Electronic Journal of Differential Equations, Vol. 2007(2007), No. 131, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A NONLINEAR TRANSMISSION PROBLEM WITH TIME DEPENDENT COEFFICIENTS

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Abstract. In this article, we consider a nonlinear transmission problem for the wave equation with time dependent coefficients and linear internal damping. We prove the existence of a global solution and its exponential decay. The result is achieved by using the multiplier technique and suitable unique continuation theorem for the wave equation.

1. Introduction

In this work, we consider the transmission problem

$$\rho_1 u_{tt} - b u_{xx} + f_1(u) = 0 \quad \text{in }]0, L_0[\times\mathbb{R}^+, \tag{1.1}$$

$$\rho_2 v_{tt} - (a(x,t)v_x)_x + \alpha v_t + f_2(v) = 0 \quad \text{in }]L_0, L[\times\mathbb{R}^+, \tag{1.2}$$

$$u(0,t) = v(L,t), \quad t > 0, \tag{1.3}$$

$$u(L_0,t) = v(L_0,t), \quad b u_x(L_0,t) = a(L_0,t)v_x(L_0,t), \quad t > 0, \tag{1.4}$$

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), \quad x \in]0, L_0[, \tag{1.5}$$

$$v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), \quad x \in]L_0, L[, \tag{1.6}$$

where ρ_1, ρ_2 are constants; α, b are positive constants, f, g are nonlinear functions and $a(x,t)$ is a positive function. Controllability and Stability for transmission problem has been studied by many authors (see for example Lions [7], Lagnese [5], Liu and Williams [8], Muñoz Rivera and Portillo Oquendo [9], Andrade, Fatori and Muñoz Rivera [1]).

The goal of this work is to study the existence and uniqueness of global solutions of (1.1)-(1.6) and the asymptotic behavior of the energy.

All the authors mentioned above established their results with constant coefficients. To the best of our knowledge this is a first publication on transmission problem with time dependent coefficients and the nonlinear terms. In general,the dependence on spatial and time variables causes difficulties,semigroups arguments are not suitable for finding solutions to (1.1)-(1.6); therefore,we make use of a Galerkin's process. Note that the time-dependent coefficient also appear in the second boundary condition, thus there are some technical difficulties that we need to overcome. To prove the exponential decay, the main difficulty is that the dissipation only

2000 *Mathematics Subject Classification.* 35B40, 35L70, 45K05.
Key words and phrases. Transmission problem; time dependent coefficients; stability.
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Submitted May 2, 2007. Published October 9, 2007.

works in $[L_0, L]$ and we need estimates over the whole domain $[0, L]$; we overcome this problem introducing suitable multiplicadors and a compactness/uniqueness argument.

2. Notation and statement of results

We denote

$$(w, z) = \int_I w(x)z(x)dx, \quad |z|^2 = \int_I |z(x)|^2 dx$$

where $I =]0, L_0[$ or $]L_0, L[$ for u's and v's respectively. Now, we state the general hypotheses.

(A1) The functions $f_i \in C^1(\mathbb{R})$, $i = 1, 2$, satisfy $f_i(s)s \geq 0$ for all $s \in \mathbb{R}$ and

$$|f_i^{(j)}(s)| \leq c(1 + |s|)^{\rho - j}, \quad \forall s \in \mathbb{R},\ j = 0, 1$$

for some $c > 0$ and $\rho \geq 1$. We assume that $f_1(s) \geq f_2(s)$ and set

$$F_i(s) = \int_0^s f_i(\xi)d\xi\,.$$

(A2) We assume that the coefficient a satisfies

$$\begin{gathered} a \in W^{1,\infty}(0, \infty; C^1([L_0, L])) \cap W^{2,\infty}(0, \infty; L^\infty(L_0, L)) \\ a_t \in L^1(0, \infty; L^\infty(L_0, L)) \\ a(x, t) \geq a_0 > 0, \quad \forall (x, t) \in]L_0, L[\times]0, \infty[\,. \end{gathered}$$

We define the Hilbert space

$$V = \{(w, z) \in H^1(0, L_0) \times H^1(L_0, L) : w(0) = z(L) = 0;\ w(L_0) = z(L_0)\}\,.$$

Also we define the first-order energy functionals associated to each equation,

$$\begin{gathered} E_1(t, u) = \frac{1}{2}\Big(\rho_1|u_t|^2 + b|u_x|^2 + 2\int_0^{L_0} F_1(u)dx\Big) \\ E_2(t, v) = \frac{1}{2}\Big(\rho_2|v_t|^2 + (a, v_x^2) + 2\int_{L_0}^{L} F_2(v)dx\Big) \\ E(t) = E_1(t, u, v) = E_1(t, u) + E_2(t, v). \end{gathered}$$

We conclude this section with the following lemma which will play essential role when establishing the asymptotic behavior of solutions.

Lemma 2.1 ([2, Lemma 9.1])**.** *Let $E : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a non-increasing function and assume that there exist two constants $p > 0$ and $c > 0$ such that*

$$\int_s^{+\infty} E^{(p+1)/2}(t)dt \leq cE(s), \quad 0 \leq s < +\infty.$$

Then for all $t \geq 0$,

$$E(t) \leq \begin{cases} cE(0)(1 + t)^{-2(p-1)} & \text{if } p > 1, \\ cE(0)e^{1-wt} & \text{if } p = 1, \end{cases}$$

where c and w are positive constants.

3. Existence and uniqueness of solutions

First, we define weak solutions of problem (1.1)-(1.6).

Definition 3.1. We say that the pair $\{u, v\}$ is a weak solution of (1.1)-(1.6) when

$$\{u, v\} \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; L^2(0, L_0) \times L^2(L_0, L))$$

and satisfies

$$\begin{aligned}
&- \rho_1 \int_0^{L_0} u^1(x)\varphi(x, 0)dx + \rho_1 \int_0^{L_0} u^0(x)\varphi_t(x, 0)dx - \rho_2 \int_{L_0}^{L} v^1(x)\psi(x, 0)dx \\
&+ \rho_2 \int_{L_0}^{L} v^0(x)\psi_t(x, 0)dx + \rho_1 \int_0^T \int_0^{L_0} (u\varphi_{tt} + bu_x\varphi_x + f_1(u)\varphi)\, dx\, dt \\
&+ \rho_2 \int_0^T \int_{L_0}^{L} (v\psi_{tt} + a(x, t)v_x\psi_x + \alpha v_t\psi + f_2(v)\psi)\, dx\, dt = 0
\end{aligned}$$

for any $\{\varphi, \psi\} \in C^2(0, T; V)$ such that $\varphi(T) = \varphi_t(T) = 0 = \psi(T) = \psi_t(T)$

To show the existence of strong solutions we need a regularity result for the elliptic system associated to the problem (1.1)–(1.6) whose proof can be obtained, with little modifications, in the book by Ladyzhenskaya and Ural'tseva [3, theorem 16.2].

Lemma 3.2. *For any given functions $F \in L^2(0, L_0)$, $G \in L^2(L_0, L)$, there exists only one solution $\{u, v\}$ to the system*

$$\begin{gathered}
-bu_{xx} = F \quad \text{in }]0, L_0[, \\
-(a(x, t)v_x)_x = G \quad \text{in }]L_0, L[, \\
u(0) = v(L) = 0, \\
u(L_0) = v(L_0), \quad bu_x(L_0) = a(L_0, t)v_x(L_0),
\end{gathered}$$

with t a fixed value in $[0, T]$, with u in $H^2(0, L_0)$ and v in $H^2(L_0, L)$.

The existence result to the system (1.1)–(1.6) is summarized in the following theorem.

Theorem 3.3. *Suppose that $\{u^0, v^0\} \in V$, $\{u^1, v^1\} \in L^2(0, L_0) \times L^2(L_0, L)$ and that assumptions (A1)–(A3) hold. Then there exists a unique weak solution of (1.1)–(1.6) satisfying*

$$\{u, v\} \in C(0, T; V) \cap C^1(0, T; L^2(0, L_0) \times L^2(L_0, L)).$$

In addition, if $\{u^0, v^0\} \in H^2(0, L_0) \times H^2(L_0, L)$, $\{u^1, v^1\} \in V$, verifying the compatibility condition

$$bu_x^0(L_0) = a(L_0, 0)\ v_x^0(L_0)\,. \tag{3.1}$$

Then

$$\{u, v\} \in \bigcap_{k=0}^{2} W^{k,\infty}(0, T; H^{2-k}(0, L_0) \times H^{2-k}(L_0, L))$$

Proof. The main idea is to use the Galerkin Method. Let $\{\varphi^i, \psi^i\}$, $i = 1, 2, \dots$ be a basis of V. Let us consider the Galerkin approximation

$$\{u^m(t), v^m(t)\} = \sum_{i=1}^{m} h_{im}(t)\{\varphi^i, \psi^i\}$$

where u^m and v^m satisfy

$$\begin{aligned} &\rho_1(u^m_{tt},\varphi^i)+b(u^m_x,\varphi^i_x)+(f_1(u^m),\varphi^i)+\rho_2(v^m_{tt},\psi^i)\\ &+(a(x,t)v^m_x,\psi^i_x)+\alpha(v^m_t,\psi^i)+(f_2(v^m),\psi^i)=0 \end{aligned} \tag{3.2}$$

where $i=1,2,\dots$ With initial data

$$\begin{gathered} \{u^m(0),v^m(0)\}\to\{u^0,v^0\} \quad \text{in } V,\\ \{u^m_t(0),v^m_t(0)\}\to\{u^1,v^1\} \quad \text{in } L^2(0,L)\times L^2(L_0,L). \end{gathered} \tag{3.3}$$

Standard results about ordinary differential equations guarantee that there exists only one solution of this system on some interval $[0,T_m[$. The priori estimate that follow imply that in fact $T_m=+\infty$. $\square$

subsection*Existence of weak solutions Multiplying (3.2) by $h'_{im}(t)$ integrating by parts and summing over i, we get

$$\frac{d}{dt}\,E(t,u^m,v^m)+\alpha|v^m_t|^2\le\frac{|a_t(t)|_{L^\infty}}{a_0}\,E(t,u^m,v^m). \tag{3.4}$$

From this inequality, the Gronwall's inequality and taking into account the definition of the initial data of $\{u^m,v^m\}$ we conclude that

$$E(t,u^m,v^m)\le C,\quad \forall t\in[0,T],\ \forall m\in\mathbb{N} \tag{3.5}$$

thus we deduce that

$$\begin{gathered} \{u^m,v^m\} \text{ is bounded in } L^\infty(0,T;V)\\ \{u^m_t,v^m_t\} \text{ is bounded in } L^\infty(0,T;L^2(0,L_0)\times L^2(L_0,L)) \end{gathered}$$

which implies that

$$\begin{gathered} \{u^m,v^m\}\to\{u,v\} \text{ weak } * \text{ in } L^\infty(0,T;V)\\ \{u^m_t,v^m_t\}\to\{u_t,v_t\}\text{weak } * \text{ in } L^\infty(0,T;L^2(0,L_0)\times L^2(L_0,L)). \end{gathered}$$

In particular, by application of the Lions-Aubin's Lemma [6, Theorem 5.1], we have $\{u^m,v^m\}\to\{u,v\}$ strongly in $L^2(0,T;L^2(0,L_0)\times L^2(L_0,L))$ and consequently

$$\begin{gathered} u^m\to u \text{ a.e in }]0,L_0[\text{ and } f_1(u^m)\to f_1(u) \text{ a.e in }]0,L_0[\\ v^m\to v \text{ a.e in }]L_0,L[\text{ and } f_2(v^m)\to f_2(v) \text{ a.e in }]L_0,L[. \end{gathered}$$

Also, from the growth condition in (A1) we have

$$\begin{gathered} f_1(u^m) \text{ is bounded in } L^\infty(0,T;L^2(0,L_0))\\ f_2(v^m) \text{ is bounded in } L^\infty(0,T;L^2(L_0,L)); \end{gathered}$$

therefore,

$$\{f_1(u^m),f_2(v^m)\}\rightharpoonup\{f_1(u),f_2(v)\}\quad \text{in } L^2(0,T;L^2(0,L_0)\times L^2(L_0,L)).$$

The rest of the proof of the existence of a weak solution is matter of routine.

Regularity of solutions. To get the regularity, we take a basis $B = \{\{\varphi^i, \psi^i\}, i \in \mathbb{N}\}$ such that

$$\{u^0, v^0\},\ \{u^1, v^1\}\text{are in the span of } \{\{\varphi^0, \psi^0\}, \{\varphi^1, \psi^1\}\}.$$

Therefore, $\{u^m(0), v^m(0)\} = \{u^0, v^0\}$ and $\{u^m_t(0), v^m_t(0)\} = \{u^1, v^1\}$. Let us differentiate the approximate equation and multiply by $h''_{im}(t)$. Using a similar argument as before we obtain

$$\begin{aligned} \frac{d}{dt} E_2(t, u^m, v^m) + \alpha|v^m_{tt}|^2 &= -(f_1'(u^m)u^m_t, u^m_{tt}) - (f_2'(v^m)v^m_t, v^m_{tt}) \\ &\quad - (a_t v^m_x, v^m_{xtt}) + \frac{1}{2}(a_t, (v^m_{xt})^2) \end{aligned} \tag{3.6}$$

where

$$E_2(t, u, v) = \frac{\rho_1}{2}|u_{tt}|^2 + \frac{b}{2}|u_{xt}|^2 + \frac{\rho_2}{2}|v_{tt}|^2 + \frac{1}{2}(a, v_{xt})^2.$$

Note that

$$-(a_t v^m_x, v^m_{xtt}) = -(a_t v^m_x, v^m_{xt})_t + (a_{tt} v^m_x, v^m_{xt}) + \big(a_t, (v^m_{xt})^2\big), \tag{3.7}$$

$E_2(0, u^m, v^m)$ is bounded, because of our choice of the basis.

From the assumption (A1) and from the Sobolev imbedding we have

$$\int_0^{L_0} f_1'(u^m)u^m_t u^m_{tt} dx \le C\Big[\int_0^{L_0} (1 + |u^m_x|)^2 dx\Big]^{(p-1)/2} |u^m_{xt}||u^m_{tt}|, \tag{3.8}$$

and similarly

$$\int_{L_0}^{L} f_2'(v^m)v^m_t v^m_{tt} dx \le C\Big[\int_{L_0}^{L} (1 + |v^m_x|)^2 dx\Big]^{(p-1)/2} |v^m_{xt}||v^m_{tt}| \tag{3.9}$$

Substituting (3.7), the inequalities (3.8)–(3.9), using the estimative (3.5) in (3.6) and applying Gronwall's inequality we conclude that

$$E_2(t, u^m, v^m) \le C \tag{3.10}$$

which imply

$$\begin{gathered} \{u^m_t, v^m_t\} \to \{u_t, v_t\} \quad \text{weak} * \text{ in } L^\infty(0, T; H^1(0, L_0) \times H^1(L_0, L)) \\ \{u^m_{tt}, v^m_{tt}\} \to \{u_{tt}, v_{tt}\} \quad \text{weak} * \text{ in } L^\infty(0, T; L^2(0, L_0) \times L^2(L_0, L)). \end{gathered}$$

Therefore, $\{u, v\}$ satisfies (1.1)–(1.4) and we have

$$\begin{gathered} -bu_{xx} = -\rho_1 u_{tt} - f_1(u) \in L^2(0, L_0), \\ -(a(x,t)v_x)_x = -\rho_2 v_{tt} - f_2(v) - \alpha v_t \in L^2(L_0, L), \\ u(L_0, t) = v(L_0, t), \quad bu_x(L_0, t) = a(L_0, t)v_x(L_0, t). \\ u(0, t) = 0 = v(L, t) \end{gathered}$$

Then using Lemma 3.2 we have the required regularity for $\{u, v\}$.

4. Exponential Decay

In this section we prove that the solution of the system (1.1)–(1.6) decay exponentially as time approaches infinity. In the remainder of this paper we denote by c a positive constant which takes different values in different places. We shall suppose that $\rho_1 \le \rho_2$ and

$$a(x,t) \le b, \quad a_t(x,t) \le 0, \quad \forall (x,t) \in]L_0, L[\times]0,\infty[$$
$$a_x(x,t) \le 0\,.$$

Theorem 4.1. *Take $\{u^0, v^0\}$ in V and $\{u^1, v^1\}$ in $L^2(0,L_0)\times L^2(L_0,L)$ with*

$$u_x^0(L_0) = 0. \tag{4.1}$$

Then there exists positive constants γ and c such that

$$E(t) \le cE(0)e^{-\gamma t}, \quad \forall t \ge 0. \tag{4.2}$$

We shall prove this theorem for strong solutions; our conclusion follow by standard density arguments.

The dissipative property of (1.1)–(1.6) is given by the following lemma.

Lemma 4.2. *The first-order energy satisfies*

$$\frac{d}{dt}\, E_1(t,u,v) = -\alpha|v_t|^2 + (a_t, v_x^2). \tag{4.3}$$

Proof. Multiplying equation (1.1) by u_t, equation (1.2) by v_t and performing an integration by parts we get the result. $\square$

Let $\psi \in C_0^\infty(0,L)$ be such that $\psi = 1$ in $]L_0-\delta, L_0+\delta[$ for some $\delta > 0$, small constant. Let us introduce the following functional

$$I(t) = \int_0^{L_0} \rho_1 u_t q u_x dx + \int_{L_0}^{L} \rho_2 v_t \psi q v_x dx$$

where $q(x) = x$.

Lemma 4.3. *There exists $c_1 > 0$ such that for all $\varepsilon > 0$,*

$$\begin{aligned}
\frac{d}{dt}I(t) \le &-\frac{L_0}{2}\{(\rho_2-\rho_1)v_t^2(L_0,t) + a(L_0,t)[1-\frac{a(L_0,t)}{b}]v_x^2(L_0,t)\}\\
&- L_0(F_1(u(L_0,t)) - F_2(v(L_0,t))) - \frac{1}{2}\int_0^{L_0}(\rho_1 u_t + bu_x^2 + 2F(u))dx\\
&- \frac{1}{4}\int_{L_0}^{L_0+\delta} av_x^2 dx + c_1\Big(\int_{L_0+\delta}^{L_0}(v_t^2 + av_x^2)dx + \int_{L_0}^{L} v_t^2 dx + \int_0^{L_0} u^2 dx\\
&+ \int_{L_0}^{L} v^2 dx\Big) + \varepsilon E(t,u,v)\,.
\end{aligned}$$

Proof. Multiplying (1.1) by qu_x, equation (1.2) by $\psi q v_x$, integrating by parts and using the corresponding boundary conditions we obtain

$$\begin{aligned}\frac{d}{dt}(\rho_1 u_t, qu_x) &= \frac{L_0}{2}[\rho_1 u_t^2(L_0,t)+bu_x^2(L_0,t)] - L_0F_1(u(L_0,t))\\ &\quad -\frac{1}{2}\int_0^{L_0}\rho_1 u_t^2 + bu_x^2 + 2F_1(u)dx\end{aligned}\tag{4.4}$$

$$\begin{aligned}\frac{d}{dt}(\rho_2 v_t,\psi q v_x) &\le -\frac{L_0}{2}\big[\rho_2 v_t^2(L_0,t)+a(L_0,t)v_x^2(L_0,t)\big]\\ &\quad + L_0F_2(v(L_0,t)) + \frac{1}{2}\int_{L_0}^{L_0+\delta} x a_x\psi v_x^2 dx - \frac{1}{4}\int_{L_0}^{L_0+\delta} a v_x^2 dx\\ &\quad + c_1\Big[\int_{L_0+\delta}^{L}(v_t^2+av_x^2)dx + \int_{L_0}^{L}(v_t^2+F_2(v))dx\Big]\end{aligned}\tag{4.5}$$

Summing up (4.4) and (4.5), and taking the assumption on a_x into account, we get

$$\begin{aligned}\frac{d}{dt}I(t) &\le -\frac{L_0}{2}[(\rho_2-\rho_1)v_t^2(L_0,t)+a(L_0,t)v_x^2(L_0,t)-bu_x^2(L_0,t)]\\ &\quad - L_0[F_1(u(L_0,t))-F_2(v(L_0,t))]\\ &\quad -\frac{1}{2}\int_0^{L_0}(\rho_1 u_t^2+bu_x^2+2F_1(u))dx - \frac{1}{4}\int_{L_0}^{L_0+\delta} a v_x^2 dx\\ &\quad + c_1\Big(\int_{L_0+\delta}^{L}(v_t^2+av_x^2)dx + \int_{L_0}^{L}(v_t^2+F_2(v))dx + \int_0^{L_0}F(u)dx\Big)\end{aligned}\tag{4.6}$$

According to (A1), we have $f_i(0)=0$ and

$$|f_i(s)| \le c(|s|+|s|^\rho)\tag{4.7}$$

this implies

$$|F_i(s)| \le c(|s|^2+|s|^{\rho+1}) \le c(|s|^2+|s|^{2\rho}).\tag{4.8}$$

From the interpolation inequality

$$|y|_p \le |y|_2^\alpha |y|_q^{1-\alpha},\quad \frac{1}{p}=\frac{\alpha}{2}+\frac{1-\alpha}{q},\quad \alpha\in[0,1]$$

and the immersion $H^1(\Omega)\hookrightarrow L^{2(2p-1)}(\Omega)$, $\Omega =]0,L_0[,\ ]L_0,L[$, we obtain for all $t\ge 0$

$$|u(t)|_{2\rho}^{2\rho} \le c_\varepsilon[E(0)]^{2(\rho-1)}|u(t)|_2^2 + \frac{\varepsilon}{[E(0)]^{2(\rho-1)}}|u_x(t)|_2^{2(2\rho-1)},\ \text{for all }\ \varepsilon>0.$$

Considering that

$$|u_x(t)|_2^2 \le cE(0,u,v) \equiv c_1E(0)$$

it follows that

$$|u(t)|_{2\rho}^{2\rho} \le c_\varepsilon[E(0)]^{2(\rho-1)}|u(t)|_2^2 + \varepsilon E(t,u,v).\tag{4.9}$$

Replacing the inequalities (4.7)–(4.9) in (4.6) our conclusion follows. $\square$

Let $\varphi\in C^\infty(\mathbb{R})$ a nonnegative function such that $\varphi=0$ in $I_{\delta/2} =]L_0-\frac{\delta}{2},\ L_0+\frac{\delta}{2}[$ and $\varphi=1$ in $\mathbb{R}\backslash I_\delta$ and consider the functional

$$J(t)=\int_{L_0}^{L}\rho_2 v_t\varphi v\ dx.$$

We have the following lemma.

Lemma 4.4. *Given $\varepsilon > 0$, there exists a positive constant c_ε such that*
$$\frac{d}{dt}\ J(t) \le -\frac{1}{2}\int_{L_0+\delta}^{L} av_x^2\,dx + \varepsilon\int_{L_0}^{L_0+\delta} av_x^2\,dx + c_\varepsilon\int_{L_0}^{L}(v^2+v_t^2)dx$$

Proof. Multiplying equation (1.2) by φv and integrating by parts we get
$$\frac{d}{dt}J(t) = -(av_x,\varphi v_x) - (av_x,\varphi_x v) - \alpha(v_t,\varphi v) - (\varphi, f_2(v)v) + (v_t,\varphi v_t).$$
Applying Young's Inequality and hypothesis (A1) we concludes our assertion. □

Let us consider the functional
$$K(t) = I(t) + (2c_1+1)J(t)$$
and we take $\varepsilon = \varepsilon_1$ in lemma 4.4, where ε_1 is the solution of the equation
$$(2c_1+1)\varepsilon_1 = \frac{1}{8}\,.$$
Taking in to consideration (A1) in lemma 4.3, we obtain
$$\begin{aligned}\frac{d}{dt}K(t) &\le -E_1(t,u) - \frac{1}{8}\int_{L_0}^{L}(av_x^2+2F_2(v))dx + \varepsilon E(t,u,v)\\ &\quad + c_2\Big(\int_{L_0}^{L}(v_t^2+v^2)dx + \int_0^{L_0} u^2dx\Big).\end{aligned}\tag{4.10}$$
Now in order to estimate the last two terms of (4.10) we need the following result.

Lemma 4.5. *Let $\{u,v\}$ be a solution in theorem 3.3. Then there exists $T_0 > 0$ such that if $T \ge T_0$ we have*
$$\int_S^T(|v|^2+|u|^2)ds \le \varepsilon\Big[\int_S^T(b|u_x|^2+|u_t|^2)ds+\int_S^T|a^{1/2}v_x|^2ds\Big]+c_\varepsilon\int_S^t|v_t|^2ds \tag{4.11}$$
for any $\varepsilon > 0$ and c_ε is a constant depending on T and ε, by independent of $\{u,v\}$, for any initial data $\{u^0,v^0\},\{u^1,v^1\}$ satisfying $E(0,u,v)\le R$, where $R>0$ is fixed and $0<S<T<+\infty$.

Proof. We use a contradiction argument. If (4.11) were false, there would exist a sequence of solutions $\{u^\nu,v^\nu\}$ such that
$$\int_S^T(|v^\nu|^2+|u^\nu|^2)ds \ge \nu\int_S^t|v_t^\nu|^2ds + c_0\int_S^T(b|u_x^\nu|^2+|u_t|^2+|a^{1/2}v_x|^2)ds$$
and $E(0,u^\nu,v^\nu)\le R$ for all ν. Let
$$\lambda_\nu^2 = \int_S^T(|v^\nu|^2+|u^\nu|^2)ds,$$
$$w^\nu(x,t) = \frac{u^\nu(x,t)}{\lambda_\nu},\quad z^\nu(x,t) = \frac{v^\nu(x,t)}{\lambda_\nu},\quad 0\le t\le T.$$
Then we have
$$\nu\int_S^T|z_t^\nu|^2ds + c_0\int_S^T\Big(b|w_x^\nu|^2+|w_t^\nu|^2+|a^{1/2}z_x^\nu|^2\Big)ds \le 1$$

and consequently

$$\int_S^T |z_t^\nu|^2 ds \to 0 \quad \text{as } \nu \to \infty, \tag{4.12}$$

$$\int_S^T (b|w_x^\nu|^2 + |w_t^\nu|^2 + |a^{1/2} z_x^\nu|^2) ds \le c. \tag{4.13}$$

Also we have

$$\int_S^T \big(|z^\nu|^2 + |w^\nu|^2\big) ds = 1\,. \tag{4.14}$$

Since S is chosen in the interval $[0,T[$, we obtain from (4.12)–(4.13) that, there exists a subsequence $\{w^\nu, z^\nu\}$ which we denote in the same way, such that

$$\begin{gathered}
w^\nu \to w \quad \text{in } L^2(0,T;H^1(0,L_0)),\\
w_t^\nu \to w_t \quad \text{in } L^2(0,T;L^2(0,L_0)),\\
z^\nu \to z \quad \text{in } L^2(0,T;H^1(L_0,L)),\\
z_t^\nu \to 0 \quad \text{in } L^2(0,T;L^2(L_0,L)).
\end{gathered}$$

From which

$$\begin{gathered}
w^\nu \to w \quad \text{in } L^2(0,T;L^2(0,L_0)),\\
z^\nu \to z \quad \text{in } L^2(0,T;L^2(L_0,L)).
\end{gathered}$$

This implies

$$\int_0^T \big(|z|^2 + |w|^2\big) ds = 1\,. \tag{4.15}$$

Besides, from the uniqueness of the limit we conclude that $z_t(x,0) = 0$ and therefore

$$z(x,t) = \varphi(x)\,. \tag{4.16}$$

□

Note that $\{w^\nu, z^\nu\}$ satisfies

$$\begin{gathered}
\rho_1 w_{tt}^\nu - b w_{xx}^\nu + \frac{1}{\lambda_\nu} f_1(\lambda_\nu w^\nu) = 0 \quad \text{in }]0,L_0[\times]0,T[,\\
\rho_2 z_{tt}^\nu - (a(x,t) z_x^\nu)_x + \frac{1}{\lambda_\nu} f_2(\lambda_\nu z^\nu) + \alpha z_t^\nu = 0 \quad \text{in }]L_0,L[\times]0,T[,\\
w^\nu(0,t) = 0 = z^\nu(L,t),\\
w^\nu(L_0,t) = z^\nu(L_0,t),\\
b w_x^\nu(L_0,t) = a(L_0,t) z_x^\nu(L_0,t),\\
w^\nu(x,0) = \frac{u^{\nu,0}(x)}{\lambda_\nu}, \quad w_t^\nu(x,0) = \frac{1}{\lambda_\nu} u^{\nu,1}(x),\\
z^\nu(x,0) = \frac{1}{\lambda_\nu}\ v^{\nu,0}(x), \quad z_t^\nu(x,0) = \frac{1}{\lambda_\nu}\ v^{\nu,1}(x).
\end{gathered} \tag{4.17}$$

Now, we observe that $\{\lambda_\nu\}_{\nu\geq 1}$ is a bounded sequence,

$$\begin{aligned}\lambda_\nu &= \Big[\int_S^T (|v^\nu|^2 + |u^\nu|^2)ds\Big]^{1/2}\\ &\leq c\Big[\int_S^T (|v_x^\nu|^2 + |u_x^\nu|^2)ds\Big]^{1/2}\\ &\leq cE(0,u,v) \leq cR,\end{aligned}$$

where R is a fixed value, because the initial data are in the ball $B(\theta, R)$. Hence, there exists a subsequence of $\{\lambda_\nu\}_{\nu\geq 1}$ (still denoted by (λ_ν) such that

$$\lambda_\nu \to \lambda \in]0,+\infty[.$$

In this case passing to limit in (4.17), when $\nu \to \infty$ for $\{w, z\}$, we get

$$\begin{gathered}\rho_1 w_{tt} - bw_{xx} + \frac{1}{\lambda}\ f_1(\lambda w) = 0 \quad \text{in} \quad]0, L_0[\times]0,T[,\\ (a(x,t)z_x)_x + \frac{1}{\lambda} f_2(\lambda z) = 0 \quad \text{in }]L_0, L[\times]0,T[,\\ w(0,t) = 0 = z(L,t)\\ w(L_0,t) = z(L_0,t)\\ bw_x(L_0,t) = a(L_0,t)z_x(L_0,t),\\ z_t(x,0) = 0 \quad \text{in }]L_0, L[\times]0,T[,\end{gathered} \tag{4.18}$$

and for $y = w_t$,

$$\begin{gathered}\rho_1 y_{tt} - by_{xx} + f'(\lambda w)y = 0 \quad \text{in }]0, L_0[\times]0,T[,\\ y(0,t) = 0 = y(L_0,t),\\ by_x(L_0,t) = a_t(L_0,t)z_x(L_0,t).\end{gathered} \tag{4.19}$$

Here, we observe that

$$\frac{w_{xt}(L_0,t)}{w_x(L_0,t)} = \frac{a_t(L_0,t)}{a(L_0,t)}\,.$$

Then after an integration, $w_x(L_0,t) = k\ a(L_0,t)$ whee k is a constant. Using the hypotheses, we obtain

$$0 = \lim_{t\to 0^+} w_x(L_0,t) = ka(L_0,0).$$

Consequently $k = 0$ and $y_x(L_0,t) = 0$. Thus, the function y satisfies

$$\begin{gathered}\rho_1 y_{tt} - by_{xx} + f'(\lambda w)y = 0 \quad \text{in }]0, L_0[\times]0,T[,\\ y(0,t) = 0 = y(L_0,t) \quad \text{on }]0,T[,\\ y_x(L_0,t) = 0 \quad \text{on} \quad]0,T[.\end{gathered}$$

Then, using the results in [4] (based on Ruiz arguments [10]) adapted to our case we conclude that $y = 0$, that is $w_t(x,t) = 0$, for T suitable big.

Returning to (4.18) we obtain the elliptic system

$$\begin{gathered}-bw_{xx} + \frac{1}{\lambda}\ f_1(\lambda w) = 0,\\ (a(x,t)z_x)_x + \frac{1}{\lambda} f_2(\lambda z) = 0\,.\end{gathered}$$

multiplying by u and v respectively and integrating, then summing up we arrive at

$$b\int_0^{L_0} w_x^2 dx + \int_{L_0}^{L} a(x,t) z_x^2 dx + \frac{1}{\lambda}\int_0^{L_0} f_1(\lambda w) w dx + \frac{1}{\lambda}\int_{L_0}^{L} f_2(\lambda z) z dx = 0\,.$$

So we have $w = 0$ and $z = 0$, which contradicts (4.15).

Suppose we are not in the above situation and there exists a subsequence satisfying $\lambda_\nu \to 0$. Applying inequality (4.10) to the solutions $\{u^\nu, v^\nu\}$ we have

$$\frac{d}{dt}K^\nu(t) \le -\delta_0 E(t,u^\nu,v^\nu) + c_3\Big(\int_{L_0}^{L}((v_t^\nu)^2 + (v^\nu)^2)dx + \int_0^{L_0}(u^\nu)^2 dx\Big),$$

integrating from s to T, we obtain

$$K^\nu(T) + \delta_0\int_S^T E(t,u^\nu,v^\nu)dt \le K(S) + c_3\Big(\int_S^T(|v_t^\nu|^2 + |v^\nu|^2 + |u^\nu|^2)\Big)dt.$$

Since K^ν satisfies

$$c_0 E(t,u^\nu,v^\nu) \le K^\nu(T) \le c_1 E(t,u^v,v^\nu)$$

and E is a decreasing function we have

$$\begin{aligned}
&E(T,u^v,v^\nu) + \delta_0'\int_S^T E(t,u^\nu,v^\nu)dt\\
&\le \frac{c_1'}{T}\int_S^T E(t,u^\nu,v^\nu)dt + c_3\int_S^T(|v_t^\nu|^2 + |v^\nu|^2 + |u^\nu|^2)dt;
\end{aligned}$$

thus, we obtain

$$E(T,u^v,v^\nu) + (\delta_0' - \frac{c_1'}{T})\int_S^T E(t,u^\nu,v^\nu)dt \le c_3\int_S^T\big(|v_t^\nu|^2 + |v^\nu|^2 + |u^\nu|^2\big)dt,$$

Dividing both sides of the above inequality by λ_ν^2, using (4.12) and (4.14), taking T large enough we conclude that $(|w_t^\nu|^2 + |z_t^\nu|^2 + |w_x^\nu|^2 + |z_x^\nu|^2)(T)$ is bounded. Now, multiplying equations $(4.17)_1$, $(4.17)_2$ by w_t^ν and z_t^ν respectively, performing an integration by parts we get

$$E(t,w^v,z^\nu) \le E(T,w^v,z^\nu) + \alpha\int_S^T |z_t^\nu|^2 dt - \int_S^T (a_t,(z_x^\nu)^2)dt.$$

From (4.12) , (4.13) and Poincare Inequality we deduce that $E(t,w^v,z^\nu)$ is bounded for all $t \in [S,T]$. Then in particular, on a subsequence we obtain

$$\begin{gathered}
w^\nu \to w \quad \text{weak star in } L^\infty(0,T;H^1(0,L_0)),\\
w_t^\nu \to w_t \quad \text{weak star in } L^\infty(0,T;L^2(0,L_0)),\\
z^\nu \to z \quad \text{weak star in } L^\infty(0,T;H^1(L_0,L)),\\
z_t^\nu \to z_t \quad \text{weak star in } L^\infty(0,T;L^2(L_0,L)),\\
w^\nu \to w \quad \text{in } L^2(0,T;L^2(0,L_0)),\\
z^\nu \to z \quad \text{in } L^2(0,T;L^2(L_0,L)).
\end{gathered}$$

On the other hand,we note that

$$\frac{1}{\lambda_\nu} f_1(\lambda_\nu w^\nu) \to f_1'(0) w \quad \text{in } L^2(0,T; L^2(0,L_0)x]0,T[), \tag{4.20}$$

$$\frac{1}{\lambda_\nu} f_2(\lambda_\nu z^\nu) \to f_2'(0) z \quad \text{in } L^2(0,T; L^2(L_0,L)x]0,T[). \tag{4.21}$$

Indeed

$$\begin{aligned}
\Delta_\nu &= |f_1'(0)w^\nu - \frac{1}{\lambda_\nu} f_1(\lambda_\nu w^\nu)|^2_{L^2((0,L_0)x]0,T[)} \\
&= \int_{|u^\nu|\le\epsilon} |f_1'(0)w^\nu - \frac{1}{\lambda_\nu} f_1(u^\nu)|^2\,dx\,dt + \int_{|u^\nu|>\epsilon} |f_1'(0)w^\nu - \frac{1}{\lambda_\nu} f_1(u^\nu)|^2\,dx\,dt \\
&\le \int_{|u^\nu|\le\epsilon} |w^\nu|^2 |f_1'(0) - \frac{1}{\lambda_\nu w^\nu} f_1(u^\nu)|^2\,dx\,dt + 2|f_1'(0)|^2 \int_{|u^\nu|>\epsilon} |w^\nu|^2\,dx\,dt \\
&\quad + 2\int_{|u^\nu|>\epsilon} \frac{1}{\lambda_\nu^2} |f_1(u^\nu)|^2\,dx\,dt \\
&\le M_\varepsilon^2 |w^\nu|^2_{L^2((0,L_0)x]0,T[)} + C\int_{|u^\nu|>\epsilon} (\frac{1}{\lambda_\nu^2}|u^\nu|^2 + \frac{1}{\lambda_\nu^2}|u^\nu|^{2\rho})\,dx\,dt \\
&\le M_\varepsilon^2 |w^\nu|^2_{L^2(0,L_0x]0,T[)} + C\int_{|u^\nu|>\epsilon} \frac{1}{\lambda_\nu^2}|u^\nu|^{2\rho}(1 + \frac{1}{\varepsilon^{2\rho-2}})\,dx\,dt \\
&\le M_\varepsilon^2 |w^\nu|^2_{L^2((0,L_0)x]0,T[)} + C_\varepsilon \lambda_\nu^{2\rho-2} |w^\nu|^{2\rho}_{L^{2\rho}((0,L_0)x]0,T[)}
\end{aligned}$$

where $M_\varepsilon = \sup_{|s|\le\varepsilon} |f_1'(0) - \frac{f_1(s)}{s}|$, $M_\varepsilon \to 0$ as $\varepsilon \to 0$.

From (4.13), $\{w^\nu\}$ is bounded in $L^\infty(0,T;H^1(0,L_0)) \hookrightarrow L^\infty(0,T;L^{2\rho}(0,L_0))$, and consequently

$$\limsup_{\nu\to\infty} \Delta_\nu \le \sup_\nu |w^\nu|^2_{L^2((0,L_0)x]0,T[)} . M_\varepsilon^2$$

Thus,taking $\varepsilon \to 0$ we obtain (4.20). Applying a similar method as that used for $\{w^\nu\}$ we get (4.21).

Now, the limit function $\{w,z\}$ satisfies

$$\begin{gathered}
\rho_1 w_{tt} - b w_{xx} + f_1'(0) w = 0 \quad \text{in }]0,L_0[\times]0,T[, \\
(a(x,t) z_x)_x + f_2'(0) z = 0 \quad \text{in }]L_0,L[\times]0,T[, \\
w(0,t) = 0 = z(L,t), \\
w(L_0,t) = z(L_0,t), \\
b w_x(L_0,t) = a(L_0,t) z_x(L_0,t), \\
z_t(x,t) = 0 \quad \text{in }]L_0,L[\times]0,T[
\end{gathered}$$

Repeating the above procedure we get $w = 0$ and $z = 0$ which is a contradiction. The proof of lemma 4.5 is now complete.

Proof of theorem 4.1. Let us introduce the functional

$$L(t) = N\ E(t) + K(t)$$

with $N > 0$. Using Young's Inequality and taking N large enough we find that

$$\theta_0 E(t) \le L(t) \le \theta_1 E(t) \tag{4.22}$$

for some positive constants θ_0 and θ_1.

Applying the inequalities (4.10) and (4.22), along with the ones in Lemma 4.5 and integrating from S to T where $0 \leq S \leq T < \infty$ we obtain

$$\int_S^T E(t)dt \leq c\ E(S).$$

In this situation, lemma 2.1 implies

$$E(t) \leq c\ E(0)e^{-rt}\ ,$$

this completes the proof. □

Remark. If in Equation (1.2) we consider a linear localized dissipation $\alpha = \alpha(x)$ in $C^2(]L_0, L[)$, with $\alpha(x) = 1$ in $]L_0, L_0 + \delta[$, $\alpha(x) = 0$ in $]L_0 + 2\delta, L[$, then our situation is very delicate and we need a new unique continuation theorem for the wave equation with variable coefficients. This is a work in preparation by the authors.

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