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A GEOMETRIC APPROACH TO INTEGRABILITY CONDITIONS FOR RICCATI EQUATIONS

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Abstract. Several instances of integrable Riccati equations are analyzed from the geometric perspective of the theory of Lie systems. This provides us a unifying viewpoint for previous approaches.

1. Introduction

The Riccati equation

$$\frac{dx}{dt} = b_0(t) + b_1(t)x + b_2(t)x^2\,, \tag{1.1}$$

which is a simple nonlinear differential equation, is very often used in many fields of mathematics, control theory and theoretical physics (see for instance [1, 2] and references therein) and its importance in this field has been increasing since Witten's introduction of supersymmetric Quantum Mechanics. It is essentially the only differential equation with one dependent variable admitting a non-linear superposition principle [1, 3]. In spite of its apparent simplicity, the general solution of a Riccati equation cannot be expressed by means of quadratures except in some very particular cases and several interesting results about this property can be found, for instance, in the books by Kamke [4] and Murphy [5]. Few years ago Strelchenya presented some integrability conditions [6] claiming to complete previous solvability criteria. However, it was shown in [7] that such criterion reduced to a very well known result, the knowledge of a particular solution. Other cases of integrable Riccati equations appearing in Kamke [4] and Murphy [5] were also analyzed in [7]. There exist other papers dealing with integrability conditions of the Riccati equation [8, 9, 10, 11] by reduction to related separable Riccati equations and also recent works as [12, 13, 14, 15].

In this paper we review the geometric approach to the Riccati equation according to the results of [7] with the aim of proving that the integrability conditions of this equation in the above mentioned cases can be understood in a very general way from the point of view of Lie systems.

This paper is organized as follows. In Section 2 we analyze some known facts about the integrability of Riccati equations. The geometric interpretation of the

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general Riccati equation as a time-dependent vector field in the one-point compactification of the real line is summarized in Section 3. More specifically, we study Riccati equations through an equation in the group $SL(2,\mathbb{R})$. In Section 4 we describe the action of the group of curves in $SL(2,\mathbb{R})$ on the set of Riccati equations and how this action can be seen in terms of transformations of the corresponding equations in $SL(2,\mathbb{R})$ (for a more geometric treatment see [16]). Finally, in Section 5 we analyze how our geometric point of view allows us to consider the integrability conditions as a way to perform a transformation process from the Riccati equations into integrable ones related, as Lie systems, with an equation in a Lie subgroup with solvable Lie algebra. In this way we obtain some known results about integrability of the Riccati equations found in [8, 9, 10, 11, 12, 13] and we give a theoretical treatment of why the transformations used there actually work.

2. Integrability of Riccati equations

A particular case for which the Riccati equation is integrable by quadratures is when $b_2(t) = 0$. In this case the equation reduces to an inhomogeneous linear one and two quadratures allow us to find the general solution. More explicitly, the general solution is given by

$$x(t) = \exp\Big(\int_0^t b_1(s)\,ds\Big)\Big(x_0 + \int_0^t b_0(t')\exp\Big(-\int_0^{t'} b_1(s)\,ds\Big)dt'\Big)\,.$$

Note that under the change of variable $w = -1/x$ the Riccati equation (1.1) becomes a new Riccati equation

$$\frac{dw}{dt} = b_0(t)w^2 - b_1(t)w + b_2(t)\,.$$

In particular, if in the original equation $b_0(t) = 0$ (Bernoulli for $n = 2$), then the mentioned change of variable transforms the given equation into a linear one.

On the other hand, the change $v = -x$ transforms the differential equation (1.1) into a new Riccati equation

$$\frac{dv}{dt} = -b_0(t) + b_1(t)v - b_2(t)v^2\,. \tag{2.1}$$

Another very well-known property on integrability of Riccati equation is that if one particular solution x_1 of (1.1) is known, then the change of variable $x = x_1 + z$ leads to a new Riccati equation for which $b_0(t) = 0$:

$$\frac{dz}{dt} = (2b_2(t)x_1(t) + b_1(t))z + b_2(t)z^2\,, \tag{2.2}$$

that can be reduced to an inhomogeneous linear equation with the change $z = -1/u$. Consequently, when one particular solution is known, the general solution can be found by means of two quadratures; $x = x_1 - 1/u$ with

$$\begin{aligned} u(t) =& \exp\Big(-\int_0^t [b_1(s) + 2b_2(s)x_1(s)]\,ds\Big) \\ &\times\Big(u_0 + \int_0^t b_2(t')\exp\Big(\int_0^{t'} [b_1(s) + 2b_2(s)x_1(s)]\,ds\Big)dt'\Big). \end{aligned}$$

When not only one but two particular solutions of (1.1) are known, x_1 and x_2, the general solution can be found by means of only one quadrature. In fact, the change of variable $x = (x_1 - zx_2)/(1 - z)$, or in an equivalent way, $z = (x - x_1)/(x - x_2)$, transforms the original equation in a homogeneous first order linear differential equation in the new variable z,

$$\frac{dz}{dt} = b_2(t)(x_1(t) - x_2(t))z\,,$$

and therefore the general solution can be immediately found:

$$z(t) = z(t=0)\exp\Big(\int_0^t b_2(s)(x_1(s) - x_2(s))\,ds\Big)\,.$$

Finally, if three particular solutions, x_1, x_2, x_3, are known, the general solution can be written, without making use of any quadrature, in the following way:

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k.$$

This is a nonlinear superposition principle which has been studied in [2] from a group theoretical perspective.

The simplest case of (1.1) being an autonomous equation (b_0, b_1 and b_2 constants), has been fully studied (see e.g. [17] and references therein) and it is integrable by quadratures. This is a consequence of the existence of a constant (maybe complex) solution, which allows us to reduce the problem to an inhomogeneous linear one. Also separable Riccati equations, of the form

$$\frac{dx}{dt} = \varphi(t)(c_0 + c_1 x + c_2 x^2)\,,$$

are integrable, because it is enough to introduce a new time variable τ such that $d\tau/dt = \varphi(t)$ and the problem reduces to an autonomous case.

It will be shown in Section 5 that there are other cases of Riccati equations related with this type of separable or autonomous ones.

3. Geometric Interpretation of Riccati equation

From the geometric viewpoint the Riccati equation (1.1) can be considered as a differential equation determining the integral curves of the time-dependent vector field

$$\Gamma = (b_0(t) + b_1(t)x + b_2(t)x^2)\frac{\partial}{\partial x}\,. \tag{3.1}$$

The simplest case is when all the coefficients $b_\alpha(t)$ are constant, because then Γ, given by (3.1), is a true vector field. Otherwise, Γ is a vector field along the projection map $\pi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, given by $\pi(t,x) = x$ (see for example [18] where it is shown that these vector fields along π also admit integral curves).

The important point is that Γ is a linear combination with time-dependent coefficients of the three vector fields

$$L_0 = \frac{\partial}{\partial x}, \quad L_1 = x\frac{\partial}{\partial x}, \quad L_2 = x^2\frac{\partial}{\partial x}, \tag{3.2}$$

which close on a 3-dimensional real Lie algebra, with defining relations

$$[L_0, L_1] = L_0, \quad [L_0, L_2] = 2L_1, \quad [L_1, L_2] = L_2\,. \tag{3.3}$$

Consequently this Lie algebra is isomorphic to $T_I SL(2,\mathbb{R})$, considered as a Lie algebra in the natural way, which is made up of traceless 2×2 matrices. A particular basis is given by

$$M_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{3.4}$$

for which

$$[M_0, M_1] = -M_0, \quad [M_0, M_2] = -2M_1, \quad [M_1, M_2] = -M_2$$

that are commutation relations analogous to (3.3), i.e. the linear correspondence $L_\alpha \mapsto M_\alpha$ is an antihomomorphism of Lie algebras.

Note also that L_0 and L_1 generate a 2-dimensional Lie subalgebra isomorphic to the Lie algebra of the affine group of transformations in one dimension, and the same holds for L_1 and L_2. The one-parameter subgroups of local transformations of $\mathbb{R}$ generated by L_0, L_1 and L_2 are

$$x \mapsto x + \epsilon, \quad x \mapsto e^\epsilon x, \quad x \mapsto \frac{x}{1 - x\epsilon}\,.$$

Note that L_2 is not a complete vector field on $\mathbb{R}$. However we can do the one-point compactification of $\mathbb{R}$ and then L_0, L_1 and L_2 can be considered as the fundamental vector fields corresponding to the action of $SL(2,\mathbb{R})$ on the completed real line $\overline{\mathbb{R}}$, given by

$$\Phi(A, x) = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \text{if } x \neq -\frac{\delta}{\gamma},$$
$$\Phi(A, \infty) = \alpha/\gamma, \quad \Phi(A, -\delta/\gamma) = \infty, \quad \text{when } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R}).$$

Let $\Phi : SL(2,\mathbb{R}) \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be the preceding effective left action of $SL(2,\mathbb{R})$ on the one-point compactification of the real line. The remarkable fact is the following: let $A(t)$ be the curve in $SL(2,\mathbb{R})$ that is the integral curve of the t-dependent vector field

$$X(t, A) = -\sum_{\alpha=0}^{2} b_\alpha(t) X^{\mathrm{R}}_\alpha(A),$$

which starts from the neutral element, i.e. $A(0) = I$. Here $\{M_\alpha \mid \alpha = 0, 1, 2\}$ is a basis of the tangent space $T_I SL(2,\mathbb{R}) \simeq \mathfrak{sl}(2,\mathbb{R})$ and X^{R}_α denotes the right-invariant vector field in $SL(2,\mathbb{R})$ such that $X^{\mathrm{R}}_\alpha(I) = M_\alpha$. This vector field satisfies that $X^{\mathrm{R}}_\alpha(A) = M_\alpha A$. In other words, $A(t)$ satisfies

$$\dot{A}(t)A^{-1}(t) = -\sum_{\alpha=0}^{2} b_\alpha(t) M_\alpha \equiv \mathrm{a}(t). \tag{3.5}$$

Then, $x(t) = \Phi(A(t), x_0)$ is the solution of the Riccati equation (1.1) with initial condition $x(0) = x_0$. Also, we remark that the r.h.s. of (3.5) is a curve in $T_I SL(2,\mathbb{R})$ that can be identified with a curve in the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of left-invariant vector fields on $SL(2,\mathbb{R})$ through the usual isomorphism: we relate a left-invariant vector field X with the element $X(I) \in T_I SL(2,\mathbb{R})$. From now on we will not distinguish explicitly between elements in $T_I SL(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$.

In summary, the general solution of the Riccati equation (1.1) can be obtained through the curve in $SL(2,\mathbb{R})$ which starts from I and it is solution of equation (3.5). Therefore we have transformed the problem of finding the general solution of (1.1) to that of determining the curve solution of (3.5) starting from the neutral element.

4. Transformation laws of Riccati equations

In this section we describe an important property of Lie systems, in the particular case of Riccati equation, which plays a very relevant rôle for establishing, as indicated in [7], integrability criteria. The group of curves in the group defining the Lie system, here $SL(2,\mathbb{R})$, acts on the set of Riccati equations.

More explicitly, each Riccati equation (1.1) can be considered as a curve in $\mathbb{R}^3$. The point now is that we can transform every function in $\overline{\mathbb{R}}$, $x(t)$, under an element of the group of smooth $SL(2,\mathbb{R})$-valued curves $\operatorname{Map}(\mathbb{R}, SL(2,\mathbb{R}))$, which from now on will be denoted as $\mathcal{G}$, as follows [7]:

$$\begin{gathered}\Theta(A,x(t))=\frac{\alpha(t)x(t)+\beta(t)}{\gamma(t)x(t)+\delta(t)},\quad \text{if } x(t)\neq-\frac{\delta(t)}{\gamma(t)},\\ \Theta(A,\infty)=\alpha(t)/\gamma(t),\quad \Theta(A,-\delta(t)/\gamma(t))=\infty,\\ \text{when } A=\begin{pmatrix}\alpha(t) & \beta(t)\\ \gamma(t) & \delta(t)\end{pmatrix}\in\mathcal{G}.\end{gathered}\tag{4.1}$$

If we transform a curve $x(t)$ solution of (1.1) by means of $x'(t)=\Theta(\bar{A}(t),x(t))$, the function x' satisfies a new Riccati equation with coefficients b_2', b_1', b_0', given by

$$\begin{aligned}b_2'&=\bar{\delta}^2b_2-\bar{\delta}\bar{\gamma}b_1+\bar{\gamma}^2b_0+\bar{\gamma}\dot{\bar{\delta}}-\bar{\delta}\dot{\bar{\gamma}},\\ b_1'&=-2\bar{\beta}\bar{\delta}b_2+(\bar{\alpha}\bar{\delta}+\bar{\beta}\bar{\gamma})b_1-2\bar{\alpha}\bar{\gamma}b_0+\bar{\delta}\dot{\bar{\alpha}}-\bar{\alpha}\dot{\bar{\delta}}+\bar{\beta}\dot{\bar{\gamma}}-\bar{\gamma}\dot{\bar{\beta}},\\ b_0'&=\bar{\beta}^2b_2-\bar{\alpha}\bar{\beta}b_1+\bar{\alpha}^2b_0+\bar{\alpha}\dot{\bar{\beta}}-\bar{\beta}\dot{\bar{\alpha}}.\end{aligned}\tag{4.2}$$

In fact this expression defines an affine action (see e.g. [19] for the definition of this concept) of the group $\mathcal{G}$ on the set of Riccati equations. More details can be found in [7]. This means that in order to transform the coefficients of a general Riccati equation by means of first a transformation given by the curve $A_1(t)$ and then another defined by the curve $A_2(t)$, it suffices to do the transformation by the product element A_2A_1 of $\mathcal{G}$.

The result of the action of $\mathcal{G}$ can also be studied from the point of view of the equations in $SL(2,\mathbb{R})$. First, $\mathcal{G}$ acts on the left on the set of curves in $SL(2,\mathbb{R})$ by left translations, i.e. a curve $\bar{A}(t)$ transform the curve $A(t)$ into a new one $A'(t)=\bar{A}(t)A(t)$, and if $A(t)$ is a solution of (3.5) then the new curve satisfies a new equation like (3.5) but with a different right hand side, and therefore it corresponds to a new equation in $SL(2,\mathbb{R})$ associated with a new Riccati equation. In this way $\mathcal{G}$ acts on the set of curves in $\mathfrak{sl}(2,\mathbb{R})$, i.e. on the set of Riccati equations. It can be shown that the relation between both curves in $T_ISL(2,\mathbb{R})$ is given by:

$$\mathrm{a}'(t)=\bar{A}(t)\mathrm{a}(t)\bar{A}^{-1}(t)+\dot{\bar{A}}(t)\bar{A}^{-1}(t)=-\sum_{\alpha=0}^{2}b_\alpha'(t)M_\alpha.\tag{4.3}$$

Therefore, $A(t)$ is a solution of (3.5) if and only if $A'(t)=\bar{A}(t)A(t)$ is a solution for the equation corresponding to the curve $\mathrm{a}'(t)$ given by (4.3). If we restrict ourselves (3.5) to curves starting from the neutral element, $\bar{A}(t)$ should also start from I. The transformed Riccati equation, obtained through (4.2) is the one related with equation (4.3) in the group $SL(2,\mathbb{R})$.

We have shown that it is possible to associate Riccati equations with equations in the Lie group $SL(2,\mathbb{R})$ and to define a group of transformations on the set of Riccati equations. Roughly speaking, we have transformed the initial problem of

Riccati equations on $\overline{\mathbb{R}}$ into a problem in the group $SL(2,\mathbb{R})$ and we have explained a way to relate both problems.

5. Interpretation of integrability conditions

In this section we analyze and reproduce the results of [8, 9, 10, 11, 12, 13] from our geometric viewpoint. Our approach provides us with additional information about why the methods used in these papers work.

In summary, in most of these papers one starts with a Riccati differential equation

$$\frac{dy}{dt} = b_0(t) + b_1(t)y + b_2(t)y^2 \tag{5.1}$$

such that under a time-dependent change of variables $y' \equiv y'(y,t)$ the initial equation is transformed into

$$\frac{dy'}{dt} = \varphi(t)(c_0 + c_1 y' + c_2 y'^2) \tag{5.2}$$

with constants c_0, c_1 and c_2. This new equation is integrable as mentioned before. Then the solution of the initial differential equation is obtained from the solution of (5.2) in terms of the initial variable.

The key point in this method is that of finding out an appropriate time-dependent change of coordinates in order to transform the initial Riccati equation into a new one that can be integrated. Then, it is interesting to know the possible forms of these time-dependent changes of variables and their mathematical meaning.

Now, we will explain how to use our geometrical formalism for finding the general solution of the given Riccati equations and in particular for explaining the results of different papers about this topic in the literature.

The process of finding out the general solution for a Riccati equation can be seen as similar to a reduction process as it appears in [2, 20]. In our geometrical formalism, given an initial Riccati equation, there is a related equation in the group $SL(2,\mathbb{R})$ characterized by a curve $\mathrm{a}(t)$ in the Lie algebra of $SL(2,\mathbb{R})$. If this equation is not integrable, we can try to transform it into a new Riccati equation related with a Lie subalgebra $\mathfrak{h} \subset \mathfrak{sl}(2,\mathbb{R})$ which may be either a 1-dimensional one given by $\langle c_0 M_0 + c_1 M_1 + c_2 M_2\rangle$, with c_0, c_1, c_2 fixed real numbers, or a 2-dimensional one, isomorphic to the Lie algebra of the affine group on the real line. This transformation process is carried out by means of the action on the given equation in the group of an appropriate curve $\bar{A}(t)$ in such group. More specifically, this $\bar{A}(t)$ must be such that:

$$\mathrm{a}'(t) = \bar{A}(t)\mathrm{a}(t)\bar{A}^{-1}(t) + \dot{\bar{A}}(t)\bar{A}^{-1}(t) = -\varphi(t)\sum_{\alpha=0}^{2} c_\alpha M_\alpha \in T_I H \tag{5.3}$$

This is a Lie system in the connected Lie subgroup $H \subset SL(2,\mathbb{R})$ with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Once such a curve $\bar{A}(t)$ has been found, and also the solution of the corresponding Riccati equation to $\mathrm{a}'(t)$ given by (5.3), we can obtain the solution of the original equation in the group $SL(2,\mathbb{R})$ and $\overline{\mathbb{R}}$ by inverting the transformation carried out by $\bar{A}(t)$. In other words, if $A'(t)$ and $x'(t)$ are the solutions of the transformed equations, then $A(t) = \bar{A}^{-1}(t)A'(t)$ and $x(t) = \Theta(\bar{A}^{-1}(t), x'(t)) = \Theta(\bar{A}^{-1}(t)A'(t), x_0)$ are the solutions of the initial equations in the group $SL(2,\mathbb{R})$ and in $\overline{\mathbb{R}}$, respectively.

The important point here is that the described action of the group $\mathcal{G}$ of curves in $SL(2,\mathbb{R})$ on the set of Lie systems in the group $SL(2,\mathbb{R})$ (see [16] and [18]) can be used to construct transformations in the set of Riccati equations that are general enough to reproduce the transformations that have been used in the literature.

We next reproduce some results of the papers [8, 9, 10, 11, 12, 15] from our geometrical approach. Consider first the example studied in [10] where the following Riccati equation

$$\frac{dy}{dt} = b_0(t) + b_1(t)y + b_2(t)y^2, \quad b_2(t) > 0 \tag{5.4}$$

is analyzed in an interval such that $W = b_2^2(t)b_0(t) + \dot b_1(t)b_2(t) - b_1(t)\dot b_2(t) > 0$, where

$$\frac{b_2(t)W' - (3\dot b_2(t) - 2b_1(t)b_2(t))W}{2b_2(t)^{1/2}W^{3/2}} \equiv K, \tag{5.5}$$

K being a constant. Defining v by putting $W = b_2(t)^3v^2$ and using the so-called Rao's transformation:

$$y = v(t)y' - b_1(t)/b_2(t),$$

the Riccati equation (5.4) is transformed into

$$\frac{dy'}{dt} = \varphi(t)(c_0 + c_1y' + c_2y'^2),$$

where c_0, c_1 and c_2 are real numbers satisfying

$$\varphi(t)c_0 = \sqrt{\frac{W(t)}{b_2(t)}},$$
$$\varphi(t)c_1 = -K\sqrt{\frac{W(t)}{b_2(t)}},$$
$$\varphi(t)c_2 = \sqrt{\frac{W(t)}{b_2(t)}}.$$

Such new Riccati equation is separable and therefore it is integrable.

In the framework of Lie systems theory we start with the associated equation in $SL(2,\mathbb{R})$ given by

$$\dot A(t)A^{-1}(t) = -\sum_{\alpha=0}^{2} b_\alpha(t)M_\alpha = \mathrm{a}(t),$$

which under the left action of a curve $\bar A(t) \in \mathrm{Map}(\mathbb{R}, SL(2,\mathbb{R}))$ becomes a similar new equation but where the curve $\mathrm{a}(t)$ is replaced by a new one $\mathrm{a}'(t)$ according to formulas (4.2) or (4.3). In particular, with the choice

$$\bar\alpha(t) = \frac{1}{\sqrt{v(t)}}, \quad \bar\beta(t) = \frac{b_1(t)}{b_2(t)\sqrt{v(t)}}, \quad \bar\gamma(t) = 0, \quad \bar\delta(t) = \sqrt{v(t)},$$

we define a transformation that under the conditions imposed in the article gives the following result:

$$\frac{dy'}{dt} = \Big(\frac{W}{b_2(t)}\Big)^{1/2}(1 - Ky' + y'^2).$$

This equation corresponds to the final Riccati equation found in [10]. A similar transformation is that of Theorem 2 of [11]. The initial Riccati equation associated with an equation in the group $SL(2,\mathbb{R})$ characterized by a curve $\mathrm{a}(t)$ in $\mathfrak{sl}(2,\mathbb{R})$, becomes a new Lie system under the proposed transformation but now characterized by a curve in the 1-dimensional Lie subalgebra $\mathfrak{h} = \langle M_0 - KM_1 + M_2\rangle$. The equation in the subgroup determined by $\mathfrak{h}$ is integrable, and once its solution is known we recover the solution of the initial problem in $SL(2,\mathbb{R})$ as $A(t) = \bar{A}^{-1}(t)A'(t)$. Finally, the solution for the initial Riccati equation with initial condition $y(0) = y_0$ is given by $y(t) = \Theta(\bar{A}^{-1}(t)A'(t), y_0)$.

Another case also studied in [10] is when $W = 0$. In this case, we can perform the transformation defined by

$$\begin{aligned}
\bar{\alpha}(t) &= \exp\Big(\frac{1}{2}\int_0^t b_1(t')dt'\Big),\\
\bar{\beta}(t) &= \exp\Big(\frac{1}{2}\int_0^t b_1(t')dt'\Big)\frac{b_1(t)}{b_2(t)},\\
\bar{\gamma}(t) &= 0\\
\bar{\delta}(t) &= \exp\Big(-\frac{1}{2}\int_0^t b_1(t')dt'\Big),
\end{aligned}$$

and then, the new equation is

$$\frac{dy'}{dt} = b_2(t)\bar{\alpha}^{-2}(t)y'^2.$$

In this way we transform the initial Riccati equation, with $W = 0$, into a new one characterized by a curve in the one dimensional Lie subalgebra $\mathfrak{h} = \langle M_2\rangle \subset \mathfrak{sl}(2,\mathbb{R})$, and we can solve the problem as before. This last equation is also obtained in [10].

Another case of integrable Riccati equation (1.1) is considered in [11], where the coefficient functions $b_0(t), b_1(t), b_2(t)$ are assumed to satisfy the relations

$$\frac{dv}{dt} = -kb_0(t) + b_1(t)v, \quad b_2(t) = \frac{b_0(t)}{cv^2(t)},$$

where $v(t)$ is a new function and c and k are two real constants. We can transform (1.1) by means of the curve $\bar{A}(t)$ of coefficients

$$\bar{\alpha}(t) = \frac{1}{\sqrt{v(t)}}, \quad \bar{\beta}(t) = 0, \quad \bar{\gamma}(t) = 0, \quad \bar{\delta}(t) = \sqrt{v(t)},$$

into

$$\frac{dy'}{dt} = \frac{b_0(t)}{v(t)}\Big(\frac{y'^2}{c} + ky' + 1\Big).$$

Another example, given in [9], is a Riccati equation of form (1.1) whose coefficient functions $b_0(t), b_1(t), b_2(t)$ satisfy

$$\frac{b_1(t) + \frac{1}{2}\left(\frac{\dot{b}_2(t)}{b_2(t)} - \frac{\dot{b}_0(t)}{b_0(t)}\right)}{\sqrt{b_0(t)b_2(t)}} = C, \tag{5.6}$$

where C is a constant. In such a case it can be transformed into the new one:

$$\frac{dy'}{dt} = \sqrt{b_0(t)b_2(t)}(1 + Cy' + y'^2)$$

by means of the transformation given by the curve $\bar{A}(t)$ in $SL(2,\mathbb{R})$ with

$$\bar{\alpha}(t) = \Big(\frac{b_2(t)}{b_0(t)}\Big)^{1/4}, \quad \bar{\beta}(t) = 0, \quad \bar{\gamma}(t) = 0, \quad \bar{\delta}(t) = \Big(\frac{b_0(t)}{b_2(t)}\Big)^{1/4}.$$

Another example that can be analyzed from our viewpoint is the following Riccati equation, [12],

$$\frac{dy}{dt} = -\frac{c_1}{F(t)}y^2 + \Big(c_2 + \frac{F'(t)}{F(t)}\Big)y + F(t).$$

In this case, if $F(t) > 0$ we can perform the transformation defined by

$$\bar{\alpha}(t) = \sqrt{1/F(t)}, \quad \bar{\beta}(t) = 0, \quad \bar{\gamma}(t) = 0, \quad \bar{\delta}(t) = \sqrt{F(t)},$$

and the following new Riccati equation is obtained:

$$\frac{dy'}{dt} = -c_1 y'^2 + c_2 y' + 1,$$

which can easily be integrated. This is a particular case of the example in [11] explained above, with $v(t) = F(t)$, $k = c_2$ and $c = -1/c_1$. Once again, we have performed a transformation from the initial problem in $SL(2, \mathbb{R})$ characterized by a curve in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ into a new equation in the 1-dimensional Lie subgroup characterized by a curve $\mathrm{a}'(t)$ in the 1-dimensional subalgebra $\mathfrak{h} = \langle M_0 + c_2 M_1 - c_1 M_2\rangle$. The solution $A'(t)$ with initial condition $A'(0) = I$ lies in the corresponding subgroup and the solution to the initial Riccati equation can be obtained from it as indicated before.

Other transformations that can be used and the corresponding integrable Riccati equations are:

$$\bar{A}(t) = \begin{pmatrix} \sqrt{-\frac{c_1}{F(t)}} & 0 \\ 0 & \sqrt{-\frac{F(t)}{c_1}} \end{pmatrix} \Longrightarrow \frac{dy'}{dt} = y'^2 + c_2 y' - c_1,$$

$$\bar{A}(t) = \begin{pmatrix} \Big(\frac{-c_1}{F^2(t)}\Big)^{1/4} & 0 \\ 0 & \Big(-\frac{F^2(t)}{c_1}\Big)^{1/4} \end{pmatrix} \Longrightarrow \frac{dy'}{dt} = \sqrt{-c_1}\,y'^2 + c_2 y' + \sqrt{-c_1}$$

The first case is a particular case of the cited example in [11] with $v(t) = -F(t)/c_1$, $k = -c_2/c_1$ and $c = -c_1$. The second case is a particular case of the example in [9] developed above, where the constant C in (5.6) takes the value $c_2/\sqrt{-c_1}$. In both cases the time-dependent changes of coordinates can be described through our set of transformations.

We next study from our perspective an example which can be found in [8]. According to its results the Riccati equation can be integrated directly if the time-dependent coefficients satisfy the relation

$$\frac{d}{dt}\log\frac{-b_0(t)}{b_2(t)} = 2b_1(t).$$

or, equivalently,

$$\log\frac{-b_0(t)}{b_2(t)} = 2\int_0^t b_1(t')dt' + \log a,$$

where $a > 0$ is a constant. In this case, the way to proceed is just by means of the action (4.1) of the curve in $SL(2, \mathbb{R})$ given by

$$\bar{\alpha}(t) = \exp\Big(-\frac{1}{2}\int_0^t b_1(t')dt'\Big), \quad \bar{\beta}(t) = 0,$$

$$\bar{\gamma}(t) = 0, \quad \bar{\delta}(t) = \exp\Big(\frac{1}{2}\int_0^t b_1(t')dt'\Big).$$

The final result is

$$\frac{dy'}{dt} = (y'^2 - a)\bar{\alpha}^{-2}(t)b_2(t),$$

which can be integrated and thus, by inverting the time-dependent change of variables, we obtain the solution for the initial Riccati equation.

The aim of the transformations on Riccati equations considered so far was to obtain separable Riccati equations. This is not the only way to integrate such differential equations, but we can also transform the given Riccati equation into another one associated, as Lie system, with a bidimensional real Lie algebra, which is solvable, and therefore the corresponding Lie system can integrated in this way [20].

A particular instance of this which can be found in [13] is

$$\frac{dy}{dt} = P(t) + Q(t)y + k(Q(t) - kP(t))y^2,$$

where k is a non-vanishing constant. If we consider the transformation (4.1) with $\bar{\alpha}(t) = 0, \bar{\beta}(t) = -1/k, \bar{\gamma}(t) = k, \bar{\delta}(t) = 1$, it gives rise to the time independent change of variables

$$y' = -\frac{1}{k^2 y + k}, \tag{5.7}$$

under which we obtain an inhomogeneous linear equation

$$\frac{dy'}{dt} = \frac{Q(t)}{k} - P(t) + (Q(t) - 2kP(t))y'. \tag{5.8}$$

Note that this transformation (5.7) corresponds to the fact that $y = -1/k$ is a solution of the original Riccati equation.

In this equation the dynamical vector field

$$X = \Big(\frac{Q(t)}{k} - P(t)\Big)\frac{\partial}{\partial y'} + (Q(t) - 2kP(t))y'\frac{\partial}{\partial y'}$$

can be written in terms of two vector fields in $\mathbb{R}$

$$X_1 = \frac{\partial}{\partial y'}, \quad X_2 = y'\frac{\partial}{\partial y'},$$

because

$$X = \Big(\frac{Q(t)}{k} - P(t)\Big)X_1 + (Q(t) - 2kP(t))X_2.$$

These vector fields are such that $[X_1, X_2] = X_1$ and thus they close on a bidimensional non-Abelian real Lie algebra, which, as it is well-known, is solvable and (5.8) can be integrated by quadratures.

Finally, all the examples given in [15] can be dealt with our geometric formalism. Next, we shall specify some of the cases of such paper. The first one is a Riccati equation (1.1) such that the time-dependent coefficients satisfy that for a certain function $D(t)$ and constants a, b, c:

$$\begin{gathered} b_2(t)b_0(t) = acD^2(t), \\ \frac{\dot{b}_2(t)}{b_2(t)} + b_1(t) = \frac{\dot{D}(t)}{D(t)} + bD(t). \end{gathered}$$

It can be noted that if a, b, c and $D(t)$ satisfy the above integrability conditions in an interval of t, then $-a, b, -c$ and $D(t)$ satisfy the same conditions in the same interval. Then, we can always choose a in such a way that $aD(t)/b_2(t) > 0$ in that interval. Thus, the transformation given by $\bar{A}(t)$ with coefficients:

$$\bar{\alpha}(t) = \sqrt{\frac{b_2(t)}{aD(t)}}, \quad \bar{\beta}(t) = 0, \quad \bar{\gamma}(t) = 0, \quad \bar{\delta}(t) = \sqrt{\frac{aD(t)}{b_2(t)}},$$

allows us to transform the initial Riccati equation into

$$\frac{dy'}{dt} = D(t)(c + by' + ay'^2).$$

We have transformed the associated equation in the Lie group $SL(2,\mathbb{R})$ to the initial Riccati equation into a new one in a 1-dimensional Lie subgroup of $SL(2,\mathbb{R})$ of Lie algebra given by $\mathfrak{h} = \langle cM_0 + bM_1 + aM_2 \rangle$.

Other integrability conditions proposed in [15] are:

$$b_2(t)\Big(- \frac{dE}{dt} + b_2(t)E^2(t) + b_1(t)E(t) + b_0(t)\Big) = acD^2(t),$$
$$\frac{\dot{b}_2(t)}{b_2(t)} + b_1(t) + 2E(t)b_2(t) = \frac{\dot{D}(t)}{D(t)} + bD(t),$$

where a, b, c are constants and $D(t)$, $E(t)$ are functions. As before, we can always choose a in such a way that $aD(t)/b_2(t) > 0$. Then, if we consider the transformation given by $\bar{A}(t)$, with coefficients:

$$\bar{\alpha}(t) = \sqrt{\frac{b_2(t)}{aD(t)}}, \quad \bar{\beta}(t) = -\sqrt{\frac{b_2(t)}{aD(t)}}E(t), \quad \bar{\gamma}(t) = 0, \quad \bar{\delta}(t) = \sqrt{\frac{aD(t)}{b_2(t)}},$$

we see that the initial Riccati equation transforms into

$$\frac{dy'}{dt} = D(t)(c + by' + ay'^2),$$

which is integrable, and then we can obtain the solution to the initial Riccati equation. Other results of [15] are summarized in Table 1.

In this table we have used:

$$L[E(t)] = -\frac{dE}{dt} + b_2(t)E^2(t) + b_1(t)E(t) + b_0(t),$$
$$L2[E(t)] = L\left[\frac{A(t)}{B(t)} + E(t)\right].$$

As it happens in the examples above of [15], the integrability conditions presented in the table allow to change $a \to -a, c \to -c$ leaving them invariant. This symmetry is used implicitly in order to get the square roots to be real.

It is to be remarked that some of the time-dependent transformations used in [15] are homographies of the type

$$y' = \frac{\alpha(t)y + \beta(t)}{\gamma(t)y + \delta(t)},$$

for which the coefficients satisfy $\alpha(t)\delta(t) - \beta(t)\gamma(t) < 0$ and thus they cannot be treated directly by the method presented here. For example, a transformation like

$$y' = \frac{A(t)b_2(t)}{aD(t)}y, \quad \text{with } \frac{A(t)b_2(t)}{aD(t)} < 0$$

TABLE 1. Some integrability conditions in [15]

Integrability condition	Transformation
$b_2(t)b_0(t) = acD^2(t),\quad \frac{\dot b_0(t)}{b_0(t)} - b_1(t) = \frac{\dot D(t)}{D(t)} - bD(t)$	$\begin{pmatrix} \sqrt{\frac{cD(t)}{b_0(t)}} & 0 \\ 0 & \sqrt{\frac{b_0(t)}{cD(t)}} \end{pmatrix}$
$b_2(t)L[E(t)] = acD^2(t),\quad \frac{\dot L[E(t)]}{L[E(t)]} - b_1(t) - 2E(t)b_2(t) = \frac{\dot D(t)}{D(t)} + bD(t)$	$\begin{pmatrix} 0 & \sqrt{\frac{L[E(t)]}{aD(t)}} \\ -\sqrt{\frac{aD(t)}{L[E(t)]}} & E(t)\sqrt{\frac{aD(t)}{L[E(t)]}} \end{pmatrix}$
$b_2(t)L[E(t)] = acD^2(t),\quad \frac{\dot b_2(t)}{b_2(t)} + b_1(t) + 2E(t)b_2(t) = \frac{\dot D(t)}{D(t)} - bD(t)$	$\begin{pmatrix} 0 & \sqrt{\frac{cD(t)}{b_2(t)}} \\ -\sqrt{\frac{b_2(t)}{cD(t)}} & E(t)\sqrt{\frac{b_2(t)}{cD(t)}} \end{pmatrix}$
$b_2(t)L[E(t)] = acD^2(t),\quad \frac{\dot L[E(t)]}{L[E(t)]} - b_1(t) - 2E(t)b_2(t) = \frac{\dot D(t)}{D(t)} - bD(t)$	$\begin{pmatrix} \sqrt{\frac{cD(t)}{L[E(t)]}} & -E(t)\sqrt{\frac{cD(t)}{L[E(t)]}} \\ 0 & \sqrt{\frac{L[E(t)]}{cD(t)}} \end{pmatrix}$
$\left(\frac{B(t)}{A(t)}\right)^2 L[E(t)]L2[E(t)] = acD^2(t),\quad \frac{\dot L[E(t)]}{L[E(t)]} - \frac{2B(t)}{A(t)}L[E(t)] - b_1(t) - 2E(t)b_2(t) = \frac{\dot D(t)}{D(t)} + bD(t)$	$\begin{pmatrix} -\sqrt{\frac{L[E(t)]}{aD(t)}}\,\frac{B(t)}{A(t)} & \sqrt{\frac{L[E(t)]}{aD(t)}}\left(1 + E(t)\frac{B(t)}{A(t)}\right) \\ -\sqrt{\frac{aD(t)}{L[E(t)]}} & \sqrt{\frac{aD(t)}{L[E(t)]}}E(t) \end{pmatrix}$
$\left(\frac{B(t)}{A(t)}\right)^2 L[E(t)]L2[E(t)] = acD^2(t),\quad \frac{\dot L2[E(t)]}{L2[E(t)]} - \frac{2\frac{d}{dt}\left(A(t)/B(t)\right)}{A(t)/B(t)} + \frac{2B(t)}{A(t)}L[E(t)] + b_1(t) + 2E(t)b_2(t) = \frac{\dot D(t)}{D(t)} - bD(t)$	$\begin{pmatrix} -\sqrt{\frac{cD(t)}{L2[E(t)]}} & \sqrt{\frac{cD(t)}{L2[E(t)]}}\left(1 + \frac{B(t)}{A(t)}E(t)\right)\frac{A(t)}{B(t)} \\ -\frac{B(t)}{A(t)}\sqrt{\frac{L2[E(t)]}{cD(t)}} & \frac{B(t)}{A(t)}E(t)\sqrt{\frac{L2[E(t)]}{cD(t)}} \end{pmatrix}$

belong to this type. However, we can consider then a transformation $y \to y''$ of the form $y'' = -y$ and after the new transformation $y'' \to y'$ given by

$$y' = -\frac{A(t)b_2(t)}{aD(t)}y'', \quad \text{with } -\frac{A(t)b_2(t)}{aD(t)} > 0,$$

can be written as a homography with $\alpha(t)\delta(t) - \beta(t)\gamma(t) = 1$, namely

$$y' = \frac{\sqrt{-\frac{A(t)b_2(t)}{aD(t)}}\,y''}{\sqrt{-\frac{aD(t)}{A(t)b_2(t)}}} = -\frac{A(t)b_2(t)}{aD(t)}y''.$$

In summary, we may use the set of time-dependent changes of variables generated by curves in the group $SL(2,\mathbb{R})$ in order to transform a given Riccati equation into another one which is a Lie system with an associated solvable Lie algebra, i.e. either 1-dimensional or non-Abelian 2-dimensional one. In this way, the transformed Riccati equation can be integrated by quadratures and thus, by the time-dependent change of variables, we can obtain the solution to the initial Riccati equation.

Conclusions and outlook. In summary it has been shown in this paper that previous works about the integrability of the Riccati equation can be explained from the unifying viewpoint of Lie systems, and so appropriate transformations of $SL(2,\mathbb{R})$ can be used to transform the Riccati equations considered in the mentioned papers into other ones that can be easily integrated. These transformations are made in such a way that the initial Riccati equation, as a Lie system related with an equation in $SL(2,\mathbb{R})$, is transformed into another Riccati equation related with an equation in a Lie subgroup of $SL(2,\mathbb{R})$ with solvable algebra. The system is transformed in this way into a simpler one and the integrability conditions arise in this framework as sufficient conditions for the existence of the convenient curve in $SL(2,\mathbb{R})$ to be used in order to carry out this transformation process.

Another interesting problem to be studied in the future is to invert this process, i.e. we start from a Riccati equation related with a solvable group as a Lie system and perform a certain kind of transformations by means of the action of curves of $SL(2,\mathbb{R})$ in order to obtain Riccati equations integrable by quadratures. The integrability conditions appear then as properties of the set of Riccati equations obtained in this process.

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