a_7 = 0$ or $a_2 = 0$. We find that the origin is a fine focus of maximum order six when $a_7 = 0$ and maximum order seven when $a_2 = 0$.

Theorem 2.1. *Let $\lambda = a_7 = 0$. The origin is a centre for system* (1.1) *if and only if one of the following conditions holds:*

(i) $a_2 = a_5 = a_7 = 0$;
(ii) $k = a_1 = a_3 = a_5 = a_7 = 0$;
(iii) $k = -\frac{1}{2}a_1$, $a_3 = -\frac{2}{3}a_1$, $a_4 = -\frac{1}{4}a_1^2$, $a_5 = -\frac{1}{3}a_1a_2$, $a_6 = a_7 = 0$;

(iv) $k=-a_1$, $a_3=-\frac{2}{3}a_1$, $a_4=0$, $a_5=-\frac{1}{3}a_1a_2$, $a_6=a_7=0$;
(v) $k=-\frac{1}{4}a_1$, $a_3=-\frac{3}{4}a_1$, $a_4=-\frac{1}{4}a_1^2$, $a_5=-\frac{1}{4}a_1a_2$, $a_6=a_7=0$;
(vi) $k=-\frac{1}{2}a_1$, $a_3=-a_1$, $a_5=a_6=a_7=0$;
(vii) $a_4=a_3(a_1+a_3)$, $a_5=-a_2(a_1+a_3)$, $(a_1+2a_3)a_6-a_3(k-a_3)(a_1+a_3)=0$, $a_7=0$;
(viii) $k=-(a_1+a_3)$, $a_6=a_1(a_1+a_3)$, $a_5=-a_2(a_1+a_3)$, $(3a_1+2a_3)a_4+a_1^2(a_1+a_3)=0$, $a_7=0$.

Proof. Calculation of the focal values for system (1.1) with $a_7=0$, up to η_{14}, and their reduction to give the corresponding Liapunov quantities is routine. We do not present the details here. We find that $L(0)=L(1)=\cdots=L(6)=0$ only if one of the conditions of Theorem 2.1 holds. The sufficiency of these conditions is confirmed as follows.

When (i) holds the system is invariant under the transformation $(x,y,t)\mapsto(x,-y,-t)$; the system is symmetric in the x-axis, hence the origin is a centre. Similarly, when condition (ii) holds the system is invariant under the transformation $(x,y,t)\mapsto(-x,y,-t)$; the system is symmetric in the y-axis, so the origin is a centre.

Conditions (iii), (iv), (v) and (vi) have $a_6=a_7=0$, in which case system (1.1) is of the form

$$\dot{x}=(1+kx)y,\quad \dot{y}=x(-1+a_1x+a_4x^2)+x(a_2+a_5x)y+a_3y^2. \tag{2.1}$$

If $k=0$ in these cases then condition (ii) is satisfied. When $k\neq 0$, we are able to transform (2.1) to a Liénard system. The required transformation (see [3]) is $(x,y,t)\mapsto(x,(1+kx)y\Psi(x),\tau)$, where

$$\Psi(x)=\frac{dt}{d\tau}=(1+kx)^{-1}\exp\Big(-\int_0^x a_3(1+ks)^{-1}ds\Big)=(1+kx)^{-1-\frac{a_3}{k}}.$$

Then system (2.1) becomes a system of the form (1.2) with

$$f(x)=-x(a_2+a_5x)(1+kx)^{-1-\frac{a_3}{k}},\quad g(x)=x(1-a_1x-a_4x^2)(1+kx)^{-1-\frac{2a_3}{k}}.$$

We compute the integrals of f, g and denote these by F, G respectively. For condition (iii) we have

$$F(x)=\frac{a_2}{a_1^2}\Big(9-\big(\frac{2}{a_1x-2}\big)^{4/3}(a_1x-3)^2\Big),$$
$$G(x)=-\frac{6}{a_1^2}\Big(3+\big(\frac{2}{a_1x-2}\big)^{2/3}\big(a_1x-3\big)\Big).$$

Let $u^3=a_1x-2$ and $v^3=a_1z-2$ then

$$F(x)-F(z)=\frac{2^{4/3}a_2}{a_1^2u^4v^4}(v-u)(u^3v^2+u^2v^3-u^2-v^2)\Omega,$$

$$G(x)-G(z)=3\frac{2^{\frac{5}{3}}a_2}{a_1^2u^2v^2}(v-u)\Omega,$$

where

$$\Omega=u^2v^2+u+v=\left((a_1x-2)\,(a_1z-2)\right)^{2/3}+(a_1x-2)^{1/3}+(a_1z-2)^{1/3}.$$

When $x=z=0,\Omega_x=\Omega_z=-2^{-\frac{2}{3}}a_1$. By the Implicit Function Theorem there is $z(x)$ with $z'(x)<0$ such that $F(x)=F(z(x))$, $G(x)=G(z(x))$. The origin is a centre by Lemma 1.1 (ii).

Similarly for condition (iv) we find

$$F(x) = \frac{a_2}{4a_1^2}\Big(9 - \frac{(a_1x-3)^2}{(1-a_1x)^{2/3}}\Big), \quad G(x) = -\frac{3}{2a_1^2}\Big(9 + \frac{(a_1x-3)}{(1-a_1x)^{1/3}}\Big)$$

and

$$\Omega = ((1-a_1x)(1-a_1z))^{1/3}\left((1-a_1x)^{1/3} + (1-a_1z)^{\frac{1}{3}}\right) - 2.$$

When condition (v) holds

$$F(x) = -8\frac{a_2x^2}{(a_1x-4)^2},$$

$$G(x) = 8\frac{x^2}{(a_1x-4)^6}\left(a_1^4x^4 - 24a_1^3x^3 + 240a_1^2x^2 - 768a_1x + 768\right)$$

and $\Omega = a_1xz - 2(x+z)$. For condition (vi)

$$F(x) = -2\frac{a_2x^2}{(a_1x-2)^2},$$

$$G(x) = -4\frac{(a_1^2a_4x^4 + 4a_1^2x^2 - 8a_1x + 4)}{(a_1x-2)^4}$$

and $\Omega = a_1xz - x - z$. In each case Ω is a common factor of $F(x) - F(z)$ and $G(x) - G(z)$; the origin is a centre by Lemma 1.1 (ii).

To prove the sufficiency of the remaining conditions we use INVAR to help us find appropriate invariant functions and to build Dulac functions. Confirmation that the functions obtained are indeed Dulac functions is routine. When condition (vii) holds we find the Dulac function

$$D = (1+kx)^{\alpha_1}e^{\alpha_2x}C^{\alpha_3},$$

where

$$C = 1 + a_3x - \gamma y, \quad \alpha_1 = \frac{(a_2-2\gamma)(a_3k-a_6) - k^2\gamma}{k^2\gamma},$$

$$\alpha_2 = \frac{a_6(a_2-2\gamma)}{k\gamma}, \quad \alpha_3 = -\frac{a_2}{\gamma},$$

and γ satisfies $\gamma^2 - a_2\gamma - a_3^2 + a_3k - a_6 = 0$. Hence, when $k\gamma \neq 0$, the origin is a centre. When $k = 0$ condition (iii) of Theorem 1.2 holds. When $\gamma = 0$, then $a_6 = a_3(k - a_3) = 0$ and the system can be transformed to Liénard form with $f(x) = -a_2g(x)$; the origin is a centre by Lemma 1.1 (i).

For condition (viii) we find the Dulac function

$$D = (1+kx)^{\alpha_1}e^{\alpha_2x}C^{\alpha_3},$$

where

$$C = 1 - a_1x + \frac{a_2}{\gamma}y - a_4x^2 + \frac{a_5}{\gamma}xy,$$

$$\alpha_1 = 1, \quad \alpha_2 = a_1(\gamma+2), \quad \alpha_3 = \gamma,$$

and γ is a root of $a_1^3\gamma^2 - (3a_1 + 2a_3)a_2^2(\gamma+1) = 0$. If $\gamma \neq 0$, the origin is a centre. When $\gamma = 0$, then one of conditions (i), (ii) or (vii) is satisfied. This completes the proof. □

When none of the conditions of Theorem 2.1 holds and $L(i) = 0$, for $i = 0, 1, 2, \ldots, 5$, then $L(6) \neq 0$; the origin is then a fine focus of maximum order six and at most six small amplitude limit cycles can be bifurcated from the origin.

We now consider the sub-class of system (1.1) with $a_2 = 0$ and $a_3 a_7 \neq 0$. We exclude the possibility that $a_3 = 0$ because, when $a_2 = a_3 = 0$, the origin is a centre for system (1.1) only if $a_7 = 0$.

Theorem 2.2. *Let $\lambda = a_2 = 0$, with $a_3 a_7 \neq 0$. The origin is a centre for system (1.1) if and only if one of the following conditions holds:*

(i) $a_2 = 0$, $k = -(2a_1 + a_3)$, $(a_1 + 2a_3)a_4 + a_1^2(a_1 + a_3) = 0$, $a_5 = -3a_7$, $(a_1+2a_3)a_6 - 2a_1(a_1+a_3)(2a_1+a_3) = 0$, $2(a_1+2a_3)^2 a_7^2 + a_1^3(a_1+a_3)^2(3a_1+2a_3) = 0$;

(ii) $a_2 = 0$, $k = -(a_1 + a_3)$, $2a_3 a_4 + a_1(a_1 + a_3)(a_1 + 3a_3) = 0$, $a_5 = -3a_7$, $2a_3 a_6 - a_1(a_1 + a_3)(3a_1 + 5a_3) = 0$, $4a_3^2 a_7^2 + a_1(a_1 + a_3)^4(a_1 + 2a_3) = 0$.

Proof. When $a_2 = 0$ and $a_3 a_7 \neq 0$ we find that $L(0) = L(1) = \cdots = L(7) = 0$ only if one of the conditions of Theorem 2.2 holds. The sufficiency of these conditions is confirmed by constructing integrating factors from invariant functions. Again we use INVAR to find these functions. When condition (i) holds there exists a Dulac function

$$D = (1 + kx)^{\alpha_1} e^{\alpha_2 x} C^{-3},$$

where

$$C = 1 - a_1 x + \frac{a_1^2(a_1 + a_3)}{(2a_1 + a_3)} x^2 + a_7 xy,$$
$$\alpha_1 = \frac{(a_1 + a_3)^2}{(2a_1 + a_3)^2}, \alpha_2 = -\frac{a_1(a_1 + a_3)}{(2a_1 + a_3)},$$

and hence the origin is a centre. We note that when $2a_1 + a_3 = 0$, then $k = 0$ and condition (v) of Theorem 1.2 is satisfied.

The Dulac function for condition (ii) is somewhat more complicated. It consists of an invariant line, an invariant conic, an invariant degree three curve and an invariant exponential. We have

$$D = (1 + kx)^{\alpha_1} e^{\alpha_2 x} C_1^{\alpha_3} C_2^{\alpha_4}, \tag{2.2}$$

with

$$C_1 = 1 - a_1 x + \frac{2a_3 a_7}{k^2} y + \frac{k\tau(2a_3 - k)}{2a_3} x^2 - \frac{2a_1 a_7}{k} xy + \frac{k^2 \tau}{2a_3} y^2,$$
$$C_2 = 1 + \frac{\Gamma_1}{12a_3^2\gamma^2 w\upsilon} x - \gamma y + \frac{\Gamma_2}{72a_3^5\gamma^2 w^2 \upsilon} x^2 + \frac{\tau\Gamma_3}{8a_3^3\gamma^2 w\upsilon} xy + \frac{\Gamma_4}{12a_3^2\gamma^2\upsilon} y^2$$
$$+ \frac{\tau(2a_3 - k)\Gamma_5}{144a_3^5\gamma^2 w^2 \upsilon} x^3 + \frac{\Gamma_6}{48a_3^5\gamma^2 w^2\upsilon} x^2 y + \frac{\Gamma_7}{36a_3^3\gamma^2 w\upsilon} xy^2 + wy^3,$$
$$\alpha_1 = \frac{9a_3k^2\rho\Phi_0 + 3k^2\rho\Phi_1\gamma - a_3\Phi_2\gamma^2 - 6a_3k^2\Phi_3\gamma^3 - 4a_3^2k^2\rho\gamma^4(\Phi_4 - a_3a_7\gamma^2)}{-48a_3^4\gamma^2w^2(3a_3a_7 + k^2\gamma)},$$
$$\alpha_2 = \frac{9a_3k^3\rho F_0 + 3k^3\rho F_1\gamma + a_3\tau F_2\gamma^2 - 6a_3k^2\tau F_3\gamma^3 - 4a_3^2k^3\rho\gamma^4(F_4 - a_3a_7\gamma^2)}{48a_3^4\gamma^2w^2(3a_3a_7 + k^2\gamma)},$$
$$\alpha_3 = -\frac{6a_3a_7\gamma}{3k^2\rho + 4a_3a_7\gamma}, \alpha_4 = -\frac{3k^2\rho}{3k^2\rho + 4a_3a_7\gamma},$$

where $\rho = a_3^2 - k^2$, $\tau = a_3 + k$ and $\upsilon = 4a_3^2 a_7\gamma - 3k^3\rho$. Here γ, w are roots of

$$4a_3^2\gamma^4 - 36a_1a_3(a_1^2 + a_1a_3 - a_3^2)\gamma^2 + 81a_1^2k^4 = 0,$$
$$64a_3^6w^4 - 16a_1^3a_3^3\big(a_1^2 + a_1a_3 - a_3^2\big) \times \big(a_1^4 - 4a_1^3a_3 - 22a_1^2a_3^2 - 20a_1a_3^3 + a_3^4\big)w^2 + a_1^6k^{12} = 0,$$

respectively and the Γ_i, Φ_i and F_i are as given in the Appendix.

To complete the proof we consider what happens when any of the denominators in the above are zero. When $k\gamma w = 0$, then $a_7 = 0$. When $\upsilon = 0$ then either $a_7 = 0$ or $a_1^2 + 7a_1a_3 + 8a_3^2 = 0$. Let $a_1 = \frac{1}{2}(\sqrt{17} - 7)a_3$. We find a Dulac function that consists of an invariant exponential function and three invariant lines. We have

$$D = (1 + kx)e^{\alpha_1 x}C_1^{\alpha_2}C_2^{\alpha_3},$$

with

$$C_1 = 1 + \frac{(4a_3^2 + \vartheta^2)(5\vartheta^4 + 329a_3^2\vartheta^2 - 4a_3^4)}{2\vartheta^2a_3\delta}x - \vartheta y,$$
$$C_2 = 1 + \frac{\vartheta^4\Phi_1}{16a_3^3\delta\beta}x - ny, \quad \alpha_1 = \frac{\vartheta^2\Phi_2}{4a_3\Phi_3},$$
$$\alpha_2 = -\frac{12a_3^2\beta}{\varpi}, \quad \alpha_3 = \frac{12\vartheta a_3^2\beta}{n\varpi}$$

where $\delta = 812a_3^2 + 33\vartheta^2$, $\beta = 4a_3^4 + 39a_3^2\vartheta^2 - 73\vartheta^4$, $\varpi = 16a_3^6 - 301a_3^2\vartheta^4 + 5\vartheta^6$,

$$n = \frac{4a_3^2\beta}{\vartheta(156a_3^4 + 9a_3^2\vartheta^2 - 5\vartheta^4)},$$

ϑ is a root of $(16a_3^4 - 32a_3^2\vartheta^2 - \vartheta^4)(4a_3^4 - 103a_3^2\vartheta^2 - \vartheta^4) = 0$ and Φ_1, Φ_2, Φ_3 are polynomials of degree six in a_3, ϑ.

When $(3a_3a_7 + k^2\gamma)(3k^2\rho + 4a_3a_7\gamma) = 0$ then either $a_7 = 0$ or $2a_1 + 3a_3 = 0$. Let $a_1 = -\frac{3}{2}a_3$. Then there exists a Dulac function

$$D = (1 + kx)C_1^{-2}C_2^{-1}$$

where

$$C_1 = 1 + \frac{3}{4}a_3x + \frac{4a_7}{a_3}y,$$
$$C_2 = 1 + \frac{3}{2}a_3x - \frac{8a_7}{a_3}y + \frac{9}{16}a_3^2x^2 - 3a_7xy.$$

This completes the proof. □

When none of the conditions of Theorem 2.2 holds and $L(i) = 0$, for $i = 0, 1, 2, \ldots, 6$, then $L(7) \neq 0$; the origin is a fine focus of maximum order seven, at most seven small amplitude limit cycles can be bifurcated from the origin.

3. Sufficient centre conditions

Now we return to the full system and derive some sufficient conditions for the origin to be a centre. We have obtained the necessary and sufficient conditions for the origin to be a centre for three sub-classes of system (1.1); with $k = 0$, with $a_2 = 0$ or with $a_7 = 0$. In these sub-classes we determined possible centre conditions by considering the focal values and then proved that the conditions we had found were sufficient. As the reduction of the focal values in the general

case requires the calculation of some resultants that cannot be obtained with the currently available hardware and software we adopt a different approach. We use the knowledge gained from consideration of the sub-classes to give us an insight into the probable centre conditions in the general case. We search for invariant functions and corresponding integrating factors for the general system without introducing a condition for which the origin may be a centre. The relationships between the coefficients in system (1.1) that must be satisfied to ensure that $\dot{C}_i = C_i L_i$ and (1.4) holds, for $D = \Pi_{i=1}^{n} C_i^{\alpha_i}$, are sufficient conditions for the origin to be a centre. We find three sufficient conditions for the origin to be a centre for system (1.1), with $ka_2a_7 \neq 0$, using this approach.

Knowledge gained from the sub-classes suggests the type of invariant functions we should seek in order to determine integrating factors. In particular, for the Kukles system and the sub-class with $a_7 = 0$ combinations of invariant exponential functions, invariant lines and invariant conics are required. The Dulac functions for the class with $a_2 = 0$ are more complicated and include invariant lines, conics and cubic functions. The line $kx = -1$ is invariant with respect to system (1.1), with $\lambda = 0$, and is included in each Dulac function we seek in the general case. Where the degrees of the equations in the system are not equal it is often found that an exponential function is also required and this is so in all cases here.

We search for functions that are invariant with respect to system (1.1). Both $f(x) = e^x$ and $g(x) = kx+1$ are invariant without any constraints on the coefficients a_i, k. Next we look for functions that are invariant only when some relationships between the coefficients are satisfied. For the three sub-classes we knew from the reduction of the focal values what these relationships were. Here we aim to find the relationships by satisfying $\dot{C}_i = C_i L_i$ and equation (1.4).

We start with the simplest invariant curve, namely a line. Let $C = 1+c_{10}x+c_{01}y$, with cofactor $L = m_{10}x + m_{01}y + m_{20}x^2 + m_{11}xy + m_{02}y^2$. We have $c_{10} = m_{01}$ and $c_{01} = -m_{10}$; then seven equations must be satisfied for $C = 0$ to be invariant with respect to (1.1). We assume that $m_{10} \neq 0$, otherwise we recover the line $kx = -1$. We determine $m_{01}, m_{20}, m_{11}, m_{02}$ in terms of m_{10} and the a_i, k. There are three remaining equations which must be satisfied. At this stage we try to build a Dulac function using this line together with f and g. Five additional equations must hold if $D = g^{\alpha_1} f^{\alpha_2} C^{\alpha_3}$ is a Dulac function that satisfies (1.4). If $a_7 \neq 0$, we must have $\alpha_3 = -3$. Then $m_{10} = \frac{1}{3}a_2$ and the other α_i are given by two of these equations. We have determined all the coefficients of C and L, and the α_i. The four relationships between the coefficients that must hold to satisfy the remaining equations are those of condition (i) of Theorem 3.1 below.

In a similar manner we search for invariant conics. Let $C = 1 + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2$, with cofactor L as above. Again $c_{10} = m_{01}$ and $c_{01} = -m_{10}$. Twelve equations in the remaining eight coefficients of the conic and its cofactor must be satisfied if $C = 0$ is invariant with respect to system (1.1). Five additional equations must hold if $D = g^{\alpha_1} f^{\alpha_2} C^{\alpha_3}$ is a Dulac function that satisfies (1.4). Consideration of all possible situations in which the conic does not reduce to a line, or become the product of two lines, leads us to conclude that either $c_{02} = 0$ or $m_{02} = 2a_7$. Let $c_{02} = 0$. If $a_7 \neq 0$, then $\alpha_3 = -3$, the coefficients of the cofactor are $m_{10} = \frac{1}{3}a_2, m_{01} = -a_1, m_{20} = \frac{1}{3}a_5, m_{11} = -a_1^2 - a_1k - 2a_7 - \frac{2}{9}a_2^2$ and C, α_1, α_2 are as given in the proof of condition (ii) of Theorem 3.1 below. As two of the equations are linearly dependent in this situation this leaves five equations

in the coefficients k, a_i that must be satisfied if $\dot{C} = CL$ and (1.4) holds. These equations lead to precisely the relationships of condition (ii) of Theorem 3.1. When $c_{02} \neq 0$ and $m_{02} = 2a_7$ these same five equations must be satisfied together with an additional equation; this is a specific instance of condition (ii).

We know that, when $a_2 = 0$, there is a Dulac function which consists of powers of f, g, an invariant conic and an invariant cubic curve. We search for this type of Dulac function in the general case. Again the linear coefficients in each invariant curve can be given in terms of the linear coefficients in the corresponding cofactor. We have thirty-five equations in the twenty-four unknowns. In this instance the invariant conic in which $c_{02} \neq 0$ and $m_{02} = 2a_7$ is used. We determine all the coefficients of the invariant conic and its cofactor, in terms of the coefficients of system (1.1), from the twelve equations that must be satisfied for the conic to be invariant with respect to (1.1). This leaves four relationships between the coefficients in (1.1) that must hold.

We then proceed to determine the coefficients of the cubic function and its cofactor. Here there are eighteen equations in twelve unknowns. We eliminate all but two of the unknowns, namely the coefficient of x in the cofactor (say γ) and the coefficient of y^3 in the invariant cubic (say w). One of the remaining equations is quadratic in γ and independent of w. Attempts to eliminate γ from all remaining equations using this equation lead to expressions being generated that result in stack overflow. We turn our attention to the five equations that must be satisfied if (1.4) holds. We find that if $a_7 \neq 0$, then $\alpha_3 = -\frac{3}{2}(\alpha_4 + 1)$ and with α_4 in terms of γ, w and the coefficients k, a_i we must have

$$(a_1a_2 + a_2a_3 + a_5 + 4a_7)(a_1a_2 + a_2a_3 + a_5 + 3a_7)(a_1a_3 + a_3^2 - a_4) = 0.$$

The two remaining equations that must hold to satisfy (1.4) give α_1, α_2. We calculate that $\eta_4 = a_1a_2 + a_2a_3 + a_5 + 3a_7$; $\eta_4 = 0$ is necessary for the origin to be a centre. This, with the four relationships from the requirement for the conic to be invariant, yield condition (iii) of Theorem 3.1 below. We use these relationships to replace k, a_4, a_5, a_6, a_7 in the remaining equations. We note that we have introduced another unknown, r, where $r^2 = a_2^2 + 4a_3^2 - 4k^2$.

We use the quadratic in γ mentioned above to eliminate r and, for consistency, we equate this expression for r with $\sqrt{a_2^2 + 4a_3^2 - 4k^2}$. This consistency condition has

$$\begin{aligned} V =&(a_2^2 + 4a_3^2)\gamma^4 - 3a_2(a_2^2 + 4a_3^2)\gamma^3 \\ &- 9(4a_1^3a_3 - 2a_1^2a_2^2 + 4a_1^2a_3^2 - 4a_1a_2^2a_3 - 4a_1a_3^3 - a_2^2a_3^2)\gamma^2 \\ &- 54a_1a_2(a_1 + a_3)^3\gamma + 81a_1^2(a_1 + a_3)^4 \end{aligned}$$

as a factor. We know from consideration of a specific example that this factor will ultimately lead to an appropriate Dulac function. This is the only remaining equation that is independent of w.

We factorise each of the equations and remove any factors that involve only the remaining coefficients a_1, a_2, a_3; we are able to show that such factors being zero lead to specific instances of conditions that are already known to us. Other than $V = 0$, the simplest of the remaining equations has over 7000 terms. We use a polynomial remainder sequence to eliminate γ (see Section 4 for more details on polynomial remainder sequences). The later stages can only be completed by further simplifying the expressions by replacing a_1 by $-(k + a_3)$, $a_2^2 + 4a_3^2$ by t and scaling

such that $k = 1$. For example, at the second stage of the polynomial remainder sequence, where a quadratic in γ is produced with approximately 30000 terms, the size of the expressions can be almost halved by these changes of variable. We note however that in order to check for factors that can be removed we need to replace t by $a_2^2 + 4a_3^2$ before attempting the factorisation. The calculations are repetitive, but formidable. In some cases, in order to multiply two expressions together, we have to split each expression into smaller units and multiply each unit then sum the results. Near the final stage we produce an expression with 191690 terms, which we need to factorise. Fortunately we can predict that one of the factors will be the coefficient of γ^2 at the quadratic stage of the polynomial remainder sequence, an expression with 9411 terms. There are four other factors, one of which is

$$\begin{aligned}
W =&(a_2^2+4a_3^2)^3w^4 + a_2(a_2^2+4a_3^2)^2(a_2^4+7a_2^2a_3^2-6a_2^2k^2+12a_3^4-24a_3^2k^2\\
&-6a_3k^3+6k^4)w^3 + (-a_2^6a_3^6+6a_2^6a_3^4k^2+6a_2^6a_3^3k^3-3a_2^6a_3^2k^4-6a_2^6a_3k^5\\
&-a_2^6k^6-12a_2^4a_3^8+72a_2^4a_3^6k^2+60a_2^4a_3^5k^3-69a_2^4a_3^4k^4-90a_2^4a_3^3k^5\\
&+9a_2^4a_3^2k^6+36a_2^4a_3k^7+6a_2^4k^8-48a_2^2a_3^{10}+288a_2^2a_3^8k^2+192a_2^2a_3^7k^3\\
&-408a_2^2a_3^6k^4-432a_2^2a_3^5k^5+120a_2^2a_3^4k^6+276a_2^2a_3^3k^7+72a_2^2a_3^2k^8\\
&-12a_2^2a_3k^9-64a_3^{12}+384a_3^{10}k^2+192a_3^9k^3-720a_3^8k^4-672a_3^7k^5\\
&+272a_3^6k^6+528a_3^5k^7+192a_3^4k^8+16a_3^3k^9)w^2\\
&=a_2k^7(a_3+k)^3(6a_2^2a_3^2+3a_2^2a_3k-a_2^2k^2+24a_3^4+12a_3^3k-36a_3^2k^2\\
&-24a_3k^3)w+k^{12}(a_3+k)^6.
\end{aligned}$$

We can show that when $V = W = 0$ all remaining equations are satisfied. We have found an appropriate Dulac function and condition (iii) of Theorem 3.1 is sufficient for the origin to be a centre.

Theorem 3.1. *Let $\lambda = 0$. The origin is a centre for system* (1.1) *if one of the following conditions holds:*

(i) $a_5 = -a_2(a_1+a_3) - 3a_7$,

$$a_6 = \frac{-2a_2^4 - 9a_2^2a_3^2 + 9a_2^2a_3k - 81a_2a_3a_7 + 27a_2a_7k - 162a_7^2}{9a_2^2},$$

$$a_4 = \frac{(-2a_2^4 - 9a_2^2a_3^2 + 9a_2^2a_3k - 54a_2a_3a_7 + 27a_2a_7k - 81a_7^2)\gamma^2}{2a_2^6},$$

$$a_1 = \frac{(-4a_2^4 - 9a_2^2a_3^2 + 9a_2^2a_3k - 54a_2a_3a_7 + 27a_2a_7k - 81a_7^2)\gamma}{2a_2^5},$$

where $\gamma = a_2a_3 + 3a_7$ and $a_2 \neq 0$;

(ii) $a_5 = a_2k - 3a_7,\ k = -(a_1+a_3)$,

$$a_7 = \frac{k^2(a_2k - a_3r)}{a_2^2+4a_3^2},$$

$$a_6 = \frac{k(a_2^2(a_1+3a_3) - 4a_1a_3(3a_1+5a_3)) - 3a_2k^2r}{2(a_2^2+4a_3^2)},$$

$$a_4 = \frac{k(a_2^2(a_1-a_3) + 4a_1a_3(a_1+3a_3)) + a_2k^2r}{2(a_2^2+4a_3^2)},$$

where $r^2 = a_2^2 + 4a_3^2 - 4k^2$ *and* $a_2^2 + 4a_3^2 \neq 0$;

(iii) $a_5 = -a_2(a_1 + a_3) - 3a_7$,

$$a_7 = \frac{9a_1^2(a_1+k) - 2a_2^2(a_1+2a_3) + 9a_4(3a_1+2k)}{12a_2},$$

$$a_6 = \frac{9(18a_4k\epsilon - 2a_1a_2^2\mu + 9a_1(a_1^2(\mu+2k) + a_4(3\mu+4k) + a_1k\epsilon)) - 8a_2^2\delta}{36a_2^2},$$

$$9(16a_2^2 + (9a_1+6k)^2)a_4^2 + 2\rho(27a_1^2 + 18a_1k + 4a_2^2)a_4 + a_1^2\rho^2 = 0,$$

$$\begin{aligned}&9(3a_1+2k)(4a_2^2 + 9\mu(3a_1+2k))a_4^2 + 2\Big(81a_1^2\mu(a_1+k)(3a_1+2k)\\ &+ 36a_1a_2^2(2a_1+k)(a_1+k) - 4a_2^4(a_1+2a_3)\Big)a_4\\ &+ a_1^2\rho\big(2a_2^2(k-a_3) + 9a_1\mu(a_1+k)\big) = 0,\end{aligned}$$

where $\rho = 9a_1^2 + 9a_1k + 2a_2^2$, $\mu = 2a_1 + a_3 + k$, $\epsilon = a_3 + k$, $\delta = a_2^2 + 9a_4$ *and* $a_2 \neq 0$.

Proof. When either condition (i) or (ii) holds we find a Dulac function which consists of the line $kx = -1$, an exponential function and either another line or a conic. The Dulac function then takes the form $D = (1+kx)^{\alpha_1}e^{\alpha_2 x}C^{-3}$, where C and the α_i are given below for each condition. For condition (iii) an invariant line, an invariant conic and an invariant cubic together with an exponential function are needed.

When condition (i) holds, and $k \neq 0$, we find

$$\begin{aligned}C &= 1 + \frac{(a_2a_3 + 3a_7)}{a_2}x - \frac{a_2}{3}y,\\ \alpha_1 &= (2a_2^4 + 54a_2a_7k - 81a_7^2 + 9(a_3^2 - k^2)a_2^2)/9a_2^2k^2,\\ \alpha_2 &= (-2a_2^4 + 27a_2a_7k + 81a_7^2 + 9(k-a_3)a_2^2a_3)/9a_2^2k.\end{aligned}$$

When $k = 0$, condition (iv) of Theorem 1.2 holds.

For condition (ii) we find

$$\begin{aligned}C &= 1 - a_1x - \frac{a_2}{3}y - a_4x^2 - \frac{a_5}{3}xy,\\ \alpha_1 &= (9a_1^2 + 2a_2^2 - 6a_3k + 18a_4 + 6a_6 - 3k^2)/3k^2,\\ \alpha_2 &= -(9a_1^2 + 9a_1k + 2a_2^2 + 18a_4 + 6a_6)/3k,\end{aligned}$$

with $k \neq 0$. When $k = 0$, condition (v) of Theorem 1.2 holds.

The Dulac function required when condition (iii) holds is by some way the most complicated we have encountered. Here, in addition to the invariant exponential function and the line $kx = -1$, we require an invariant conic, $C_1 = 0$, and an invariant cubic, $C_2 = 0$. The Dulac function is

$$D = (1+kx)^{\alpha_1}e^{\alpha_2 x}C_1^{\alpha_3}C_2^{\alpha_4}.$$

The invariant curves are not of high degree but have thousands of terms, and the powers α_i are non-trivial. The expressions are too lengthy to be given here. We note that when $a_2 = 0$, condition (iii) becomes condition (ii) of Theorem 2.2 and the Dulac function reduces to that of equation (2.2). □

4. Focal values

Having established a set of sufficient conditions for the origin to be a centre for system (1.1) with $ka_2a_7 \neq 0$ we endeavour to ascertain if we have found the necessary conditions. If we could find a basis for the focal values for system (1.1) we would be able to determine the necessary and sufficient conditions for the origin to be a centre. However the computations soon become too large for the currently available software and hardware systems. We reduce the focal values as far as is possible and, by using examples, determine whether or not the sufficient conditions we have found are indeed the only conditions for the origin to be a centre. We conjecture that the conditions given in Theorem 3.1 are both necessary and sufficient for the origin to be a centre for system (1.1).

We calculate the focal values up to η_{16} and in order to simplify them we set $a_1 = m - a_3$. We assume throughout this section that $ka_2a_7 \neq 0$. We aim to establish under what conditions the $L(i)$ are zero simultaneously. We have

$$L(1) = a_2m + a_5 + 3a_7.$$

Let $a_5 = -a_2m - 3a_7$. Then $L(1) = 0$ and

$$L(2) = Aa_6 + B,$$

where

$$A = a_2(a_3 + m) - 3a_7$$

and

$$\begin{aligned} B = &- 2a_2^2a_7 + 2a_2a_3^2m - a_2a_3a_4 - 3a_2a_3km - 5a_2a_3m^2 + 2a_2a_4k \\ &+ 5a_2a_4m + 6a_3a_7m - 9a_4a_7 - 9a_7km - 15a_7m^2. \end{aligned}$$

Assume that $A \neq 0$ and let $a_6 = -B/A$. Then

$$\begin{aligned} L(3) &= M_0 + M_1a_4 + M_2a_4^2, \\ L(4) &= N_0 + N_1a_4 + N_2a_4^2 + N_3a_4^3, \\ L(5) &= P_0 + P_1a_4 + P_2a_4^2 + P_3a_4^3 + P_4a_4^4, \\ L(6) &= Q_0 + Q_1a_4 + Q_2a_4^2 + Q_3a_4^3 + Q_4a_4^4 + Q_5a_4^5, \end{aligned}$$

where the M_i, N_i, P_i, Q_i are polynomials in k, a_1, a_2, a_3, a_7.

In this instance calculating resultants to eliminate a_4 is not feasible because of the degrees to which the variables occur and the number of terms in the polynomials involved. We employ a polynomial remainder sequence approach, the main advantage being that we can work with the individual coefficients of the variable being eliminated rather than the entire polynomial. Also factors of the reduced polynomials can be removed at each stage in the process and some such factors can be predicted. We use the following result to establish what these factors are.

Lemma 4.1. *Suppose we have two univariate polynomials α_1, α_2. We can determine a sequence of polynomials $\alpha_3, \dots, \alpha_j$, of decreasing degree, such that*

$$\text{remainder}(\epsilon_i^{\delta_{i-1}+1}\alpha_{i-1}, \alpha_i) = \beta_{i+1}\alpha_{i+1}; \quad i \geqslant 2,$$

where δ_i is the difference in degrees between α_i and α_{i+1}; ϵ_i is the leading coefficient of α_i; $\beta_3 = 1, \beta_{i+1} = \epsilon_{i-1}^{\delta_{i-2}+1}$. Hence we have that β_{i+1} divides the pseudo remainder of α_{i-1} and α_i.

The Proof of the above lemma can be found in [4].

Assume that $M_2 \neq 0$. Let $a_4^2 = -(M_0 + M_1 a_4)/M_2$ such that $L(3) = 0$. Then

$$L(4) = A^2(\rho_0 + \rho_1 a_4),$$
$$L(5) = A^3(\nu_0 + \nu_1 a_4),$$
$$L(6) = A^4(\tau_0 + \tau_1 a_4),$$

where the ρ_i, ν_i, τ_i are polynomials in m, a_2, a_3, a_7, k. In particular ρ_0, ρ_1 are polynomials with $936,654$ terms respectively.

Assume that $\rho_1 \neq 0$ and let $a_4 = -\rho_0/\rho_1$. Then

$$L(3) = M_2^2 A^2 a_7 \digamma \Omega,$$
$$L(5) = M_2^2 A a_7 \digamma \Gamma,$$
$$L(6) = M_2^2 A a_7 \digamma \Phi,$$

where

$$\begin{aligned}\digamma =& -81a_2a_7^2k - 54a_2^2a_3a_7k - 9a_2^3a_3^2k + 243a_7^3 + 12a_2^4a_7 \\ &+ 2a_2^5a_1 + 243a_2a_3a_7^2 + 4a_2^5a_3 + 81a_2^2a_3^2a_7 + 9a_2^3a_3^3\end{aligned}$$

and Ω, Γ, Φ are polynomials in m, a_2, a_3, a_7.

When $\digamma = 0$, the focal values $\eta_8, \dots, \eta_{14}$ have a common factor

$$\begin{aligned}\Psi =& 2a_2^6a_3^2 + 2a_2^6a_4 + 12a_2^5a_3a_7 + 9a_2^4a_3^4 - 9a_2^4a_3^3k + 18a_2^4a_7^2 + 108a_2^3a_3^3a_7 \\ &- 81a_2^3a_3^2a_7k + 486a_2^2a_3^2a_7^2 - 243a_2^2a_3a_7^2k + 972a_2a_3a_7^3 - 243a_2a_7^3k + 729a_7^4.\end{aligned}$$

Let

$$a_1 = \frac{(-4a_2^4 - 9a_2^2a_3^2 + 9a_2^2a_3k - 54a_2a_3a_7 + 27a_2a_7k - 81a_7^2)\gamma}{2a_2^5},$$
$$a_4 = \frac{(-2a_2^4 - 9a_2^2a_3^2 + 9a_2^2a_3k - 54a_2a_3a_7 + 27a_2a_7k - 81a_7^2)\gamma^2}{2a_2^6},$$

where $\gamma = a_2a_3 + 3a_7$. Then $\digamma = \Psi = 0$ and

$$a_6 = \frac{-2a_2^4 - 9a_2^2a_3^2 + 9a_2^2a_3k - 81a_2a_3a_7 + 27a_2a_7k - 162a_7^2}{9a_2^2}.$$

These, together with $a_5 = -a_2m - 3a_7$, are condition (i) of Theorem 3.1; the origin is a centre for system (1.1). We note that in the special case when $A = B = 0$ this condition is still satisfied and there are no other conditions with $ka_2a_7 \neq 0$.

The polynomials Ω, Γ, Φ have $2294, 2895$ and 7674 terms respectively. The degrees to which each of the remaining variables occur in Ω, Γ, Φ are as shown in the following table:

	a_2	a_3	m	a_7	k
Ω	12	13	19	11	11
Γ	13	14	20	12	12
Φ	18	19	25	15	16

Clearly any further progress in the reduction of the focal values is going to be difficult, if not impossible, but we note that $\Omega = \Gamma = \Phi = 0$ if either of the conditions (ii) or (iii) of Theorem 3.1 holds.

Suppose that we could calculate the resultants of Ω, Γ and Ω, Φ with respect to a_3. Any common factor of the leading coefficients of a_3 in $\Omega,\ \Gamma,\ \Phi$ will be a factor

of the resultants, but this common factor being zero may not be sufficient for the vanishing of the polynomials. We have that $a_2a_7m(k+m)$ is such a factor. In particular, when $k=-m$, we find

$$\Upsilon_1 = (a_2^2+4a_3^2)a_7^2 + 2a_2m^3a_7 + a_1m^4(a_1+2a_3)$$

is a common factor of Ω, Γ and Φ. Let

$$\omega = a_7^2 = \frac{-2a_2m^3a_7 - a_1m^4(a_1+2a_3)}{a_2^2+4a_3^2}.$$

Then $L(3) = q_0 + q_1a_7$. Assume $q_1 \neq 0$ and let $a_7 = -q_0/q_1$. For consistency we must have $\Upsilon = q_1^2\omega - q_0^2 = 0$. We find

$$\Upsilon_2 = (a_2^2+4a_3^2)a_4^2 + (4a_1a_3(a_1+3a_3)+a_2^2(a_1-a_3))ma_4 + a_1m^2(a_1(a_1+3a_3)^2 - a_2^2a_3)$$

is a common factor of $L(4)$ and Υ. Now $k=-m$, $a_5 = -a_2m - 3a_7$, $a_6 = -B/A$, $\Upsilon_1 = \Upsilon_2 = 0$ is precisely condition (iii) of Theorem 3.1; the origin is a centre.

We have seen how conditions (i) and (iii) of Theorem 3.1 emerge from the reduction of the focal values. However we are unable to locate condition (ii) by a similar argument. It is possible that conditions (ii) and (iii) are specific instances of more general conditions. We show that this is not the case by considering a particular example. Each of these conditions has five relationships between the eight coefficients a_i, k. We can choose values for three of the variables without imposing new relationships.

Let $a_1 = 1$, $a_2 = 1$, $a_3 = -2$. Now Ω, Γ, Φ are polynomials in a_7, k, and a_4, a_5, a_6 are given in terms of a_7, k also. Let $R(f,g,x)$ denote the resultant of f and g with respect to x and # represent a (large) integer. We calculate resultants with respect to a_7 and find

$$R(\Omega,\Gamma,a_7) = \#(k-1)^3(2k-3)^6(k^2+6k+10)^4\phi K_1K_2,$$
$$R(\Omega,\Phi,a_7) = \#(k-1)^3(2k-3)^4(k^2+6k+10)^6\phi K_1K_3,$$

where $\phi = 162k^3 + 152k^2 - 28k - 84$ and K_1, K_2, K_3 are irreducible polynomials of degrees $52, 64, 95$ in k, respectively. When $K_1 = 0$, then $\rho_0 = \rho_1 = 0$ and when $2k-3=0$, then $a_7 = 0$; both situations are excluded under current assumptions. The leading coefficients of a_7 in Ω, Γ, Φ have $k^2+6k+10$, which is positive definite, as a common factor. For the general case this factor is $a_2^2 + (a_1 - a_3 + k)^2$, which is non-zero when $a_2 \neq 0$. Clearly K_2, K_3 cannot be zero simultaneously. So we must have $k = 1 (= -m)$ or $\phi = 0$; the former is true when condition (iii) holds, the latter when condition (ii) holds. Furthermore there are no other centre conditions with five or fewer relationships that satisfy $\Omega = \Gamma = \Phi = 0$.

After extensive consideration of all the possible situations in which the Liapunov quantities up to $L(6)$ are zero simultaneously we have not found any centre conditions other than those given in Theorems 1.2, 2.1, 2.2 and 3.1. However there are several instances where we are unable to calculate resultants or eliminate variables using a polynomial remainder sequence, so we cannot be certain that we have a complete set of necessary conditions. We have shown, by considering an example, that when $\Omega = \Gamma = \Phi = 0$ there are no other such conditions which contain five or fewer relationships between the coefficients. We can also demonstrate that when one of k, a_2 or a_7 is zero in Ω, Γ, Φ then all the centre conditions found, by considering $\Omega = \Gamma = \Phi = 0$, can be obtained from one of the conditions in Theorem 3.1

with the appropriate variable set to zero. This wealth of evidence leads us to the following claim.

Conjecture 4.2. *The origin is a centre for system* (1.1) *when* $ka_2a_7 \neq 0$ *if and only if one of the conditions of Theorem 3.1 holds.*

5. Coexisting fine foci

One of the significant features of a planar dynamical system is the possible configuration of limit cycles. Information is sought on the number of critical points that can be encircled by closed orbits and on the number of closed orbits that can encircle one such critical point. This may be phrased as asking how many nests of limit cycles can there be and how many limit cycles make up each nest.

Suppose that $\lambda = 0$ in system (1.1), so that the origin is a fine focus. From the equation for $\dot{x}$, critical points can occur only on the x-axis and on $kx = -1$. However $kx = -1$ is invariant, so any critical point on it cannot be of focus type. Thus any fine focus must have $y = 0$ and $x(a_4x^2 + a_1x - 1) = 0$. The condition for a fine point is $x(a_5x + a_2) = 0$. Thus only one critical point other than the origin can be a fine focus.

Lemma 5.1. *System* (1.1) *with* $\lambda = 0$ *can have at most two fine foci.*

Suppose that there are two fine foci. We can scale coordinates so that they are $(0, 0)$ and $(1, 0)$. Then $a_4 = 1 - a_1, a_5 = -a_2$ and the system is

$$\begin{aligned} \dot{x} &= y(1 + kx), \\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + (1 - a_1)x^3 - a_2x^2y + a_6xy^2 + a_7y^3. \end{aligned} \tag{5.1}$$

The point $(1, 0)$ is a fine focus, as opposed to a fine col, if $(k + 1)(a_1 - 2) > 0$. We denote the Liapunov quantities associated with the origin by $L(i)$ and those for the point $(1, 0)$ by $M(i)$.

Theorem 5.2. *Suppose that the origin and* $(1, 0)$ *coexist as fine foci in system* (5.1). *If* $(1, 0)$ *is of order one then the origin is of order at most six.*

Proof. We calculate that

$$L(1) = a_2(a_1 + a_3 - 1) + 3a_7, \tag{5.2}$$

$$M(1) = 3(a_1 - 2)^2a_7 + a_2((2 - a_1)a_6 + (a_1 - 1)(k - a_3 + 1) + a_3). \tag{5.3}$$

For $(1, 0)$ to be a fine focus of order one we must ensure that $M(1) \neq 0$. Let $a_7 = \frac{1}{3}a_2(1 - a_1 - a_3)$, then $L(1) = 0$ and

$$L(2) = a_2(3a_6(2a_1 + 3a_3 - 1) + \varphi),$$

where

$$\begin{aligned} \varphi =& 15a_1^3 + 24a_1^2a_3 + 9a_1^2k - 39a_1^2 + 2a_1a_2^2 + 9a_1a_3^2 + 9a_1a_3k - 45a_1a_3 \\ & - 15a_1k + 33a_1 + 2a_2^2a_3 - 2a_2^2 - 9a_3^2 - 9a_3k + 21a_3 + 6k - 9. \end{aligned}$$

If $a_2 = 0$, then $a_5 = a_7 = 0$ and the origin is a centre by Theorem 2.1 (i). We require $a_2 \neq 0$ for the origin to be a fine focus of order greater than two. If $2a_1 + 3a_3 = 1$ then $\varphi = a_2\Phi$, where

$$\Phi = (a_1 - 2)(2a_2^2 + 9(a_1 - 1)^2) + 9(k + 1)(a_1 - 1)^2, \tag{5.4}$$

which is non-zero if $a_2 \neq 0$ and $(1, 0)$ is a fine focus.

So, for the origin to be a fine focus of order greater than two, we also need $2a_1+3a_3\neq 1$. Let $a_6=\frac{\varphi}{3(1-2a_1-3a_3)}$. Now we require $M(1)=a_2(a_1+a_3-1)\Phi$ to be non-zero and we have

$$L(3)=M(1)(\Gamma-2a_2^2(a_1+a_3-1)),$$

where

$$\begin{aligned}\Gamma=&3(25a_1^3+97a_1^2a_3+7a_1^2k-32a_1^2+117a_1a_3^2+22a_1a_3k-88a_1a_3\\&-4a_1k+14a_1+45a_3^3+15a_3^2k-54a_3^2-8a_3k+22a_3-2).\end{aligned}$$

Let $a_2^2=\frac{\Gamma}{2(a_1+a_3-1)}$. Then

$$L(4)=a_2(Ak^2+Bk+C)\Omega,$$

where

$$\begin{aligned}A&=15(a_1+a_3)^2-20(a_1+a_3)+8>0,\\B&=15(a_1+a_3)^2(11a_1+13a_3)-5(a_1+a_3)(59a_1+67a_3)+4(45a_1+49a_3-8),\\C=&90(a_1+a_3)^3(4a_1+5a_3)-45(a_1+a_3)^2(19a_1+23a_3)\\&+30(a_1+a_3)(25a_1+29a_3)-8(35a_1+38a_3-5),\\\Omega=&15(a_1-2)a_3^2+(a_1-2)(29a_1+5k-13)a_3+14a_1^3+5a_1^2k-40a_1^2\\&-11a_1k+28a_1+3k-7.\end{aligned}$$

When a_3 is either root of $\Omega=0$, we have $a_2^2<0$. We require a_2 to be real, so $\Omega\neq 0$. As A is non-zero let $k=\frac{(-B+r)}{2A}$, where

$$r=\sqrt{B^2-4AC}. \tag{5.5}$$

Then

$$L(5)=a_2\Omega(\alpha r+\beta),$$

where α,β are polynomials of degrees nine and twelve in a_1,a_3. Assume for the time being that $\alpha\neq 0$. Let $r=-\frac{\beta}{\alpha}$. Then

$$L(6)=a_2\Omega(3a_1+3a_3-2)^2\Theta,$$

where Θ is a polynomial of degree sixteen in a_1 and a_3. For consistency, r as given by (5.5) must also be equal to $-\dfrac{\beta}{\alpha}$. This is true if $(3a_1+3a_3-2)\Lambda=0$, where Λ is a polynomial of degree twelve in a_1,a_3. When $3a_1+3a_3=2$ then $k=-1$, and $(1,0)$ is not a fine focus. The origin can be a fine focus of order greater than six only if $\Theta=\Lambda=0$. We calculate the resultants of Θ and Λ with respect to a_1 and a_3. We find

$$\begin{aligned}R(\Theta,\Lambda,a_3)=&\#(a_1-2)^3(a_1-1)^3(a_1-5)^2(2a_1-1)(6a_1^3-58a_1^2+183a_1-191)\\&\times(8a_1^2-41a_1+53)(27a_1^4-180a_1^3+478a_1^2-620a_1+343)^2F\Upsilon,\end{aligned}$$

where F,Υ are polynomials of degrees eighteen and forty in a_1. The quadratic and degree four factors are positive definite. The one real root of the cubic factor, together with the corresponding value for a_3, are such that $a_1+a_3=1$, which is excluded if $(1,0)$ is a fine focus of order one. When $a_1=\frac{1}{2}$ (or $a_1=5$) then $a_3=0$ (or $a_3=-3$) and $2a_1+3a_3=1$, which is also excluded. Similarly, when $a_1=1$ or $a_1=2$ then $(k+1)(a_1-2)\leq 0$.

Using Sturm sequences [13] we find that $\Upsilon = 0$ has six distinct real roots. We also locate the corresponding roots a_3, and find that all the root pairs are such that $(k+1)(a_1-2) \leq 0$. When $F = 0$, we have $\alpha = \beta = 0$. A similar analysis to that for the case $\alpha \neq 0$ leads us to conclude that, when $\alpha = 0$, the origin can be of maximum order five when $(1,0)$ is of order one.

We conclude that $L(6) \neq 0$ under current assumptions; the origin is of maximum order six and at most seven small amplitude limit cycles can be bifurcated simultaneously from the two fine foci. □

Theorem 5.3. *Suppose that the origin and* $(1,0)$ *coexist as fine foci for system* (5.1). *If* $(1,0)$ *is of order greater than one then both the fine foci are of maximum order two, or both are centres.*

Proof. We have $L(1), M(1)$ given by (5.2), (5.3) respectively. Again we make a substitution for a_7 from $L(1) = 0$. Then

$$M(1) = a_2(a_6(2-a_1) + \phi),$$

where

$$\phi = (1-a_1)(a_1^2 + a_1a_3 - 4a_1 - 2a_3 - k + 3).$$

If $(1,0)$ is a fine focus then $a_1 \neq 2$. Let $a_6 = \frac{\phi}{a_1-2}$. Then

$$L(2) = M(2) = a_2(a_1-2)(a_1+a_3-1)\Phi,$$

where Φ is given by (5.4). If $a_2 = 0$, then $a_5 = a_7 = 0$ and both critical points are centres by Theorem 2.1 (i). Similarly, when $a_1 + a_3 = 1$ then $a_4 = a_3, a_6 = \frac{a_3(k-a_3)}{a_3+1}, a_7 = 0$ and both critical points are centres by Theorem 2.1 (iv). If $a_2 \neq 0$ and $(1,0)$ is a fine focus then $(a_1-2)\Phi \neq 0$. If $L(2)$ is non-zero, so is $M(2)$. Hence both points are fine foci of maximum order two or they are both centres. □

We can demonstrate that four small amplitude limit cycles can be bifurcated from the two fine foci of order two of system (1.1). We begin with system (1.1) with

$$\lambda = 0, \quad a_4 = 1 - a_1, \quad a_5 = -a_2, \quad a_7 = -\frac{1}{3}a_2(a_1+a_3-1), \quad a_6 = \frac{\phi}{a_1-2}.$$

Hence the origin and $(1,0)$ are fine foci of order two, and $L(0) = M(0) = L(1) = M(1) = 0, M(2) = L(2) \neq 0$. First we perturb a_6 such that $M(1)$ becomes non-zero and of opposite sign to $M(2)$. If $(2-a_1)(a_1+a_3-1) > 0$ we decrease a_6, otherwise we increase a_6. The stability of $(1,0)$ is reversed and a limit cycle bifurcates. Next we perturb a_7 such that $L(1)$ becomes non-zero and of opposite sign to $L(2)$, so reversing the stability of the origin. If $a_2(a_1+a_3-1) > 0$ then we decrease a_7, otherwise we increase a_7. A second limit cycle bifurcates, but this time from the origin. Another limit cycle can be bifurcated from the origin, by increasing λ if $a_2(a_1+a_3-1) > 0$ or decreasing λ otherwise. To bifurcate a fourth limit cycle from $(1,0)$ we require the stability of $(1,0)$ to be reversed, so then it has the same stability as the origin. Hence we require $\lambda M(1) < 0$, which is the case when $M(1)M(2) < 0$.

Similarly seven small amplitude limit cycles can be bifurcated from the two fine foci when one is of order one and the other is of order six.

6. Large amplitude limit cycles

We have proved that the origin can be a fine focus of order seven and we have investigated the possibility of small amplitude limit cycles bifurcating from two coexisting fine foci. In this case we found that the maximum number of small amplitude limit cycles that can exist simultaneously is seven. By considering the global phase portrait within a particular parameter range, we shall demonstrate that a large amplitude limit cycle can surround two fine foci in system (1.1). It is known that the Kukles system, with two fine foci of order two, can have a large amplitude limit cycle surrounding both critical points [14].

Theorem 6.1. *If the fine foci at the origin and* $(1,0)$ *are both of order two for system* (1.1) *then at least five limit cycles exist under certain conditions.*

Proof. We begin with two fine foci, each of maximal order two. Therefore system (1.1) is of the form

$$\begin{aligned}\dot{x} &= y\,(1+kx)\,,\\ \dot{y} &= -x + a_1x^2 + a_2xy + (1-a_1+\delta)y^2 + (1-a_1)x^3 - a_2x^2y + Axy^2 - \frac{a_2\delta}{3}y^3,\end{aligned} \tag{6.1}$$

where

$$\delta = a_1 + a_3 - 1, \quad A = (1-a_1)\Big(\delta - \frac{(a_1+k-1)}{(a_1-2)}\Big), \quad (k+1)(a_1-2) > 0.$$

One can consider system (6.1) as a perturbation of the system

$$\begin{aligned}\dot{x} &= y\,(1+kx)\,,\\ \dot{y} &= -x + a_1x^2 + (1-a_1+\delta)y^2 + (1-a_1)x^3 + Axy^2,\end{aligned} \tag{6.2}$$

with the introduction of the term

$$\beta = -a_2y\Big((x-\frac{1}{2})^2 + \frac{\delta}{3}y^2 - \frac{1}{4}\Big),$$

when a_2 is perturbed from zero. System (6.2) has centres at the origin and $(1,0)$, and a col at $(\frac{1}{a_1-1},0)$. We consider a particular global phase portrait of system (6.2). We can arrange for there to be no critical points on the line $kx=-1$ by choosing values of the parameters such that

$$k(a_1-2)(\delta-a_1) - k + a_1^2(\delta-1) + a_1(2-3\delta) + 2\delta - 1 \geq 0.$$

For example, we can take $k=1, a_1=3, \delta=3$. We then use polar coordinates to consider the critical points at infinity. Provided that $A \leq 0$, that is $\delta \geq \frac{a_1+k-1}{a_1-2}$, the only critical points at infinity lie on $\theta = \pm\frac{\pi}{2}$.

At infinity $\dot{\theta} \leq 0$, so the motion is clockwise in the region $kx > -1$ and the outward separatrices of the col cannot tend to a critical point at infinity. The system is symmetric in the x-axis, so the separatrices form homoclinic loops and the orbits outside the 'figure of eight' so formed are closed. Take one of these closed orbits; Γ, say. Increase a_2 so that system (6.2) becomes system (6.1). For system (6.1) the flow is inwards across Γ because the vector product of the two fields is

$$-y^2(1+kx)a_2\Big((x-\frac{1}{2})^2 + \frac{\delta}{3}y^2 - \frac{1}{4}\Big).$$

If $a_2 > 0$, the two fine foci are both unstable and hence there is a large amplitude limit cycle inside Γ.

In the previous section we demonstrated that a total of four small amplitude limit cycles can be bifurcated from the origin and $(1,0)$ simultaneously. Our current assumptions are consistent with the argument therein. Therefore system (1.1) can have five limit cycles. □

This leads us to the following conjecture.

Conjecture 6.2. *When system* (1.1) *has two fine foci of orders six and one than at least eight limit cycles can exist.*

Concluding remarks. We have presented various properties of system (1.1). In particular we have found sufficient conditions for the origin to be a centre by finding complicated invariant functions that can be combined to form a Dulac function. We conjecture that we have found the necessary and sufficient conditions for the origin to be a centre for system (1.1) even though we were unable to complete the reduction of the focal values because of the size of the expressions generated. We also proved some results on the possible configurations of limit cycles.

7. Appendix

The polynomials required in the proof of condition (ii) of Theorem 2.2 are as follows:

$$\begin{aligned}
\Phi_0 =& a_3^4a_7k^4 + 2a_3^3a_7k^5 + 2a_3^3k^4w + 4a_3^2a_7w^2 + 2a_3^2k^5w \\
&- 2a_3a_7k^7 - 2a_3k^6w - a_7k^8 - 2k^7w, \\
\Phi_1 =& 2a_3^6k^4 + 20a_3^5a_7w + 2a_3^5k^5 - 5a_3^4k^6 + 12a_3^4w^2 - 24a_3^3a_7k^2w - 6a_3^3k^7 \\
&- 12a_3^3kw^2 - 4a_3^2a_7k^3w + 2a_3^2k^8 + 4a_3k^9 + k^{10}, \\
\Phi_2 =& - 4a_3^8a_7k^2 - 12a_3^7k^2w + 16a_3^6a_7k^4 + 12a_3^6k^3w + 4a_3^5a_7k^5 + 30a_3^5k^4w \\
&- 21a_3^4a_7k^6 - 72a_3^4a_7w^2 - 24a_3^4k^5w - 10a_3^3a_7k^7 + 216a_3^3a_7kw^2 \\
&- 24a_3^3k^6w + 8a_3^2a_7k^8 + 12a_3^2k^7w + 6a_3a_7k^9 + 6a_3k^8w + a_7k^{10}, \\
\Phi_3 =& a_3^5k^4 - 8a_3^4a_7w + a_3^4k^5 - 2a_3^3k^6 + 8a_3^3w^2 + 8a_3^2a_7k^2w \\
&- 2a_3^2k^7 + a_3k^8 + k^9, \\
\Phi_4 =& 2a_3^3a_7 + 3a_3^2w - 3a_3a_7k^2 - 3a_3kw - a_7k^3,
\end{aligned}$$

$$\begin{aligned}
F_0 =& a_3^4a_7k^4 + 2a_3^3a_7k^5 + 2a_3^3k^4w + 4a_3^2a_7w^2 + 2a_3^2k^5w - 2a_3a_7k^7 - 2a_3k^6w \\
&- a_7k^8 - 2k^7w, \\
F_1 =& 2a_3^6k^4 + 20a_3^5a_7w + 2a_3^5k^5 + 12a_3^4a_7kw - 5a_3^4k^6 + 12a_3^4w^2 - 12a_3^3a_7k^2w \\
&- 6a_3^3k^7 - 4a_3^2a_7k^3w + 2a_3^2k^8 + 4a_3k^9 + k^{10}, \\
F_2 =& 4a_3^7a_7k^3 - 4a_3^6a_7k^4 + 12a_3^6k^3w - 12a_3^5a_7k^5 - 12a_3^5k^4w + 8a_3^4a_7k^6 \\
&+ 144a_3^4a_7w^2 - 18a_3^4k^5w + 13a_3^3a_7k^7 - 216a_3^3a_7kw^2 + 12a_3^3k^6w \\
&- 3a_3^2a_7k^8 + 6a_3^2k_7w - 5a_3a_7k^9 - a_7k^{10}, \\
F_3 =& a_3^4k^5 - 8a_3^3a_7kw + 4a_3^3w^2 + 8a_3^2a_7k^2w - 2a_3^2k^7 + k^9, \\
F_4 =& 2a_3^3a_7 + 3a_3^2w - 3a_3a_7k^2 - a_7k^3,
\end{aligned}$$

$$\Gamma_1 = 96a_3^6a_7k^3\gamma^2 - 32a_3^6k^4\gamma^3 + 12a_3^5a_7k^4\gamma^2 + 144a_3^5a_7\gamma^3w - 9a_3^5k^7\gamma$$

$$
\begin{aligned}
&- 108a_3^5k^3\gamma^2w - 243a_3^4a_7k^7 - 282a_3^4a_7k^5\gamma^2 - 36a_3^4a_7k^3\gamma w + 72a_3^4k^8\gamma \\
&+ 98a_3^4k^6\gamma^3 + 12a_3^4k^4\gamma^2w - 486a_3^3a_7k^8 - 114a_3^3a_7k^6\gamma^2 + 180a_3^3k^9\gamma \\
&+ 34a_3^3k^7\gamma^3 + 108a_3^3k^5\gamma^2w + 186a_3^2a_7k^7\gamma^2 + 36a_3^2a_7k^5\gamma w + 18a_3^2k^{10}\gamma \\
&- 66a_3^2k^8\gamma^3 - 12a_3^2k^6\gamma^2w + 486a_3a_7k^{10} + 102a_3a_7k^8\gamma^2 - 171a_3k^{11}\gamma \\
&- 34a_3k^9\gamma^3 + 243a_7k^{11} - 90k^{12}\gamma,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_2 =&512a_3^{11}a_7k^4\gamma^3 - 384a_3^{11}k^7\gamma^2 - 36a_3^{10}a_7k^7\gamma + 2016a_3^{10}a_7k^3\gamma^2w \\
&- 96a_3^{10}k^8\gamma^2 - 672a_3^{10}k^4\gamma^3w - 1152a_3^9a_7k^8\gamma - 2624a_3^9a_7k^6\gamma^3 \\
&+ 228a_3^9a_7k^4\gamma^2w + 768a_3^9a_7\gamma^3w^2 + 837a_3^9k^{11} + 2256a_3^9k^9\gamma^2 \\
&- 225a_3^9k^7\gamma w + 96a_3^9k^5\gamma^3w - 576a_3^9k^3\gamma^2w^2 - 2106a_3^8a_7k^9\gamma \\
&- 1052a_3^8a_7k^7\gamma^3 - 5211a_3^8a_7k^7w - 5958a_3^8a_7k^5\gamma^2w - 540a_3^8a_7k^3\gamma w^2 \\
&+ 1917a_3^8k^{12} + 1206a_3^8k^{10}\gamma^2 + 1575a_3^8k^8\gamma w + 2058a_3^8k^6\gamma^3w \\
&+ 180a_3^8k^4\gamma^2w^2 + 2556a_3^7a_7k^{10}\gamma + 4164a_3^7a_7k^8\gamma^3 - 10071a_3^7a_7k^8w \\
&- 2316a_3^7a_7k^6\gamma^2w + 1368a_3^7a_7k^4\gamma w^2 + 792a_3^7a_7k^2\gamma^3w^2 - 288a_3^7a_7\gamma^2w^3 \\
&- 1917a_3^7k^{13} - 4572a_3^7k^{11}\gamma^2 + 3726a_3^7k^9\gamma w + 402a_3^7k^7\gamma^3w \\
&- 783a_3^7k^7w^2 - 126a_3^7k^5\gamma^2w^2 + 216a_3^7k^3\gamma w^3 + 8226a_3^6a_7k^{11}\gamma \\
&+ 2936a_3^6a_7k^9\gamma^3 + 702a_3^6a_7k^9w + 4008a_3^6a_7k^7\gamma^2w + 2484a_3^6a_7k^5\gamma w^2 \\
&- 168a_3^6a_7k^3\gamma^3w^2 - 7695a_3^6k^{14} - 3462a_3^6k^{12}\gamma^2 - 216a_3^6k^{10}\gamma w \\
&- 1488a_3^6k^8\gamma^3w - 1458a_3^6k^8w^2 - 54a_3^6k^6\gamma^2w^2 + 3528a_3^5a_7k^{12}\gamma \\
&- 1672a_3^5a_7k^{10}\gamma^3 + 10422a_3^5a_7k^{10}w + 2088a_3^5a_7k^8\gamma^2w + 576a_3^5a_7k^6\gamma w^2 \\
&- 2187a_3^5k^{15} + 3600a_3^5k^{13}\gamma^2 - 3861a_3^5k^{11}\gamma w - 480a_3^5k^9\gamma^3w \\
&+ 108a_3^5k^9w^2 + 702a_3^5k^7\gamma^2w^2 - 216a_3^5k^5\gamma w^3 - 4806a_3^4a_7k^{13}\gamma \\
&- 1884a_3^4a_7k^{11}\gamma^3 + 4509a_3^4a_7k^{11}w - 66a_3^4a_7k^9\gamma^2w + 8667a_3^4k^{16} \\
&+ 3690a_3^4k^{14}\gamma^2 - 1377a_3^4k^{12}\gamma w + 102a_3^4k^{10}\gamma^3w + 1458a_3^4k^{10}w^2 \\
&- 126a_3^4k^8\gamma^2w^2 - 4932a_3^3a_7k^{14}\gamma - 380a_3^3a_7k^{12}\gamma^3 - 351a_3^3a_7k^{12}w \\
&+ 6777a_3^3k^{17} - 588a_3^3k^{15}\gamma^2 + 360a_3^3k^{13}\gamma w - 18a_3^3k^{11}\gamma^3w + 675a_3^3k^{11}w^2 \\
&- 1278a_3^2a_7k^{15}\gamma - 1917a_3^2k^{18} - 1338a_3^2k^{16}\gamma^2 + 18a_3^2k^{14}\gamma w - 3510a_3k^{19} \\
&- 312a_3k^{17}\gamma^2 - 972k^{20},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_3 =&192a_3^8k^4\gamma^2 - 576a_3^7a_7\gamma^2w - 192a_3^7k^5\gamma^2 - 64a_3^6a_7k^3\gamma^3 + 576a_3^6a_7k\gamma^2w \\
&- 432a_3^6k^8 - 780a_3^6k^6\gamma^2 + 56a_3^5a_7k^4\gamma^3 + 1296a_3^5a_7k^4w + 576a_3^5a_7k^2\gamma^2w \\
&- 432a_3^5k^9 + 390a_3^5k^7\gamma^2 + 48a_3^5k^3\gamma^3w + 162a_3^4a_7k^7\gamma + 132a_3^4a_7k^5\gamma^3 \\
&+ 1296a_3^4a_7k^5w - 48a_3^4a_7k^3\gamma^2w + 1323a_3^4k^{10} + 1134a_3^4k^8\gamma^2 \\
&- 32a_3^4k^4\gamma^3w + 162a_3^3a_7k^8\gamma - 56a_3^3a_7k^6\gamma^3 + 48a_3^3a_7k^4\gamma^2w + 1782a_3^3k^{11} \\
&- 54a_3^3k^9\gamma^2 - 16a_3^3k^5\gamma^3w - 162a_3^2a_7k^9\gamma - 68a_3^2a_7k^7\gamma^3 - 432a_3^2k^{12} \\
&- 546a_3^2k^{10}\gamma^2 - 162a_3a_7k^{10}\gamma - 1350a_3k^{13} - 144a_3k^{11}\gamma^2 - 459k^{14},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_4 =&128a_3^6a_7\gamma^3 - 96a_3^6k^3\gamma^2 + 36a_3^5a_7k^3\gamma - 12a_3^5k^4\gamma^2 - 288a_3^4a_7k^4\gamma \\
&- 264a_3^4a_7k^2\gamma^3 - 48a_3^4a_7\gamma^2w + 243a_3^4k^7 + 282a_3^4k^5\gamma^2 + 36a_3^4k^3\gamma w \\
&- 684a_3^3a_7k^5\gamma - 136a_3^3a_7k^3\gamma^3 + 486a_3^3k^8 + 114a_3^3k^6\gamma^2 - 360a_3^2a_7k^6\gamma \\
&- 186a_3^2k^7\gamma^2 - 36a_3^2k^5\gamma w - 486a_3k^{10} - 102a_3k^8\gamma^2 - 243k^{11},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_5 =&512a_3^{10}a_7k^4\gamma^3 - 96a_3^9k^8\gamma^2 - 480a_3^9k^4\gamma^3w - 1116a_3^8a_7k^8\gamma \\
&- 2624a_3^8a_7k^6\gamma^3 + 1020a_3^8a_7k^4\gamma^2w + 192a_3^8a_7\gamma^3w^2 - 96a_3^8k^9\gamma^2 \\
&+ 96a_3^8k^5\gamma^3w - 2232a_3^7a_7k^9\gamma - 1088a_3^7a_7k^7\gamma^3 + 144a_3^7a_7k^5\gamma^2w \\
&+ 243a_3^7k^{12} + 390a_3^7k^{10}\gamma^2 + 1197a_3^7k^8\gamma w + 1470a_3^7k^6\gamma^3w \\
&- 324a_3^7k^4\gamma^2w^2 + 2412a_3^6a_7k^{10}\gamma + 4128a_3^6a_7k^8\gamma^3 - 1944a_3^6a_7k^8w \\
&- 2748a_3^6a_7k^6\gamma^2w + 828a_3^6a_7k^4\gamma w^2 + 648a_3^6a_7k^2\gamma^3w^2 - 288a_3^6a_7\gamma^2w^3 \\
&+ 729a_3^6k^{13} + 492a_3^6k^{11}\gamma^2 + 2169a_3^6k^9\gamma w + 216a_3^6k^7\gamma^3w \\
&+ 108a_3^6k^5\gamma^2w^2 + 8352a_3^5a_7k^{11}\gamma + 3008a_3^5a_7k^9\gamma^3 - 3888a_3^5a_7k^9w \\
&- 996a_3^5a_7k^7\gamma^2w + 1944a_3^5a_7k^5\gamma w^2 - 24a_3^5a_7k^3\gamma^3w^2 + 243a_3^5k^{14} \\
&- 390a_3^5k^{12}\gamma^2 - 450a_3^5k^{10}\gamma w - 1092a_3^5k^8\gamma^3w + 324a_3^5k^6\gamma^2w^2 \\
&+ 3708a_3^4a^7k^{12}\gamma - 1600a_3^4a_7k^{10}\gamma^3 + 1728a_3^4a_7k^8\gamma^2w + 1116a_3^4a_7k^6\gamma w^2 \\
&- 1215a_3^4k^{15} - 696a_3^4k^{13}\gamma^2 - 2394a_3^4k^{11}\gamma w - 312a_3^4k^9\gamma^3w \\
&- 108a_3^4k^7\gamma^2w^2 - 4824a_3^3a_7k^{13}\gamma - 1920a_3^3a_7k^{11}\gamma^3 + 3888a_3^3a_7k^{11}w \\
&+ 852a_3^3a_7k^9\gamma^2w - 1215a_3^3k^{16} - 6a_3^3k^{14}\gamma^2 - 747a_3^3k^{12}\gamma w \\
&+ 102a_3^3k^{10}\gamma^3w - 5004a_3^2a_7k^{14}\gamma - 416a_3^2a_7k^{12}\gamma^3 + 1944a_3^2a_7k^{12}w \\
&+ 243a_3^2k^{17} + 300a_3^2k^{15}\gamma^2 + 225a_3^2k^{13}\gamma w - 1296a_3a_7k^{15}\gamma \\
&+ 729a_3k^{18} + 102a_3k^{16}\gamma^2 + 243k^{19},
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_6 =& - 3072a_3^{13}a_7k^4\gamma^2 + 2880a_3^{12}k^4\gamma^2w + 6912a_3^{11}a_7k^8 + 21888a_3^{11}a_7k^6\gamma^2 \\
&- 1152a_3^{11}a_7\gamma^2w^2 + 256a_3^{11}k^7\gamma^3 - 576a_3^{11}k^5\gamma^2w + 13824a_3^{10}a_7k^9 \\
&+ 9408a_3^{10}a_7k^7\gamma^2 - 960a_3^{10}a_7k^3\gamma^3w + 64a_3^{10}k^8\gamma^3 - 6480a_3^{10}k^8w \\
&- 14580a_3^{10}k^6\gamma^2w - 28512a_3^9a_7k^{10} - 55704a_3^9a_7k^8\gamma^2 - 488a_3^9a_7k^4\gamma^3w \\
&+ 2592a_3^9a_7k^4w^2 - 1584a_3^9a_7k^2\gamma^2w^2 - 558a_3^9k^{11}\gamma - 1504a_3^9k^9\gamma^3 \\
&- 11664a_3^9k^9w - 2514a_3^9k^7\gamma^2w + 96a_3^9k^3\gamma^3w^2 - 85050a_3^8a_7k^{11} \\
&- 44580a_3^8a_7k^9\gamma^2 + 2394a_3^8a_7k^7\gamma w + 2844a_3^8a_7k^5\gamma^3w + 5184a_3^8a_7k^5w^2 \\
&+ 648a_3^8a_7k^3\gamma^2w^2 - 1278a_3^8k^{12}\gamma - 804a_3^8k^{10}\gamma^3 + 15957a_3^8k^{10}w \\
&+ 24042a_3^8k^8\gamma^2w + 216a_3^8k^4\gamma^3w^2 - 7614a_3^7a_7k^{12} + 51984a_3^7a_7k^{10}\gamma^2 \\
&+ 5634a_3^7a_7k^8\gamma w + 2264a_3^7a_7k^6\gamma^3w + 11340a_3^7a_7k^6w^2 \\
&+ 7440a_3^7a_7k^4\gamma^2w^2 + 192a_3^7a_7\gamma^3w^3 + 1278a_3^7k^{13}\gamma + 3048a_3^7k^{11}\gamma^3 \\
&+ 42930a_3^7k^{11}w + 9954a_3^7k^9\gamma^2w + 414a_3^7k^7\gamma w^2 + 156a_3^7k^5\gamma^3w^2 \\
&- 144a_3^7k^3\gamma^2w^3 + 137862a_3^6a_7k^{13} + 66036a_3^6a_7k^{11}\gamma^2 + 1692a_3^6a_7k^9\gamma w
\end{aligned}
$$

$$
\begin{aligned}
&- 1520a_3^6a_7k^7\gamma^3w + 17172a_3^6a_7k^7w^2 + 2952a_3^6a_7k^5\gamma^2w^2 + 5130a_3^6k^{14}\gamma \\
&+ 2308a_3^6k^{12}\gamma^3 + 8991a_3^6k^{12}w - 12690a_3^6k^{10}\gamma^2w + 972a_3^6k^8\gamma w^2 \\
&- 228a_3^6k^6\gamma^3w^2 + 112914a_3^5a_7k^{14} - 2520a_3^5a_7k^{12}\gamma^2 - 4788a_3^5a_7k^{10}\gamma w \\
&- 1776a_3^5a_7k^8\gamma^3w + 8100a_3^5a_7k^8w^2 - 1104a_3^5a_7k^6\gamma^2w^2 + 1458a_3^5k^{15}\gamma \\
&- 2400a_3^5k^{13}\gamma^3 - 29808a_3^5k^{13}w - 6462a_3^5k^{11}\gamma^2w + 144a_3^5k^9\gamma w^2 \\
&- 252a_3^5k^7\gamma^3w^2 + 144a_3^5k^5\gamma^2w^3 - 30942a_3^4a_7k^{15} - 29436a_3^4a_7k^{13}\gamma^2 \\
&- 4086a_3^4a_7k^{11}\gamma w - 364a_3^4a_7k^9\gamma^3w - 324a_3^4a_7k^9w^2 - 5778a_3^4k^{16}\gamma \\
&- 2460a_3^4k^{14}\gamma^3 - 17901a_3^4k^{14}w + 774a_3^4k^{12}\gamma^2w - 972a_3^4k^{10}\gamma w^2 \\
&+ 12a_3^4k^8\gamma^3w^2 - 78570a_3^3a_7k^{16} - 12576a_3^3a_7k^{14}\gamma^2 - 846a_3^3a_7k^{12}\gamma w \\
&- 4518a_3^3k^{17}\gamma + 392a_3^3k^{15}\gamma^3 - 486a_3^3k^{15}w - 402a_3^3k^{13}\gamma^2w \\
&- 558a_3^3k^{11}\gamma w^2 - 35694a_3^2a_7k^{17} - 1428a_3^2a_7k^{15}\gamma^2 + 1278a_3^2k^{18}\gamma \\
&+ 892a_3^2k^{16}\gamma^3 - 567a_3^2k^{16}w - 426a_3^2k^{14}\gamma^2w - 5130a_3a_7k^{18} \\
&+ 2340a_3k^{19}\gamma + 208a_3k^{17}\gamma^3 - 972a_3k^{17}w + 648k^{20}\gamma,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_7 =&768a_3^9a_7k^3\gamma^2 - 256a_3^9k^4\gamma^3 + 192a_3^8a_7k^4\gamma^2 + 384a_3^8a_7\gamma^3w + 18a_3^8k^7\gamma \\
&- 288a_3^8k^3\gamma^2w - 1674a_3^7a_7k^7 - 3744a_3^7a_7k^5\gamma^2 + 180a_3^7a_7k^3\gamma w \\
&+ 576a_3^7k^8\gamma + 1312a_3^7k^6\gamma^3 - 492a_3^7k^4\gamma^2w - 3834a_3^6a_7k^8 \\
&- 2220a_3^6a_7k^6\gamma^2 - 936a_3^6a_7k^4\gamma w - 792a_3^6a_7k^2\gamma^3w - 144a_3^6a_7\gamma^2w^2 \\
&+ 1053a_3^6k^9\gamma + 526a_3^6k^7\gamma^3 + 783a_3^6k^7w + 846a_3^6k^5\gamma^2w + 108a_3^6k^3\gamma w^2 \\
&+ 2160a_3^5a_7k^9 + 5400a_3^5a_7k^7\gamma^2 - 2304a_3^5a_7k^5\gamma w - 264a_3^5a_7k^3\gamma^3w \\
&+ 108a_3^5a_7k^3w^2 - 1278a_3^5k^{10}\gamma - 2082a_3^5k^8\gamma^3 + 2484a_3^5k^8w \\
&+ 1698a_3^5k^6\gamma^2w - 36a_3^5k^4\gamma w^2 + 11556a_3^4a_7k^{10} + 4704a_3^4a_7k^8\gamma^2 \\
&- 1044a_3^4a_7k^6\gamma w - 4113a_3^4k^{11}\gamma - 1468a_3^4k^9\gamma^3 + 1836a_3^4k^9w \\
&- 114a_3^4k^7\gamma^2w - 108a_3^4k^5\gamma w^2 + 6534a_3^3a_7k^{11} - 1800a_3^3a_7k^9\gamma^2 \\
&+ 180a_3^3a_7k^7\gamma w - 144a_3^3a_7k^5\gamma^3w - 108a_33a_7k^5w^2 - 1764a_3^3k^{12}\gamma \\
&+ 836a_3^3k^{10}\gamma^3 - 1566a_3^3k^{10}w - 1206a_3^3k^8\gamma^2w + 36a_3^3k^6\gamma w^2 \\
&- 5778a_3^2a_7k^{12} - 2676a_3^2a_7k^{10}\gamma^2 + 36a_3^2a_7k^8\gamma w + 2403a_3^2k^{13}\gamma \\
&+ 942a_3^2k^{11}\gamma^3 - 2619a_3^2k^{11}w - 444a_3^2k^9\gamma^2w - 7020a_3a_7k^{13} \\
&- 624a_3a_7k^{11}\gamma^2 + 2466a_3k^{14}\gamma + 190a_3k^{12}\gamma^3 - 918a_3k^{12}w - 1944a_7k^{14} \\
&+ 639k^{15}\gamma.
\end{aligned}
$$

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