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STEPANOV-LIKE ALMOST AUTOMORPHIC SOLUTIONS FOR NONAUTONOMOUS EVOLUTION EQUATIONS

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Abstract. We study the convolution of Stepanov-like almost automorphic functions and L^1 functions. Also we consider nonautonomous evolution equations, with a periodic operator coefficient and Stepanov-like almost automorphic forcing, and show that, under certain assumptions, any bounded mild solution is almost automorphic.

1. Introduction

The notion of almost periodic function was introduced by Bohr in 1925. Shortly after, in 1926, Stepanov found a wider class of almost periodic functions that are commonly known now as Stepanov almost periodic functions. This last notion is especially useful in the theory of evolution equations, both linear and nonlinear, because spaces of Stepanov almost periodic functions are natural counterparts of classical L^p spaces (see, e.g., [1, 15] and references therein). On the other hand, in 1955, Bochner [3] suggested another generalization of the concept of almost periodicity - almost automorphy. This notion was also used extensively in the theory of differential equations (see [10], [12], and references therein). Therefore, it seems to be natural to generalize the notion of almost automorphy in the spirit of Stepanov. Surprisingly enough, this has been done only very recently by N'Guérékata and Pankov [14], where the concept of Stepanov-like (S^p-) almost automorphy was introduced. Such a notion was, subsequently, utilized to study the existence of weak Stepanov-like almost automorphic solutions to some parabolic evolution equations. Then Diagana and N'Guérékata have studied in [6] the existence and uniqueness of an almost automorphic solution to the semilinear equation

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a densely defined closed linear operator in a Banach space $\mathbb{X}$, which is also the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$ and $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ is S^p-almost automorphic for $p > 1$ and jointly continuous. This result generalizes the existence results obtained in N'Guérékata [11].

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In the present paper, we study first the convolution of Stepanov-like almost automorphic functions and some applications to evolution equations. Then we present the conditions under which any bounded mild solution to the nonautonomous equation

$$x'(t) = A(t)x(t) + h(t), \quad t \in \mathbb{R},$$

where $A(t)$ generates a periodic evolutionary process and h is a Stepanov-like forcing term, is almost automorphic. This main result generalizes [9, Theorem 3.2].

2. Almost Automorphy

Throughout the rest of this paper, the spaces $(\mathbb{X}, \|\cdot\|)$, $C(\mathbb{R},\mathbb{X})$ and $BC(\mathbb{R},\mathbb{X})$ stands for a Banach space, the collection of all strongly continuous functions from $\mathbb{R}$ into $\mathbb{X}$, and the collection of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$, respectively. Note that $(BC(\mathbb{R},\mathbb{X}), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the sup norm

$$\|\varphi\|_\infty := \sup_{t\in\mathbb{R}} \|\varphi(t)\|$$

for each $\varphi \in BC(\mathbb{R},\mathbb{X})$, is a Banach space.

Definition 2.1 (Bochner). A function $f \in C(\mathbb{R},\mathbb{X})$ is said to be almost automorphic in Bochner's sense if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the classical Bochner's sense. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R} \to \mathbb{X}$.

Among other things, almost automorphic functions satisfy the following properties.

Theorem 2.2 ([10, Theorem 2.1.3]). *If $f, f_1, f_2 \in AA(\mathbb{X})$, then*

(i) $f_1 + f_2 \in AA(\mathbb{X})$,
(ii) $\lambda f \in AA(\mathbb{X})$ *for any scalar* λ,
(iii) $f_\alpha \in AA(\mathbb{X})$ *where* $f_\alpha : \mathbb{R} \to \mathbb{X}$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$,
(iv) *the range* $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$, *thus* f *is bounded in norm*,
(v) *if* $f_n \to f$ *uniformly on* $\mathbb{R}$ *where each* $f_n \in AA(\mathbb{X})$, *then* $f \in AA(\mathbb{X})$ *too.*

Theorem 2.3 ([5]). *If $g \in L^1(\mathbb{R})$, then $f * g \in AA(\mathbb{R})$, where $f * g$ is the convolution of f with g on $\mathbb{R}$.*

Note that $(AA(\mathbb{X}), \|\cdot\|_\infty)$ turns out to be a Banach space.

Remark 2.4. The function g in the Definition 2.1 above is measurable, but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous, see details in [13, Theorem 2.6].

Example 2.5. A classical example of an almost automorphic function, which is not almost periodic is the function defined by

$$\varphi(t) = \cos\Big(\frac{1}{2+\sin\sqrt{2}t+\sin t}\Big), \quad t \in \mathbb{R}.$$

It can be shown that φ is not uniformly continuous, and hence is not almost periodic.

Let $l^\infty(\mathbb{X})$ denote the space of all bounded (two-sided) sequence in $\mathbb{X}$. It is equipped with its corresponding sup norm defined for each sequence $x = (x_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{X})$ by: $\|x\|_\infty := \sup_{n\in\mathbb{Z}} \|x_n\|$.

Definition 2.6. A sequence $x = (x_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{X})$ is said to be almost automorphic if for every sequence of integers (k'_n), there exists a subsequence (k_n) such that

$$y_p := \lim_{n\to\infty} x_{p+k_n}$$

is well defined for each $p \in \mathbb{Z}$, and

$$\lim_{n\to\infty} y_{p-k_n} = x_p$$

for each $p \in \mathbb{Z}$. The collection of all these almost automorphic sequences is denoted by $aa(\mathbb{X})$.

In the sequel, we will denote by $AA_u(\mathbb{X})$ (u-a.a. for short) the closed subspace of all functions $f \in AA(\mathbb{X})$ with $g \in C(\mathbb{R},\mathbb{X})$. Equivalently, $f \in AA_u(\mathbb{X})$ if and only if f is a.a. and all convergences in Definition 2.1 are uniform on compact intervals; i.e., in the Fréchet space $C(\mathbb{R},\mathbb{X})$. Indeed, if f is a.a., then, by [10, Theorem 2.1.3 (iv)], its range is relatively compact.

Remark 2.7. Note that Definition 2.1 as well as the above-mentioned definition of u-a.a. functions makes sense for functions with values in any metric space (see [8]).

Obviously, the following inclusions hold:

$$AP(\mathbb{X}) \subset AA_u(\mathbb{X}) \subset AA(\mathbb{X}) \subset BC(\mathbb{X})\,,$$

where $AP(\mathbb{X})$ stands for the collection of all $\mathbb{X}$-valued almost periodic functions.

Definition 2.8. The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in $\mathbb{X}$, is defined by

$$f^b(t,s) := f(t+s).$$

Remark 2.9. A function $\varphi(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, is the Bochner transform of a certain unction $f(t)$,

$$\varphi(t,s) = f^b(t,s)\,,$$

if and only if

$$\varphi(t+\tau, s-\tau) = \varphi(s,t)$$

for all $t \in \mathbb{R}$, $s \in [0,1]$ and $\tau \in [s-1,s]$.

Definition 2.10 (see [15])**.** Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b \in L^\infty\big(\mathbb{R}, L^p(0,1;\mathbb{X})\big)$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p\,d\tau\Big)^{1/p}.$$

Definition 2.11 ([14]). The space $AS^p(\mathbb{X})$ of S^p-almost automorphic functions (s^p-a.a. for short) consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA\big(L^p(0,1;\mathbb{X})\big)$.

In other words, a function $f \in L^p_{\rm loc}(\mathbb{R};\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers (s'_n), there exists a subsequence (s_n) and a function $g \in L^p_{\rm loc}(\mathbb{R};\mathbb{X})$ such that

$$\Big(\int_0^1 \|f(t+s_n+s) - g(t+s)\|^p ds\Big)^{1/p} \to 0,$$
$$\Big(\int_0^1 \|g(t-s_n+s) - f(t+s)\|^p ds\Big)^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

Remark 2.12. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{\rm loc}(\mathbb{R};\mathbb{X})$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$. It is easily seen that $f \in AA_u(\mathbb{X})$ if and only if $f^b \in AA(L^\infty(0,1;\mathbb{X}))$. Thus, $AA_u(\mathbb{X})$ can be considered as $AS^\infty(\mathbb{X})$.

Example 2.13 ([14]). Let $x = (x_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{X})$ be an almost automorphic sequence and let $\varepsilon_0 \in (0, \frac{1}{2})$. Let $f(t) = x_n$ if $t \in (n-\varepsilon_0, n+\varepsilon_0)$ and $f(t) = 0$, otherwise. Then $f \in AS^p(\mathbb{X})$ for $p \ge 1$ but $f \notin AA(\mathbb{X})$, as f is discontinuous.

Theorem 2.14 ([14]). *The following statements are equivalent:*

(i) $f \in AS^p(\mathbb{X})$;
(ii) $f^b \in AA_u(L^p(0,1;\mathbb{X}))$;
(iii) *for every sequence (s'_n) of real numbers there exists a subsequence (s_n) such that*
$$g(t) := \lim_{n\to\infty} f(t+s_n) \tag{2.1}$$
exists in the space $L^p_{\rm loc}(\mathbb{R};\mathbb{X})$ and
$$f(t) = \lim_{n\to\infty} g(t-s_n) \tag{2.2}$$
in the sense of $L^p_{\rm loc}(\mathbb{R};\mathbb{X})$.

Let now $f, h : \mathbb{R} \to \mathbb{R}$ and consider the convolution

$$(f \star h)(t) := \int_{\mathbb{R}} f(s)h(t-s)ds, \quad t \in \mathbb{R},$$

if the integral exists.

Remark 2.15. The operator $J : AS^p(\mathbb{X}) \to AS^p(\mathbb{X})$ such that $(Jx)(t) := x(-t)$ is well-defined and linear. Moreover it is an isometry and $J^2 = I$.

Remark 2.16. The operator T_a defined by $(T_a x)(t) := x(t+a)$ for a fixed $a \in \mathbb{R}$ leaves $AS^p(\mathbb{X})$ invariant.

Theorem 2.17. *A linear combination of S^p-almost automorphic functions ($p \ge 1$) is a S^p-almost automorphic function. Moreover if $\mathbb{X}$ is a Banach space over the field $K = \mathbb{R}$, or $\mathbb{C}$ and $f \in AS^p(\mathbb{X})$, $\nu \in AA_u(K)$, then $\nu f \in AS^p(\mathbb{X})$.*

The proof of this theorem is an immediate consequence of the results above.

Theorem 2.18. *If a sequence $(f_k)_{k=1}^\infty$ of S^p almost automorphic functions is such that $\|f_k - f\|_{S^p} \to \infty$, as $k \to \infty$, then $f \in AS^p$.*

As in [5], denote by $LM(\mathbb{R})$ the set of all Lebesgue measurable functions $\mathbb{R} \to \mathbb{R}$. We also denote by $S^p(\mathbb{R}, \mathbb{X})$ the subspace of $BS^p(\mathbb{R}, \mathbb{X})$ that consists of all S^p-almost periodic functions [15].

3. S^p-almost automorphy of the convolution

Let us now discuss conditions which do ensure the S^p-almost automorphy of the convolution function $f \star h$ of f with h where f is S^p-almost automorphic and h is a Lebesgue mesurable function satisfying additional assumptions.

Let $f : \mathbb{R} \to X$ and $h : \mathbb{R} \to \mathbb{R}$; the convolution function (if it does exist) of f with h denoted $f \star h$ is defined by:

$$(f \star h)(t) := \int_{\mathbb{R}} f(\sigma)h(t-\sigma)d\sigma = \int_{\mathbb{R}} f(t-\sigma)h(\sigma)d\sigma = (h \star f)(t), \quad \text{for all } t \in \mathbb{R}.$$

Hence, if $f \star h$ is well-defined, then $f \star h = h \star f$.

Let $\varphi \in L^1$ and $\lambda \in \mathbb{C}$. It is well-known that the operator $A_{\varphi,\lambda}$ defined by

$$A_{\varphi,\lambda} u = \lambda u + \varphi \star u \tag{3.1}$$

acts continuously in BS^p for each $1 \le p < \infty$; i.e., there exists $K > 0$ such that

$$\|A_{\varphi,\lambda} u\|_{S_p} \le K\|u\|_{S^p}, \forall u \in BS^p. \tag{3.2}$$

Moreover $A_{\varphi,\lambda}$ leaves S^p invariant (see [4]).

Now denote $\mathcal{M} := \{AA(X), AA_u(X), AS^p(X)\}$.

Theorem 3.1. *For every $1 \le p < \infty$, and $\Omega \in \mathcal{M}$,*

$$A_{\varphi,\lambda}(\Omega) \subset \Omega.$$

Proof. The case $\Omega = AA(X)$ is considered in [5]. The two other cases follow from the previous one, the identity $(\varphi \star f)^b = \varphi \star (f^b)$ and the definitions of spaces AA_u and AS^p, respectively. □

Now we present a result on invertibility of convolution operators in spaces of almost automorphic functions that complements [4, Theorem 1]. Let $a(\xi) = \lambda + \hat{\varphi}(\xi)$, where $\hat{\varphi}(\xi)$ is the Fourier transform of φ, be the *symbol* of the operator $A_{\varphi,\lambda}$, with $\varphi \in L^1(\mathbb{R})$. Since $\lim_{\xi\to\infty} \varphi(\xi) = 0$, the symbol $a(\xi)$ is a well-defined continuous function on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $a(\infty) = \lambda$. Then we have the following

Theorem 3.2. *Suppose that $\varphi \in L^1(\mathbb{R})$. The following two statements are equivalent:*

(i) *$a(\xi) \ne 0$ for all $\xi \in \overline{\mathbb{R}}$;*
(ii) *the operator $A_{\varphi,\lambda}$ is invertible in any space $\Omega \in \mathcal{M}$.*

Proof. Suppose that $a(\xi) \ne 0$ for all $\xi \in \overline{\mathbb{R}}$. The function $\frac{1}{a(\xi)}$ is defined on $\overline{\mathbb{R}}$ and, by classical Wiener's theorem, is of the form

$$\frac{1}{a(\xi)} = \frac{1}{\lambda} + \hat{\psi}(\xi),$$

where $\psi \in L^1(\mathbb{R})$. Now it is easy to verify that the operator $A_{\psi,\frac{1}{\lambda}}$ is the inverse operator to $A_{\varphi,\lambda}$ and, by Theorem 3.1, acts in all spaces $\Omega \in \mathcal{M}$.

Conversely, suppose that $A = A_{\varphi,\lambda}$ is invertible in some space $\Omega \in \mathcal{M}$. Then we have that, with some $\alpha > 0$,

$$\alpha\|u\|_\Omega \le \|Au\|_\Omega$$

for all $u \in \Omega$. The function $u(t) = u_\xi(t) = \exp(i\xi t)$ belongs to Ω and it is easily seen that $\|u\|_\Omega = 1$ and $Au = a(\xi)u$. Hence, $|a(\xi)| \geq \alpha$ and we conclude. □

Now we can complement [4, Theorems 2 and 3] as follows. Let $A = A_{\varphi,\lambda}$ and Ω be a functional space in which the operator A acts. We denote by $\|A|_\Omega\|$ the norm of A as a linear operator in Ω.

Theorem 3.3. *Let $\varphi \in L^1(\mathbb{R})$, $A = A_{\varphi,\lambda}$ and $p \geq 1$. Then*

$$\|A|_{S^p}\| = \|A|_{AS^p}\| = \|A|_{BS^p}\|,$$
$$\|A|_{AP}\| = \|A|_{AA_u}\| = \|A|_{AA}\| = \|A|_{BC}\|.$$

Proof. We have the following chain of closed subspaces $S^p \subset AS^p \subset BS^p$. Hence,

$$\|A|_{S^p}\| \leq \|A|_{AS^p}\| \leq \|A|_{BS^p}\|.$$

By [4, Theorem 2], we have that $\|A|_{S^p}\| = \|A|_{BS^p}\|$. This implies the first statement of the theorem.

The second statement is similar. We need only to refer to [4, Theorem 3]. □

Application: A Volterra-like Equation. Consider the equation

$$x(t) = g(t) + \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma,\ t \in \mathbb{R}, \tag{3.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $a \in L^1(\mathbb{R})$.

Theorem 3.4. *Suppose $g \in AS^p(\mathbb{R})$ and $\|a\|_{L^1} < 1$. Then* (3.3) *above has a unique S^p-almost automorphic solution.*

Proof. It is clear that the operator

$$x \in AS^p(X) \to \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma \in AS^p(X)$$

is well-defined. Now consider $\Gamma : AS^p(X) \to AS^p(X)$ such that

$$(\Gamma x)(t) = g(t) + \int_{-\infty}^{+\infty} a(t-\sigma)x(\sigma)d\sigma, \quad t \in \mathbb{R}.$$

We can easily show that

$$\|(\Gamma x) - (\Gamma y)\| \leq \|a\|_{L^1}\|x-y\|_{S^p}.$$

The conclusion is immediate by the principle of contraction. □

4. Almost Automorphic Solutions

In this section, X will be a Banach space which does not contain any subspace isomorphic to c_0. We consider the equation

$$x'(t) = A(t)x(t) + h(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where $h \in AS^p(X) \cap C(\mathbb{R}, X)$, and $A(t)$ generates a 1-periodic exponentially bounded evolutionary process $(U(t,s))_{t\geq s}$ in X, that is, a two-parameter family of bounded linear operators that satisfies the following conditions:

(1) $U(t,t) = I$ for all $t \in \mathbb{R}$,
(2) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,
(3) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in X$,
(4) $U(t+1,s+1) = U(t,s)$ for all $t \geq s$ (1-*periodicity*),

(5) $\|U(t,s)\| \leq Ke^{\omega(t-s)}$ for some $K>0$, $\omega>0$ independent of $t \geq s$.

An X-valued continuous function u on $\mathbb{R}$ is said to be a mild solution of (4.1) if

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\xi)h(\xi)d\xi, \quad \forall t \geq s;\ t,s \in \mathbb{R}. \tag{4.2}$$

Lemma 4.1. *Let x be a bounded mild solution of (4.2) on $\mathbb{R}$ and let h be in $AS^p(X) \cap C(\mathbb{R},X)$. Then, $x \in AA(X)$ if and only if the sequence $\{x(n)\}_{n\in\mathbb{Z}} \in aa(X)$.*

Proof. The proof is similar to [9, Lemma 3.1], with the necessary adaptations. The necessity is obvious. For the Sufficiency, let the sequence $\{x(n)\}_{n\in\mathbb{Z}} \in aa(X)$. We need to prove that $x \in AA(X)$. We divide the proof into two steps:
Step 1: We first suppose that $\{n'_k\}$ is a given sequence of integers. Then there exist a subsequence $\{n_k\}$ and a function $g \in L^p_{\rm loc}(\mathbb{R},X)$ such that

$$y(n) := \lim_{k\to\infty} x(n+n_k)$$

exists for each $n \in \mathbb{Z}$ and

$$\lim_{k\to\infty} y(n-n_k) = x(n)$$

for each $n \in \mathbb{Z}$, and

$$\begin{gathered}
\lim_{k\to\infty} \Big(\int_0^1 \|h(t+n_k+s) - g(t+s)\|^p ds\Big)^{1/p} = 0;\\
\lim_{k\to\infty} \Big(\int_0^1 \|g(t-n_k+s) - h(t+s)\|^p ds\Big)^{1/p} = 0,
\end{gathered}$$

for each $t \in \mathbb{R}$.

For every fixed $t \in \mathbb{R}$, let us denote by $[t]$ the integer part of t. Then, define

$$y(\eta) := U(\eta,[t])y([t]) + \int_{[t]}^{\eta} U(\eta,\xi)g(\xi)d\xi, \quad \eta \in [[t],[t]+1).$$

In this way, we can define y on the whole line $\mathbb{R}$. Now we show that

$$\lim_{k\to\infty} x(t+n_k) = y(t).$$

In fact,

$$\lim_{k\to\infty} \|x(t+n_k) - y(t)\| \leq \lim_{k\to\infty} I_1(t) + \lim_{k\to\infty} I_2(t)$$

where

$$\begin{gathered}
I_1(t) = \|U(t+n_k,[t]+n_k)x([t]+n_k) - U(t,[t])y([t])\|,\\
I_2(t) = \int_{[t]}^t \|U(t,\eta)\|\|h(\eta+n_k) - g(\eta)\|d\eta\,.
\end{gathered}$$

Now using the 1-periodicity of $U(t,s)$ since $n_k \in \mathbb{Z}$ and boundedness of $U(t,[t])$, we get

$$\begin{aligned}
\lim_{k\to\infty} I_1(t) &= \lim_{k\to\infty} \|U(t,[t])x([t]+n_k) - U(t,[t])y([t])\|\\
&\leq C_t \lim_{k\to\infty} \|x([t]+n_k) - y([t])\| = 0,
\end{aligned}$$

for each $t \in \mathbb{R}$. Now for k sufficiently large, we get

$$I_2(t) \leq K \int_{[t]}^t e^{\omega(t-\eta)}\|h(\eta+n_k) - g(\eta)\|d\eta \leq C'_{t,\omega}\epsilon,$$

which shows that

$$\lim_{k\to\infty} I_2(t) = 0, \ for \ each \ t \in \mathbb{R}.$$

Thus

$$\lim_{k\to\infty} \|x(t+n_k) - y(t)\| = 0.$$

Similarly, we can show that

$$\lim_{k\to\infty} \|y(t-n_k) - x(t)\| = 0.$$

Step 2: Now we consider the general case where $\{s'_k\}_{k\in\mathbb{Z}}$ may not be an integer sequence. The main lines are similar to those in Step 1 combined with the strong continuity of the process.

Set $n'_k = [s'_k]$ for every k. Since $\{t_k\}_{k\in\mathbb{Z}}$, where $t_k := s'_k - [s'_k]$, is a sequence in $[0,1)$ we can choose a subsequence $\{n_k\}$ from $\{n'_k\}$ such that $\lim_{k\to\infty} t_k = t_0 \in [0,1]$ and

$$y(n) := \lim_{k\to\infty} x(n+n_k)$$

exists for each $n \in \mathbb{Z}$ and

$$lim_{k\to\infty} y(n-n_k) = x(n)$$

for each $n \in \mathbb{Z}$, for a function y, as shown in Step 1.

Let us first consider the case $0 < t_0 + t - [t_0+t]$. We show that

$$\lim_{k\to\infty} x(t_k+t+n_k) = \lim_{k\to\infty} x(t_0+t+n_k) = y(t_0+t). \tag{4.3}$$

In fact, for sufficiently large k, from the above assumption we have $[t_0+t] = [t_k+t]$. Using the 1-periodicity of the process $(U(t,s))_{t\geq s}$ we have

$$\|x(t_k+t+n_k) - x(t_0+t+n_k)\| \leq I_3(k) + I_4(k), \tag{4.4}$$

where $I_3(t)$, $I_4(k)$ are defined and estimated as below. By the 1-periodicity of the process $(U(t,s))_{t\geq s}$ we have

$$\begin{aligned}
I_3(t) &:= \|U(t_k+t+n_k, [t_k+t]+n_k)x([t_k+t]+n_k) \\
&\quad - U(t_0+t+n_k, [t_0+t]+n_k)x([t_0+t]+n_k)\| \\
&= \|U(t_k+t, [t_0+t])x([t_0+t]+n_k) - U(t_0+t, [t_0+t])x([t_0+t]+n_k)\|.
\end{aligned}$$

Using the strong continuity of the process $(U(t,s))_{t\geq s}$ and the boundedness of the range of the sequence $\{x(n)\}_{n\in\mathbb{Z}}$ we have $\lim_{k\to\infty} I_3(k) = 0$. Next, we define

$$I_4(k) := \| \int_{[t_k+t]+n_k}^{t_k+t+n_k} U(t_k+t+n_k, \eta)h(\eta)d\eta - \int_{[t_0+t]+n_k}^{t_0+t+n_k} U(t_0+t+n_k, \eta)h(\eta)d\eta\|.$$

Using the Holder inequality we have

$$\begin{aligned}
\| \int_{[t_k+t]+n_k}^{t_k+t+n_k} U(t_k+t+n_k, \eta)h(\eta)d\eta\| &\leq K \int_{[t_k+t]+n_k}^{t_k+t+n_k} e^{\omega(t_k+t+n_k-\eta)}\|h(\eta)\|d\eta \\
&\leq K(\int_0^1 e^{q\omega(t_k+t+n_k-\eta)}d\eta)^{\frac{1}{q}}(\|h\|_{S^p}) \\
&= \frac{K}{q\omega}(e^{q\omega(t_k+t+n_k-1)} - e^{q\omega(t_k+t+n_k)})^{\frac{1}{q}}(\|h\|_{S^p}).
\end{aligned}$$

By letting $k\to\infty$, we observe that the latter tends to zero since $\omega < 0$. The same treatment can be used for the second integral in I_4, so that $\lim_{k\to\infty} I_4 = 0$. So, in view of Step 1, we see that (4.3) holds.

Next, we consider the case when $t_0+t-[t_0+t]=0$, that is, t_0+t is an integer. If $t_k+t\geq t_0+t$, we can repeat the above argument. So, we omit the details. Now suppose that $t_k+t<t_0+t$. Then

$$\|x(t_k+t+n_k)-x(t_0+t+n_k)\|\leq I_5(k)+I_6(k), \tag{4.5}$$

where $I_5(k)$ and $I_6(k)$ are defined and estimated as below.

$$\begin{aligned}
I_5(k) &:= \|U(t_k+t+n_k,[t_k+t]+n_k)x([t_k+t]+n_k)\\
&\quad -U(t_0+t+n_k,t_0+t-1+n_k)x(t_0+t-1+n_k)\|\\
&=\|U(t_k+t,t_0+t-1)x(t_0+t-1+n_k)\\
&\quad -U(t_0+t,t_0+t-1)x(t_0+t-1+n_k)\|.
\end{aligned}$$

Now using the strong continuity of the process $(U(t,s))_{t\geq s}$ and the precompactness of the range of the sequence $\{x(n)\}_{n\in\mathbb{Z}}$ we obtain $\lim_{k\to\infty} I_5(k)=0$. Finally we have

$$I_6(k):=\|\int_{[t_k+t]+n_k}^{t_k+t+n_k} U(t_k+t+n_k,\eta)h(\eta)d\eta-\int_{[t_0+t]+n_k-1}^{t_0+t+n_k} U(t_0+t+n_k,\eta)h(\eta)d\eta\|$$

This can be treated as in the case of I_4; i.e., $\lim_{k\to\infty} I_6(k)=0$. The proof is now complete. □

Theorem 4.2. *Let $A(t)$ in (4.1) generate an exponentially bounded 1-periodic strongly continuous evolutionary process, and let $h\in AS^p(X)\cap C(\mathbb{R},X)$. Assume further that the space $\mathbb{X}$ does not contain ny subspace isomorphic to c_0 and the part of spectrum of the monodromy operator $U(1,0)$ on the unit circle is countable. Then, every bounded mild solution of (4.1) on the real line is almost automorphic.*

Proof. The theorem is an immediate consequence of [9, Lemmas 2.12 and 2.13] and Lemma 4.1 above. In fact, we need only to prove the sufficiency. Let us consider the discrete equation

$$x(n+1)=U(n+1,n)x(n)+\int_n^{n+1} U(n+1,\xi)h(\xi)d\xi,\quad n\in\mathbb{Z}.$$

From the 1-periodicity of the process $(U(t,s))_{t\geq s}$, this equation can be re-written in the form

$$u(n+1)=Bu(n)+y_n,\quad n\in\mathbb{Z}, \tag{4.6}$$

where

$$B:=U(1,0);\ y_n:=\int_n^{n+1} U(n+1,\xi)h(\xi)d\xi,\quad n\in\mathbb{Z}.$$

Note that y_n is well-defined.

We are going to show that the sequence $\{y_n\}_{n\in\mathbb{Z}}$ defined as above is almost automorphic. In fact, since $h\in AS^p(X)$, for every sequence $\{n_k'\}$ there exists a subsequence $\{n_k\}$ and a function $g\in L^p_{\rm loc}(\mathbb{R};\mathbb{X})$ such that

$$\begin{gathered}
\Big(\int_0^1 \|h(t+\{n_k\}+s)-g(t+s)\|^p ds\Big)^{1/p}\to 0,\\
\Big(\int_0^1 \|g(t-\{n_k\}+s)-h(t+s)\|^p ds\Big)^{1/p}\to 0
\end{gathered}$$

as $n \to \infty$ pointwise on $\mathbb{R}$. Now let

$$z_n = \int_n^{n+1} U(n,\xi)g(\xi)d\xi, \quad n \in \mathbb{Z}\,.$$

Then, by the 1-periodicity of $(U(t,s))_{t\geq s}$ and the Holder inequality we have

$$\begin{aligned}
&\|y_{n+n_k} - z_n\| \\
&= \| \int_{n+n_k}^{n+n_k+1} U(n,\xi)h(\xi)d\xi - \int_n^{n+1} U(n,\xi)g(\xi)d\xi\| \\
&= \| \int_0^1 U(n+n_k,\xi+n+n_k)h(\xi+n+n_k)d\xi - \int_0^1 U(n,\xi+n)g(\xi)d\xi\| \\
&= \| \int_0^1 U(n,\xi+n)(h(\xi+n+n_k - g(\xi)d\xi\| \\
&\leq \int_0^1 \|U(n,\xi+n)\|h(\xi+n+n_k) - g(\xi)d\xi\| \\
&\leq K \int_0^1 e^{\omega t}\|h(\xi+n+n_k) - g(\xi)\|d\xi \\
&\leq K(\int_0^1 e^{-q\omega t}d\xi)^{\frac{1}{q}} \Big(\int_0^1 (\|h(\xi+n+n_k) - g(\xi)\|)^p d\xi\Big)^{1/p} \to 0, \quad \text{as } k \to \infty
\end{aligned}$$

By [9, Lemma 2.13], since $\{x(n)\}$ is a bounded solution of (4.6), $\mathbb{X}$ does not contain any subspace isomorphic to c_0, and the part of spectrum of $U(1,0)$ on the unit circle is countable, $\{x(n)\} \in aa(X)$. By Lemma 4.1, this yields that the solution $x \in AA(X)$. The proof is now complete. $\square$

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