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EXPLOSION TIME IN STOCHASTIC DIFFERENTIAL EQUATIONS WITH SMALL DIFFUSION

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ABSTRACT. We consider solutions of a one dimensional stochastic differential equations that explode in finite time. We prove that, under suitable hypotheses, the explosion time converges almost surely to the one of the ODE governed by the drift term when the diffusion coefficient approaches zero.

1. Introduction

Explosions in one dimensional ODEs is a very well known phenomena. Let $u(t)$ be the solution of
$$\dot{u} = b(u), \quad u(0) = x_0. \tag{1.1}$$
If $b(\cdot) > 0$, there exists a finite time T such that $\lim_{t \nearrow T} u(t) = +\infty$ if and only if $\int^\infty 1/b < +\infty$. In this case we have an explicit formula for the explosion time T in terms of b and x_0,
$$T = \int_{x_0}^\infty \frac{1}{b(s)}\, ds. \tag{1.2}$$
On the other hand, let us consider the stochastic differential equation
$$dX = b(X)\, dt + \sigma(X)\, dW, \quad X(0) = x_0 > 0, \tag{1.3}$$
where b and σ are smooth positive functions and W is a (one dimensional) Wiener process defined on a given probability space $(\Omega, \mathbb{P})$.

As happens with (1.1), solutions of (1.3) may explode in finite time, that is, trajectories may diverge to infinity as t goes to some finite time S that in general depends on the particular sample path.

This phenomena has been considered, for example, in fatigue cracking (fatigue failures in solid materials) with b and σ of power type, see [5]. In this case the explosion time corresponds to the time of ultimate damage or fatigue failure in the material.

The *Feller Test for explosions* (see [3, 4]) gives a precise description in terms of b, σ and x_0 of whether explosions in finite time occur with probability zero, positive or one. For example, if b and σ behave like powers at infinity; i.e., $b(s) \sim s^p$, $\sigma(s) \sim s^q$ as $s \to \infty$, applying the Feller test one obtains that solutions to (3.1) explode with probability one if and only if $p > 2q - 1$ and $p > 1$.

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There is no simple formula for the explosion time S as (1.2) (although there exists some expressions for a version of S that involve the scale function which can be found in [3]). Hence, to estimate S is a nontrivial task. In order to get information about the stochastic explosion time one can use adaptive numerical approximations like the ones described in [1] where the authors provide a numerical method that can be used to compute a convergent approximation of S.

In this article we find, by theoretical arguments, estimates on the explosion time S when the diffusion σ is small. That is, we look at (1.3) as a stochastic perturbation of the ODE (1.1). We prove that, under adequate hypotheses on b and σ, the stochastic explosion time, $S = S(\sigma)$, converges to the deterministic one, T, almost surely when σ goes to zero. This means that the stochastic explosion times converge to a constant, T given by (1.2), that can be explicitly computed.

In the statement of the theorem we use the Stratonovich integral since the proofs are simpler. This is not a restriction thanks to the well known conversion formula (see below). We consider a family of SDE

$$dX = b(X)\,dt + \sigma(X,\varepsilon)\circ dW, \quad X(0) = x_0 > 0, \tag{1.4}$$

where $\varepsilon > 0$ is a parameter and $\sigma(\cdot,\varepsilon) \to 0$ as $\varepsilon \to 0$. We introduce a function $H\colon \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}$ ($\mathbb{R}_+ = [0,+\infty)$) defined in this way: Let $\phi = \phi_\varepsilon(t,x)$ the flux associated to the ODE

$$\dot y = \sigma(y,\varepsilon), \quad y(0) = x. \tag{1.5}$$

We assume that $\sigma(\cdot,\varepsilon)$ is globally Lipschitz and smooth and therefore ϕ_ε is globally defined. Then we define

$$H(s,x,\varepsilon) = \frac{b(\phi_\varepsilon(s,x))\sigma(x,\varepsilon)}{\sigma(\phi_\varepsilon(s,x),\varepsilon)}.$$

Theorem 1.1. *Assume*

(1) $b > 0$ *in* $\mathbb{R}_+$ *and* $\sigma(\cdot,\varepsilon) > 0$ *has continuous bounded derivatives in* $\mathbb{R}_+$*;*
(2) *Given* $s \in \mathbb{R}$*, there exists* $g_s \in L^1(\mathbb{R}_+)$ *such that for every* $x \in \mathbb{R}_+$*,*
$$\frac{1}{H(s,x,\varepsilon)} \le g_s(x); \tag{1.6}$$
(3) $H(s,x,\varepsilon) \ge H(t,x,\varepsilon)$ *if* $s \ge t$*,*
(4) $\lim_{\varepsilon\to 0} H(s,x,\varepsilon) = b(x)$*;*

then for almost every ω *the (strong) solution of* (1.4) *explodes in finite time* $S_\varepsilon(\omega)$ *for every* $\varepsilon > 0$ *and*

$$\lim_{\varepsilon\to 0} S_\varepsilon(\omega) = T. \tag{1.7}$$

If in addition H *satisfies*

(5) *For every* $s \in \mathbb{R}$*, there exists* $f_s \in L^1(\mathbb{R}_+)$ *such that* $\frac{\partial}{\partial\varepsilon}\frac{1}{H(s,x,\varepsilon)} \le f_s(x)$ *for every* $x \in \mathbb{R}_+$ *and* $0 < \varepsilon < \varepsilon_0$*,*

then $S_\varepsilon(\omega)$ *is Lipschitz continuous at* $\varepsilon = 0$ *almost surely, that is, there exist a random variable* $C = C(\omega)$ *such that with total probability*

$$|T - S_\varepsilon(\omega)| \le C\varepsilon.$$

Remark 1.2. If a SDE is given in Itô form, we can apply the conversion formula: $X(t)$ solves $dX = f(X)dt + g(X)dW$ if and only if it solves (1.4) with $b = f - \frac{1}{2}\sigma'\sigma$, $\sigma = g$. In this case we obtain that also b depends on ε but similar results can be obtained (see the second part of Example 2.3).

Remark 1.3. If $b(x)/\sigma(x,\varepsilon)$ is increasing in x then the monotonicity of $H(s,x,\varepsilon)$ in s, hypothesis (3), holds.

Remark 1.4. If $H(s,x,\varepsilon)$ is increasing (or decreasing) in ε then we can get rid of hypothesis (2), using the Monotone Convergence Theorem instead of the Dominated Convergence Theorem in the proof.

2. Some simple examples

In this section we consider some simple examples to illustrate the main ideas used in the proof of Theorem 1.1 and the principal features of the problem. We do not invoke Theorem 1.1 to deal with these examples, we prove the results "by hand". We are going to make use of Theorem 1.1 in the examples of the last section.

The main idea is to change variables in order to transform the SDE into a random differential equation. Then we obtain bounds for the explosion time by using sub and supersolutions given by ODEs.

Example 2.1 (Aadditive noise)**.** Let $u(t)$ be the solution of (1.1) with b increasing and $\int^{\infty} 1/b < +\infty$. Let X be a solution of the Itô SDE

$$dX = b(X)dt + \varepsilon dW, \quad X(0) = x_0.$$

Note that in this particular case Itô and Stratonovich interpretations are identical. Let $Z = X - \varepsilon W$, then Z solves

$$dZ = dX - \varepsilon dW = b(Z + \varepsilon W)dt, \quad Z(0) = x_0.$$

This gives a non-autonomous ODE for each ω such that $W(\cdot,\omega)$ is continuous,

$$\dot{Z}_\omega(t) = b(Z_\omega(t) + \varepsilon W(t,\omega)), \quad Z_\omega(0) = x_0. \tag{2.1}$$

In this equation ω is regarded as a parameter.

Given $M > 0$, we consider $\overline{z}$ and $\underline{z}$ the solutions of

$$\dot{\overline{z}}(t) = b(\overline{z}(t) + \varepsilon M), \quad \overline{z}(0) = x_0$$

and

$$\dot{\underline{z}}(t) = b(\underline{z}(t) - \varepsilon M), \quad \underline{z}(0) = x_0.$$

These solutions explode in finite time given by

$$\overline{T}_\varepsilon = \int_{x_0}^{\infty} \frac{1}{b(s + \varepsilon M)}\, ds, \quad \underline{T}_\varepsilon = \int_{x_0}^{\infty} \frac{1}{b(s - \varepsilon M)}\, ds,$$

respectively. Since b is increasing, by the Monotone Convergence Theorem we get

$$\lim_{\varepsilon \to 0} \overline{T}_\varepsilon = \lim_{\varepsilon \to 0} \underline{T}_\varepsilon = T. \tag{2.2}$$

Let

$$A_M = \big\{\omega \colon W(\cdot,\omega) \text{ is continuous and } \max_{0 \le t \le T+1} |W(\cdot,\omega)| \le M\big\}.$$

For $\omega \in A_M$, $\overline{z}$ and $\underline{z}$ are super and subsolutions of (2.1) for $0 < t < T+1$. Using (2.2), a comparison argument gives

$$\underline{z}(t) \le Z_\omega(t) \le \overline{z}(t),$$

as long as all of them are defined. Hence, for $\omega \in A_M$,

$$\overline{T}_\varepsilon \le S_\varepsilon(\omega) \le \underline{T}_\varepsilon.$$

Therefore, by (2.2),

$$\lim_{\varepsilon\to 0} S_\varepsilon(\omega) = T.$$

As

$$\mathbb{P}\Big(\bigcup_{M=1}^{\infty} A_M\Big) = 1$$

we get the desired result.

Remark 2.2. In this example the function H involved in Theorem 1.1 is given by

$$H(s,x,\varepsilon) = b(x+\varepsilon s),$$

and verifies the hypotheses stated there.

Observe also that in the ODE (1.1), the function b does not need to be increasing in order to have explosions. In this example, the monotonicity of b is only used to take limits in (2.2), but we can get rid of this hypothesis if we can ensure that those limits hold.

Example 2.3 (Multiplicative noise)**.** Let $u(t)$ be as in Example 1. Let X be the solution of the Stratonovich SDE

$$dX = b(X)dt + \varepsilon X \circ dW, \quad X(0) = x_0.$$

As in the preceding example, we want to get an ODE for each ω. To do that, let $Z = Xe^{-\varepsilon W}$. Hence we get that Z solves

$$dZ = \left(e^{-\varepsilon W} b(Ze^{\varepsilon W})\right) dt, \quad Z(0) = x_0.$$

As before, this gives a non-autonomous ODE for each ω such that $W(\cdot,\omega)$ is continuous,

$$\dot{Z}_\omega(t) = e^{-\varepsilon W(t,\omega)} b(Z_\omega(t)e^{\varepsilon W(t,\omega)}), \quad Z_\omega(0) = x_0. \tag{2.3}$$

Given $M > 0$, we consider $\overline{z}$ and $\underline{z}$ the solutions of

$$\dot{\overline{z}}(t) = e^{\varepsilon M} b(\overline{z}(t)e^{\varepsilon M}), \quad \overline{z}(0) = x_0$$

and

$$\dot{\underline{z}}(t) = e^{-\varepsilon M} b(\underline{z}(t)e^{-\varepsilon M}), \quad \underline{z}(0) = x_0.$$

These solutions explode in finite time given by

$$\overline{T}_\varepsilon = \int_{x_0}^{\infty} \frac{1}{e^{\varepsilon M} b(se^{\varepsilon M})}\, ds, \quad \underline{T}_\varepsilon = \int_{x_0}^{\infty} \frac{1}{e^{-\varepsilon M} b(se^{-\varepsilon M})}\, ds,$$

respectively. We have

$$\lim_{\varepsilon\to 0} \overline{T}_\varepsilon = \lim_{\varepsilon\to 0} \underline{T}_\varepsilon = T. \tag{2.4}$$

Let A_M as before. Since b is increasing, for $\omega \in A_M$, $\overline{z}$ and $\underline{z}$ are super and subsolutions of (2.3) for $0 < t < T+1$ and hence, using (2.4), we can compare their explosion times

$$\overline{T}_\varepsilon \le S_\varepsilon(\omega) \le \underline{T}_\varepsilon.$$

Therefore

$$\lim_{\varepsilon\to 0} S_\varepsilon(\omega) = T.$$

and we get the desired result. In this case $H(s,x,\varepsilon) = e^{-\varepsilon s} b(xe^{\varepsilon s})$.

Now, let us consider the same equation but in Itô sense. Let X be the solution of the Itô SDE

$$dX = b(X)dt + \varepsilon X dW, \quad X(0) = x_0.$$

As before, we want to get an ODE for each ω. To do that, let $Z = Xe^{-\varepsilon W}$. Using Itô's rule we get

$$dZ = \Big(e^{-\varepsilon W} b(Ze^{\varepsilon W}) - \frac{1}{2}\varepsilon^2 Z\Big)dt, \quad Z(0) = x_0.$$

Again this gives a non-autonomous ODE for each ω such that $W(\cdot,\omega)$ is continuous,

$$\dot{Z}_\omega(t) = e^{-\varepsilon W(t,\omega)} b(Z_\omega(t)e^{\varepsilon W(t,\omega)}) - \frac{1}{2}\varepsilon^2 Z_\omega(t), \quad Z_\omega(0) = x_0. \tag{2.5}$$

Given $M > 0$, we consider $\overline{z}$ and $\underline{z}$ the solutions of

$$\dot{\overline{z}}(t) = e^{\varepsilon M} b(\overline{z}(t)e^{\varepsilon M}) - \frac{1}{2}\varepsilon^2 \overline{z}(t), \quad \overline{z}(0) = x_0$$

and

$$\dot{\underline{z}}(t) = e^{-\varepsilon M} b(\underline{z}(t)e^{-\varepsilon M}) - \frac{1}{2}\varepsilon^2 \underline{z}(t), \quad \underline{z}(0) = x_0.$$

These solutions explode in finite time given by

$$\overline{T}_\varepsilon = \int_{x_0}^{\infty} \frac{1}{e^{\varepsilon M} b(se^{\varepsilon M}) - \frac{1}{2}\varepsilon^2 s}\, ds, \quad \underline{T}_\varepsilon = \int_{x_0}^{\infty} \frac{1}{e^{-\varepsilon M} b(se^{-\varepsilon M}) - \frac{1}{2}\varepsilon^2 s}\, ds,$$

respectively. Since $1/b$ is integrable these times are finite and we can apply dominated convergence we obtain

$$\lim_{\varepsilon\to 0} \overline{T}_\varepsilon = \lim_{\varepsilon\to 0} \underline{T}_\varepsilon = T. \tag{2.6}$$

From this point the limit

$$\lim_{\varepsilon\to 0} S_\varepsilon(\omega) = T$$

follows exactly as before.

In this example the function H is

$$H(s,x,\varepsilon) = e^{-\varepsilon s} b(xe^{\varepsilon s}) - \frac{1}{2}\varepsilon^2 x.$$

Observe that since b is superlinear H is increasing in time. However this hypothesis is not required in this case. The result can also be obtained since we can bound H from above and from below by functions that converge to b as $\varepsilon \to 0$.

3. Proof of the main result

Pathwise solutions of the SDE. We want to apply the same ideas used in the previous examples, that is, to transform the SDE in a non-autonomous ODE where ω plays the role of a parameter. This is easier when the equation is understood in Stratonovich sense.

The study of pathwise solutions to stochastic differential equations via an appropriate reduction to an ODE was first done in [2, 6]. We refer to those works and to [3] for details.

Consider a solution of the Stratonovich SDE

$$dX = b(X)dt + \sigma(X) \circ dW. \tag{3.1}$$

This solution may explode in finite time or may be globally defined.

Let y be a solution of the ODE

$$\dot{y} = \sigma(y), \quad y(0) = x, \tag{3.2}$$

and let $\phi(t,x)$ the flux associated to (3.2) which is globally defined and has continuous derivatives, since σ is smooth and globally Lipschitz.

Consider $Z_\omega = Z_\omega(t)$ the solution of the random differential equation

$$\begin{gathered} dZ_\omega(t) = \frac{b(\phi(W(t,\omega), Z_\omega(t)))}{\phi_x(W(t,\omega), Z_\omega(t))}\, dt, \\ Z_\omega(0) = x_0. \end{gathered} \tag{3.3}$$

Then $X(t,\omega) = \phi(W(t,\omega), Z_\omega(t))$ is a strong solution of (3.1) up to a possible explosion time S_ε. In fact, since (3.1) is interpreted in Stratonovich sense, we have

$$\begin{gathered} dX = \phi_t(W, Z_\omega)\, dW + \phi_x(W, Z_\omega)\, dZ_\omega = \sigma(X)\, dW + b(X)\, dt, \\ X(0) = x_0. \end{gathered}$$

Note that the explosion time $S_\varepsilon(\omega)$ is the maximal existence time of (3.3) for each ω. We are going to use this fact to prove Theorem 1.1.

Proof of Theorem 1.1. First of all observe that assumptions (1) and (2) ensure on the one hand that solutions to (1.1),(1.4) are positive and on the other hand that solutions to (1.1) explodes in finite time T given by (1.2). Applying the Feller Test for explosions one can see that these hypotheses also ensure that (1.4) explodes in finite time with probability one. Nevertheless we are going to show this fact in the course of the proof.

For each ω such that $W(\cdot,\omega)$ is continuous, consider the ODE

$$\dot Z_\omega(t) = \frac{b(\phi_\varepsilon(W(t,\omega), Z_\omega(t)))}{(\phi_\varepsilon)_x(W(t,\omega), Z_\omega(t))}, \quad Z_\omega(0) = x_0. \tag{3.4}$$

Here ϕ_ε is the flux associated to the ODE (1.5). The equation (3.4) can be written in terms of H as

$$\dot Z_\omega(t) = H(W(t,\omega), Z_\omega(t), \varepsilon), \quad Z_\omega(0) = x_0. \tag{3.5}$$

In fact, integrating (1.5) we get

$$\int_x^{\phi_\varepsilon(t,x)} \frac{d\tau}{\sigma(\tau,\varepsilon)} = t.$$

Differentiating with respect to x we obtain

$$\frac{(\phi_\varepsilon)_x(t,x)}{\sigma(\phi_\varepsilon(t,x),\varepsilon)} - \frac{1}{\sigma(x,\varepsilon)} = 0,$$

hence

$$(\phi_\varepsilon)_x(t,x) = \frac{\sigma(\phi_\varepsilon(t,x),\varepsilon)}{\sigma(x,\varepsilon)}$$

and so

$$H(s,x,\varepsilon) = \frac{b(\phi_\varepsilon(s,x))}{(\phi_\varepsilon)_x(s,x)}.$$

Given $M > 0$, we consider $\overline{z}$ and $\underline{z}$ the solutions of

$$\dot{\overline{z}}(t) = H(M, \overline{z}(t), \varepsilon), \quad \overline{z}(0) = x_0$$

and

$$\dot{\underline{z}}(t) = H(-M, \underline{z}(t), \varepsilon), \quad \underline{z}(0) = x_0.$$

These solutions explode in finite time given by

$$\overline{T}_\varepsilon = \int_{x_0}^\infty \frac{1}{H(M,x,\varepsilon)}\, dx, \quad \underline{T}_\varepsilon = \int_{x_0}^\infty \frac{1}{H(-M,x,\varepsilon)}\, dx,$$

respectively. By assumption (1.6) we can apply the Dominated Convergence Theorem to get

$$\lim_{\varepsilon\to 0}\overline{T}_\varepsilon = \lim_{\varepsilon\to 0}\underline{T}_\varepsilon = T. \tag{3.6}$$

Let A_M be as in the examples. Since $H(s,x,\varepsilon)$ is increasing in the s variable, for any $\omega \in A_M$, $\overline{z}$ and $\underline{z}$ are super and subsolutions of (3.5) for $0 < t < T+1$. Using this fact and (3.6), their explosion times can be compared. Since $X(t) = \phi_\varepsilon(W(t), Z_\omega(t))$ and ϕ_ε is globally defined, the explosion times of X and Z_ω coincide a.s. Then we obtain

$$\overline{T}_\varepsilon \le S_\varepsilon(\omega) \le \underline{T}_\varepsilon.$$

Therefore

$$\lim_{\varepsilon\to 0} S_\varepsilon(\omega) = T.$$

As

$$\mathbb{P}\big(\bigcup_{M=1}^{\infty} A_M\big) = 1,$$

we have proved (1.7). It remains to show the Lipschitz continuity. To do this observe that the Taylor expansion of $1/H(\pm M,x,\varepsilon)$ at $\varepsilon = 0$ gives for some η_ε with $0 < \eta_\varepsilon < \varepsilon$,

$$\begin{aligned}
|S_\varepsilon(\omega) - T| &\le \Big|\int_{x_0}^\infty \frac{1}{H(\pm M,x,\varepsilon)}\,dx - T\Big| \\
&= \Big|\int_{x_0}^\infty \frac{1}{b(x)}\,dx + \int_{x_0}^\infty \varepsilon\frac{\partial}{\partial\varepsilon}\frac{1}{H(\pm M,x,\eta_\varepsilon)}\,dx - T\Big| \\
&= \Big|\int_{x_0}^\infty \varepsilon\frac{\partial}{\partial\varepsilon}\frac{1}{H(\pm M,x,\eta_\varepsilon)}\,dx\Big| \\
&\le \varepsilon\Big|\int_{x_0}^\infty f_M(x)\,dx\Big| \\
&\le C\varepsilon.
\end{aligned}$$

This completes the proof. □

4. More Examples

In this section we present two additional examples where the result can be applied.

Example 4.1 (Unbounded diffusion)**.** Let $u(t)$ be the solution of (1.1) and consider the SDE

$$dX = b(X)dt + \varepsilon\sigma(X)\circ dW, \quad X(0) = x_0,$$

with

$$b(x) \sim x^p, \quad \sigma(x) \sim x^q, \quad 0 < q < 1 < p,$$

for large x and bounded below away from zero. In this case we have

$$\phi_\varepsilon(t,x) \sim \big(x^{1-q} + (1-q)\varepsilon t\big)^{\frac{1}{1-q}},$$

for x large and $t > 0$. Hence, the behavior of $H(s,x,\varepsilon)$ at infinity is given by

$$H(s,x,\varepsilon) = \frac{b(\phi_\varepsilon(s,x))\sigma(x,\varepsilon)}{\sigma(\phi_\varepsilon(s,x),\varepsilon)} \sim \big(x^{1-q} + (1-q)\varepsilon s\big)^{\frac{p-q}{1-q}}\,x^q \sim Cx^p.$$

From these expressions it is easy to check hypotheses (1), (2) and (4). If we assume (3), then we can apply our theorem to get $S_\varepsilon \to T$ almost surely. Note that (3) holds if we take, for example, $b(x) = (1+|x|)^p$, $\sigma(x) = \varepsilon(1+|x|)^q$. In fact, for $x \geq 0$ we have

$$H(s,x,\varepsilon) = \left(\varepsilon s(1-q) + (1+x)^{1-q}\right)^{\frac{p-q}{1-q}} (1+x^q).$$

In this case ,(5) also holds and so S_ε is Lipschitz at $\varepsilon = 0$ almost surely.

Example 4.2 (Bounded diffusion)**.** In this example we consider

$$dX = b(X)dt + \varepsilon\sigma(X) \circ dW, \quad X(0) = x_0,$$

with a bounded σ, $0 < c_1 \leq \sigma \leq C_2$ and b such that $\int^\infty 1/b < +\infty$. We have

$$\frac{1}{H(s,x,\varepsilon)} \leq g_s(x) := \frac{C}{b(x)}.$$

If we assume that (3) holds (the rest of the hypotheses can be easily checked) we obtain again that $S_\varepsilon \to T$ almost surely.

Example 4.3. In this example we consider

$$dX = e^{aX}dt + \varepsilon e^{bX} \circ dW, \quad X(0) = x_0,$$

with $a > b > 0$. In this case we have that the solution of

$$\dot{y} = \sigma(y,\varepsilon), \quad y(0) = x$$

is given by

$$y(s) = \phi_\varepsilon(s,x) = \frac{\ln(-b\varepsilon s + e^{-bx})}{-b}.$$

Therefore, we obtain

$$H(s,x,\varepsilon) = e^{ax}|1 - bs\varepsilon e^{bx}|^{1-\frac{a}{b}}$$

and we can conclude as before that $S_\varepsilon \to T$ almost surely.

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