Electronic Journal of Differential Equations, Vol. 2007(2007), No. 144, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NON-SYMMETRIC ELLIPTIC OPERATORS ON BOUNDED LIPSCHITZ DOMAINS IN THE PLANE

DAVID J. RULE

ABSTRACT. We consider divergence form elliptic operators $L = \operatorname{div} A\nabla$ in $\mathbb{R}^2$ with a coefficient matrix $A = A(x)$ of bounded measurable functions independent of the t-direction. The aim of this note is to demonstrate how the proof of the main theorem in [4] can be modified to bounded Lipschitz domains. The original theorem states that the L^p Neumann and regularity problems are solvable for $1 < p < p_0$ for some p_0 in domains of the form $\{(x,t) : \phi(x) < t\}$, where ϕ is a Lipschitz function. The exponent p_0 depends only on the ellipticity constants and the Lipschitz constant of ϕ. The principal modification of the argument for the original result is to prove the boundedness of the layer potentials on domains of the form $\{X = (x,t) : \phi(\mathbf{e}\cdot X) < \mathbf{e}^\perp \cdot X\}$, for a fixed unit vector $\mathbf{e} = (e_1, e_2)$ and $\mathbf{e}^\perp = (-e_2, e_1)$. This is proved in [4] only in the case $\mathbf{e} = (1, 0)$. A simple localisation argument then completes the proof.

1. Definitions and Known Results

An open bounded connected set $\Omega \subset \mathbb{R}^2$ is said to be a *bounded Lipschitz domain* if there exists numbers r_i, Lipschitz functions ϕ_i, points $Z_i \in \mathbb{R}^2$ and unit vectors $\mathbf{e}_i \in \mathbb{R}^2$ $(i = 1, 2, \dots, N)$ such that

$$\partial\Omega = \bigcup_{i=1}^{N} B_{2r_i}(Z_i) \cap \{X : \phi(\mathbf{e}_i \cdot X) = \mathbf{e}_i^\perp \cdot X\},$$

where $\mathbf{e}^\perp = (-e_2, e_1)$ for $\mathbf{e} = (e_1, e_2)$, $B_{2r_i}(Z_i) \cap \{X : \phi(\mathbf{e}_i \cdot X) < \mathbf{e}_i^\perp \cdot X\} \subset \Omega$ for each $i = 1, \dots, N$, and $B_{r_i}(Z_i) \cap B_{r_j}(Z_j) = \emptyset$ for $i \neq j$. Along with bounded Lipschitz domains we will also consider domains of the form

$$\Omega = \{X \in \mathbb{R}^2 : \phi(\mathbf{e}\cdot X) < \mathbf{e}^\perp \cdot X\} \tag{1.1}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is again a Lipschitz function, $\mathbf{e} = (e_1, e_2)$ is a fixed unit vector and $\mathbf{e}^\perp = (-e_2, e_1)$. In the sequel we will denote by τ the tangent $(\mathbf{e} + \phi'\mathbf{e}^\perp)/(1 + (\phi')^2)^{1/2}$ to $\partial\Omega$ and $\partial_\tau = \tau \cdot \nabla$ the derivative along the boundary.

2000 *Mathematics Subject Classification.* 35J25, 31A25.
Key words and phrases. $T(b)$ Theorem; layer potentials; L^p Neumann problem; L^p regularity problem; non-symmetric elliptic equations.

Submitted October 10, 2007. Published October 30, 2007.

With Ω being either one of the domains above, we will consider the Dirichlet problem

$$\begin{aligned} Lu = 0, &\quad \text{in } \Omega \\ u = f_0, &\quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

with boundary data f_0 and the Neumann problem

$$\begin{aligned} Lu = 0, &\quad \text{in } \Omega \\ \nu \cdot A\nabla u = g_0, &\quad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

with boundary data g_0. Here ν is the outward unit normal vector to $\partial\Omega$ and $L = \operatorname{div} A\nabla\cdot$ is an elliptic operator in divergence form with coefficient matrix $A = (a_{ij})_{ij}$. The matrix A is assumed to have real-valued bounded measurable entries ($\max_{i,j} \|a_{ij}\|_{L^\infty(\Omega)} = \Lambda < \infty$) and satisfy the uniform ellipticity condition

$$\lambda|\xi|^2 \le \xi \cdot A\xi \tag{1.4}$$

for some $\lambda > 0$ and all $\xi \in \mathbb{R}^2$, but A is not necessarily symmetric. The conormal derivative will be $\nu \cdot A\nabla$.

Much of the notation used here is standard and is defined in detail in [4] and [5]; in particular we have the following. Recall that $(X, Y) \mapsto \Gamma_X(Y)$ is the fundamental solution for the elliptic operator L with pole at X and taking the gradient in the parenthetical variable is denoted $\nabla\Gamma_X(Y)$ while in the subscript variable it is denoted $\nabla_X\Gamma_X(Y)$. A *non-tangential approach region* is the set

$$\Gamma(Q) = \{X \in \Omega : |X - Q| \le (1 + a)\operatorname{dist}(X, \partial\Omega)\}$$

for a given $Q \in \partial\Omega$ ($a > 0$ fixed). Here $\operatorname{dist}(X, \partial\Omega) = \inf_{Q\in\partial\Omega} |X - Q|$. Recall the *non-tangential maximal function* for a function u on Ω is a function $N(u) \colon \partial\Omega \to \mathbb{R}$ given by

$$N(u)(Q) = \sup_{\Gamma(Q)} |u|$$

and the related version

$$\widetilde{N}(u)(Q) = \sup_{X\in\Gamma(Q)} \Big(\frac{1}{|B_{\delta(X)/2}(X)|} \int_{B_{\delta(X)/2}(X)} |u|^2\Big)^{1/2}.$$

[4, Lemmata 1.1 and 1.2] provide us with the existence and uniqueness of a solution to (1.2) and (1.3) when Ω is of the form (1.1). The following serve the same role for bounded Lipschitz domains.

Lemma 1.1. *Let Ω be a bounded Lipschitz domain. For each $f_0 \in W^{1,2}(\partial\Omega)$, there exists a unique $u \in W^{1,2}(\Omega)$ such that $\operatorname{Tr}(u) = f_0$ and*

$$\int_\Omega A\nabla u \cdot \nabla\varphi = 0$$

for all $\varphi \in W_0^{1,2}(\Omega)$. Moreover, there exists a constant C, depending only on λ, Λ and Ω, such that

$$\|u\|_{W^{1,2}(\Omega)} \le C\|f_0\|_{W^{1,2}(\partial\Omega)}.$$

Proof. The proof is essentially the same as [4, Lemma 1.1]. We first construct a function $w \colon \Omega \to \mathbb{R}$ with $\operatorname{Tr}(w) = f_0$ whose $W^{1,2}(\Omega)$-norm is no more than $C\|f_0\|_{W^{1,2}(\partial\Omega)}$. In [4] we used the Poisson extension, and we can do the same here locally in each $B_{r_i}(Z_i)$, first flattening out the boundary and using appropriate cut-off functions. The sum of these local extensions is then the w we require. Secondly, we can apply the Lax-Milgram theorem as before, since

$$(\psi,\varphi)\mapsto\int_\Omega A\nabla\psi\cdot\nabla\varphi \tag{1.5}$$

is coercive on $W_0^{1,2}(\Omega)$. $\square$

Lemma 1.2. *Let Ω be a bounded Lipschitz domain. For each $g_0\in L^2(\partial\Omega)$, there exists a unique $u\in W^{1,2}(\Omega)$ (modulo constants) such that*

$$\int_\Omega A\nabla u\cdot\nabla\varphi=\int_{\partial\Omega} g_0\,\mathrm{Tr}(\varphi)\,d\sigma$$

for all $\varphi\in W^{1,2}(\Omega)$. Moreover, there exists a constant C, depending only on λ, Λ and Ω, such that

$$\|u-\int_\Omega u\|_{W^{1,2}(\Omega)}\le C\|g_0\|_{L^2(\partial\Omega)}.$$

Proof. Since (1.5) is coercive on the space $\{u\in W^{1,2}(\Omega): \int_\Omega u=0\}$ with norm $\|\cdot-\int_\Omega\cdot\|_{W^{1,2}(\Omega)}$, the proof of [4, Lemma 1.2] can be repeated once we have shown $\mathrm{Tr}\colon W^{1,2}(\Omega)\to L^2(\partial\Omega)$ is a bounded operator. Fix $\xi\colon\mathbb{R}^2\to\mathbb{R}$ to be a smooth cut-off function equal to one on $B_{r_i}(Z_i)$, supported in $B_{2r_i}(Z_i)$ and such that $|\nabla\xi|\le C/r_i$. Then, say, if $\mathbf{e}_i=(1,0)$,

$$\begin{aligned}
\int_{\partial\Omega\cap B_{r_i}(Z_i)}|\varphi|^2\,d\sigma &\le \int_{\partial\Omega}\xi|\varphi|^2\\
&=-\int_{\Omega\cap B_{2r_i}(Z_i)}\partial_t(\xi|\varphi|^2)\\
&=-\int_{\Omega\cap B_{2r_i}(Z_i)}(\partial_t\xi)|\varphi|^2-\int_{\Omega\cap B_{2r_i}(Z_i)}\xi(\partial_t\varphi)\varphi(\mathrm{sgn}(\varphi))\\
&\le\frac{C}{r_i}\int_{\Omega\cap B_{2r_i}(Z_i)}|\varphi|^2+\frac{1}{r_i}\int_{\Omega\cap B_{2r_i}(Z_i)}|\nabla\varphi||\varphi|\\
&\le\frac{C}{r_i}\|\varphi\|^2_{W^{1,2}(\Omega)},
\end{aligned}$$

where the last inequality follows from Hölder's and Cauchy's inequalities. Summing in i gives the desired result. $\square$

It is well known that existence of the estimates in the following definition (which replaces [4, Definition 1.3]) enable a certain non-tangential convergence to the boundary data to be established (see, for example, [3]).

Definition 1.3. Let Ω be a bounded Lipschitz domain.

(i) We say that the Dirichlet problem holds for p, or $(D)_p^A=(D)_p$ holds, if for any u solving (1.2) with boundary data $f_0\in L^p(\partial\Omega)\cap W^{1,2}(\partial\Omega)$ we have
$$\|N(u)\|_{L^p(\partial\Omega)}\le C(p)\|f_0\|_{L^p(\partial\Omega)}.$$

(ii) We say that the Neumann problem holds for p, or $(N)_p^A=(N)_p$ holds, if for any u solving (1.3) with boundary data $g_0\in L^p(\partial\Omega)\cap L^2(\partial\Omega)$ we have
$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)}\le C(p)\|g_0\|_{L^p(\partial\Omega)}.$$

(iii) We say that the regularity problem holds for p, or $(R)_p^A = (R)_p$ holds, if for any u solving (1.2) with boundary data $f_0 \in W^{1,p}(\partial\Omega) \cap W^{1,2}(\partial\Omega)$ we have
$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \le C(p)\|\partial_\tau f_0\|_{L^p(\partial\Omega)}.$$

In each case, the constant $C(p) > 0$ must depend only on λ, Λ, Ω and p.

The following theorem was proved by Kenig, Koch, Pipher and Toro [2]. This will be used to prove our main result, Theorem 2.1.

Theorem 1.4. *Let $L = \operatorname{div} A\nabla$ be an elliptic operator in a bounded Lipschitz domain Ω, where $A = A(x)$ is independent of the t-variable. Then there exists a (possibly large) p such that $(D)_p$ holds in Ω, with bound depending only on λ, Λ, p and the Lipschitz constant of ϕ.*

2. The Main Result

Our aim is to prove the following analogue to [4, Theorem 1.4].

Theorem 2.1. *Let $L = \operatorname{div} A\nabla$ be an elliptic operator with coefficient matrix $A = A(x)$ independent of the t-direction in a bounded Lipschitz domain Ω. Then $(N)_p$ and $(R)_p$ hold for some (possibly small) $p > 1$.*

The proof follows that in [4] and we lay out the main ingredients in its proof below, emphasising the differences. The details are contained in [5]. The main idea is to prove a reverse of the duality statement proved in [3]. We will show that under our hypothesis when $(D)_p^{A^t}$ holds then $(R)_{p'}^A$ and $(N)_{p'}^{\widetilde{A}}$ hold, where $\widetilde{A} = A^t/\det A$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Once this is done we may use Theorem 1.4 to obtain Theorem 2.1. The proof of this duality is split into three parts. Firstly, Theorem 2.3 shows the required estimates for the gradient hold at the boundary, then Theorem 2.4 shows that $\widetilde{N}(\nabla u)$ can be controlled in $L^p(\partial\Omega)$-norm by certain potentials of the boundary value of ∇u, and finally we go on to show in several steps these potentials are bounded operators on $L^p(\partial\Omega)$. The proof of Theorem 2.4 is where the main difference from [4] occurs.

We will work under the a priori assumptions that $A = I$ for large x, A and ϕ are smooth functions, $\|\phi'\|_{L^\infty(\mathbb{R})} \le k$, $\phi' \equiv \alpha_0$ for large x and $x \mapsto \phi(x) - \alpha_0 x \in C_0^\infty(\mathbb{R})$. Once our theorems have been proved under our a priori assumptions, it is a simple matter to obtain the general case. Note that, under our a priori assumptions, if u solves (1.2) with data $f_0 \in C_0^\infty(\partial\Omega)$ and Ω is of the form (1.1), then $u \in C^\infty(\overline{\Omega})$, and $u(X) = O(|X|^{\delta-1})$ and $\nabla u(X) = O(|X|^{\delta-2})$ for all $\delta > 0$ as $|X| \to \infty$. (See [5, Appendix B].)

We will make use of the following lemma from [2]. We denote by $\Lambda^{k/2}(\varepsilon_0)$ the set of all Lipschitz functions ϕ such that $\|\phi' - \alpha_0\| \le \varepsilon_0$, with $\alpha_0 \in [-k, k]$. We also require that $0 < \varepsilon_0 \le k$, so the Lipschitz constant of such functions is no more than $2k$.

Lemma 2.2. *Given a unit vector $\mathbf{e}$, suppose $\Omega = \{X = (x,t) \in \mathbb{R}^2 : \phi(\mathbf{e}\cdot X) < \mathbf{e}^\perp \cdot X\}$ is the domain above the graph of a Lipschitz function $\phi \in \Lambda^k(\varepsilon_0)$. Let $A = A(x)$ be any matrix satisfying the ellipticity condition (1.4) and with coefficients independent of the vertical direction. Also suppose that $\operatorname{div} A\nabla u = 0$ in Ω. Then, for sufficiently small ε_0 depending only on λ and Λ, there exists a change of variables $\Phi\colon \Omega' \to \Omega$ such that*

(i) *If* $v = u \circ \Phi$ *then* $\operatorname{div} B\nabla v = 0$ *in* Ω'*, where* B *is lower triangular, satisfies* (1.4)*, is independent of the* t*-variable and of the form*

$$B = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \tag{2.1}$$

(ii) *The domain* Ω' *is the domain above the graph of a Lipschitz function.*

When $\mathbf{e} = (1,0)$ *there is no restriction on* ε_0*.*

One of the main ingredients in [4] was a *conjugate* $\widetilde{u}$ to a solution u to the elliptic equation $Lu = \operatorname{div} A\nabla u = 0$. This is defined (up to a constant) by the system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\widetilde{u} = A\nabla u.$$

Recall firstly that $\widetilde{u}$ satisfies an elliptic equation with coefficient matrix $\widetilde{A} = A^t/\det A$, and secondly that the conormal derivative of u is the tangential derivative of $\tilde{u}$ and vice versa. The following theorem can be proved exactly as in [4, Theorem 2.9].

Theorem 2.3. *Let* Ω *be a bounded Lipschitz domain, let* Ω *and* A *verify the a priori assumptions, and let* u *solve (1.2). Suppose* $p' \in (1,\infty)$ *is such that* $(D)_{p'}^{A^t}$ *holds. Then there exists a constant* $C(p)$*, depending only on* λ*,* Λ*,* k*,* p *and the* $(D)_{p'}^{A^t}$ *constant of* A^t*, such that*

$$\|\nabla u\|_{L^p(\partial\Omega)} \le C(p)\|\partial_\tau f_0\|_{L^p(\partial\Omega)}.$$

Also, if u *solves* (1.3) *with coefficient matrix* A *replaced by* $\widetilde{A} = A^t/\det(A)$*, then there exists a constant* $C(p)$*, depending on the same quantities, such that*

$$\|\nabla u\|_{L^p(\partial\Omega)} \le C(p)\|g_0\|_{L^p(\partial\Omega)}.$$

As usual $\frac{1}{p} + \frac{1}{p'} = 1$*.*

[4, Theorem 3.1] must be replaced by the theorem below. We fix a unit vector $\mathbf{e}$ and define the conjugate $\widetilde{\Gamma}_X$ of Γ_X to be

$$\widetilde{\Gamma}_X(Y) = \int_{\gamma(Y_0,Y)} \nu(Z)\cdot A^t(Z)\nabla\Gamma_X(Z)\,dl(Z)$$

on the complement of the set $\{Y = (y,s) : \mathbf{e}^\perp\cdot Y \ge \mathbf{e}^\perp\cdot X, \mathbf{e}\cdot Y = \mathbf{e}\cdot X\}$. Here $\gamma(Y_0,Y)$ is a path from a fixed point Y_0 to Y parametrised by arc length via the function $t \mapsto (l_1(t), l_2(t))$ and remaining in the complement of $\{Z : \mathbf{e}^\perp\cdot Z \ge \mathbf{e}^\perp\cdot X, \mathbf{e}\cdot Z = \mathbf{e}\cdot X\}$. Also $\nu(Z) = (l_2'(t), -l_1'(t))$ is the unit normal to $\gamma(Y_0,Y)$ at $Z = (l_1(t), l_2(t))$ and dl is arc length. It is easy to see $\widetilde{\Gamma}_X(Y)$ solves the system

$$A^t(Y)\nabla\Gamma_X(Y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\nabla\widetilde{\Gamma}_X(Y). \tag{2.2}$$

The function $Y \mapsto \widetilde{\Gamma}_X(Y)$ is well-defined up to a constant (which depends on the choice of Y_0). The two vector-valued potentials $\mathcal{I}$ and $\mathcal{J}$ are defined by

$$\mathcal{I}(f)(X) = \lim_{h\searrow 0}\int_{\partial\Omega} \nabla\Gamma_Y^t(x\mathbf{e} + (\phi(x)+h)\mathbf{e}^\perp)f(Y)\,d\sigma(Y),$$
$$\mathcal{J}(f)(X) = \lim_{h\searrow 0}\int_{\partial\Omega} \nabla_X\widetilde{\Gamma}_{(x\mathbf{e}+(\phi(x)+h)\mathbf{e}^\perp)}(Y)f(Y)\,d\sigma(Y),$$

where $X = x\mathbf{e} + \phi(x)\mathbf{e}^\perp \in \partial\Omega$.

Theorem 2.4. *Let* $\Omega = \{X \in \mathbb{R}^2 : \phi(\mathbf{e} \cdot X) < \mathbf{e}^\perp \cdot X\}$ *for some Lipschitz function* $\phi \in \Lambda^{k/2}(\varepsilon_0)$*. Let* $L = \operatorname{div} A\nabla$ *be an elliptic operator satisfying* (1.4) *with coefficient matrix* $A = A(x)$ *of measurable functions bounded by* Λ *independent of the* t*-variable. Then for each* $p > 1$ *there exists a constant* $C(p)$*, depending only on* λ*,* Λ*,* k *and* p*, such that any function* $u\colon \Omega \to \mathbb{R}$ *such that* $Lu = 0$*, and* $u(X) = O(|X|^{\delta-1})$ *and* $|\nabla u(X)| = O(|X|^{\delta-2})$ *for all* $\delta > 0$ *as* $|X| \to \infty$*, we have*

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \le C(p)(\|\nabla u\|_{L^p(\partial\Omega)} + \|\mathcal{I}(\nu \cdot A\nabla u)\|_{L^p(\partial\Omega)} + \|\mathcal{J}(\tau \cdot \nabla u)\|_{L^p(\partial\Omega)}).$$

Proof. We will just give an outline of the proof, as the details are contained in [5]. Recall Green's second identity: Let us write $L = \operatorname{div} A\nabla$ and $L^t = \operatorname{div} A^t\nabla$, then we have

$$\int_\Omega (Lu)v - u(L^t v) = \int_{\partial\Omega} (\nu \cdot A\nabla u)v - (\nu \cdot A^t\nabla v)u \, d\sigma$$

so, for u such that $Lu = 0$ and replacing v with the fundamental solution Γ_X for L, so that $L^t\Gamma_X = \delta_X$, the Dirac mass at X, we obtain

$$u(X) = \int_{\partial\Omega} (\nu \cdot A^t\nabla\Gamma_X)u - (\nu \cdot A\nabla u)\Gamma_X \, d\sigma.$$

Using (2.2) and integration by parts we discover

$$\begin{aligned} u(X) &= \int_{\partial\Omega} (\tau \cdot \nabla\widetilde{\Gamma}_X)u - (\nu \cdot A\nabla u)\Gamma_X \, d\sigma \\ &= -\int_{\partial\Omega} \widetilde{\Gamma}_X(\tau \cdot \nabla u) + (\nu \cdot A\nabla u)\Gamma_X \, d\sigma, \end{aligned}$$

and then taking the gradient in X we find

$$\nabla u(X) = -\int_{\partial\Omega} (\nabla_X\widetilde{\Gamma}_X)(\tau \cdot \nabla u) + (\nu \cdot A\nabla u)(\nabla_X\Gamma_X) \, d\sigma. \tag{2.3}$$

Thus we can see that to control $\|N(\nabla u)\|_{L^p(\partial\Omega)}$ it would suffice to show, via standard Calderón-Zygmund theory, both terms on the right-hand side of (2.3) are singular integral operators, acting on $\tau \cdot \nabla u$ and $\nu \cdot A\nabla u$ respectively.

To show the two right-hand terms in (2.3) are indeed singular integrals we can follow the same procedure as [4]. It is convenient here to form matrix-valued operators from our potentials $\mathcal{I}$ and $\mathcal{J}$. The potentials are of a slightly different form to [4]. This leads us to consider transformations Φ, from Lemma 2.2, which lead to lower triangular coefficient matrices rather than upper triangular, as was the case in [4]. In addition, one more significant modification must be made. At this point we are not assuming A is as in (2.1), so in order to obtain the correct decay and smoothness estimates we must insert the appropriate Jacobian factor from the change of variables of Lemma 2.2. Although we are not assuming the Lipschitz constant of ϕ is small, we can still apply the transformation to obtain an elliptic equation in non-divergence form, however, the boundary of the resulting domain may not be the graph of a function. See [2, Lemma 3.46] for details of the transformation. Thus, since $\nabla_X\Gamma_X(Y) = \nabla\Gamma_Y(X)$, the operator T acting on matrix-valued functions formed from $\mathcal{I}$ (or rather we should say, from the transpose of $\mathcal{I}$) has the matrix kernel $K\colon \mathbb{R}^2 \to \mathcal{M}$ with both rows being

$$(\Phi' \circ \Phi^{-1})(y\mathbf{e} + \phi(y)\mathbf{e}^\perp)\nabla\Gamma^t_{(x\mathbf{e}+(\phi(x)+h)\mathbf{e}^\perp)}(y\mathbf{e} + \phi(y)\mathbf{e}^\perp)$$

and the operator $\widetilde{T}$ corresponding to $\mathcal{J}$ has the kernel $\widetilde{K}\colon \mathbb{R}^2 \to \mathcal{M}$ with rows

$$(\Phi' \circ \Phi^{-1})(x\mathbf{e} + (\phi(x)+h)\mathbf{e}^\perp)\nabla_X \widetilde{\Gamma}_{(x\mathbf{e}+(\phi(x)+h)\mathbf{e}^\perp)}(y\mathbf{e} + \phi(y)\mathbf{e}^\perp).$$

One can then show that both K and $\widetilde{K}$ are a Calderón-Zygmund kernel using Green's identities and standard tools for elliptic equations. Continuing in the same manner, one can go on to show the operator T is a continuous linear operator from $B_1\mathcal{S}$ to $(B_2\mathcal{S})'$, where B_1 is the matrix-valued function with columns $(1 + (\phi')^2)^{1/2}((\Phi^{-1})')^t A^t \nu$ and $(1+(\phi')^2)^{1/2}((\Phi^{-1})')^t\tau$, and B_2 is any bounded matrix-valued function. Finally, one can also show the operator $\widetilde{T}$ is a continuous linear operator from $B_3\mathcal{S}$ to $(B_1\mathcal{S})'$, where B_3 is the diagonal matrix-valued function with diagonal entries both being $(1+(\phi')^2)^{1/2}\tau\cdot\kappa$, where $\kappa = \mathbf{e} + \alpha_0\mathbf{e}^\perp$. The details are contained in [5, Chapter 3]. □

We now wish to show the operators T and $\widetilde{T}$ are bounded on $L^p(\mathbb{R})$, which easily leads to the $L^p(\partial\Omega)$-boundedness of $\mathcal{I}$ and $\mathcal{J}$. The first step in doing this is the following theorem.

Theorem 2.5. *For each $k > 0$ and A of the form (2.1) there exists an $\varepsilon_0 > 0$, depending only on k, λ and Λ, such that, for any $\phi \in \Lambda^{\frac{k}{4}}(\varepsilon_0)$, the singular integral operators T and $\widetilde{T}$ admit continuous extensions to $L^2(\mathbb{R},\mathcal{M})$ and therefore also to $L^p(\mathbb{R},\mathcal{M})$ for all $1 < p < \infty$ with norm depending only on p, λ, Λ and k. Consequently the potentials $\mathcal{I}$ and $\mathcal{J}$ are bounded linear operators on $L^p(\partial\Omega,\mathbb{R}^2)$ ($1 < p < \infty$).*

Proof. This is proved by applying the matrix formulation of the $T(B)$-Theorem [1]. It suffices to show $M_{B_2^t} T M_{B_1}$ and $M_{B_1^t}\widetilde{T}M_{B_3}$ are weakly bounded and $T(B_1)$, $T^t(B_2)$, $\widetilde{T}(B_3)$ and $\widetilde{T}^t(B_1)$ are in BMO, where now B_2 is the diagonal matrix-valued function with diagonal entries both being $(1+(\phi')^2)^{1/2}\nu\cdot A^t\kappa^\perp$. This is a repeat of the work in [4, Section 4] (for the exact details see [5, Chapter 4]). □

We now wish to remove the restrictions that A is of the form (2.1) and that ε_0 is small. First of all we can remove the restriction on ε_0 by applying David's build-up scheme, as in [4]. With this at hand, we now consider a domain Ω as in (1.1) and a matrix $A = A(x)$ satisfying (1.4) and our a priori smoothness assumptions, but not necessarily (2.1). To apply Lemma 2.2 we must again assume $\phi \in \Lambda^{k/4}(\varepsilon_0)$ and ε_0 is small. Once we have applied the transformation from Lemma 2.2 we will obtain an elliptic operator in a domain Ω' with a matrix of the form (2.1), but no guarantee that the Lipschitz constant of the boundary is small. However, given our application of David's build-up scheme above, we can conclude L^p-boundedness. Now, a second application of David's build-up scheme on the ϕ above allows us to remove the assumption that ε_0 is small.

With this result in hand, we may conclude the proof of Theorem 2.1. First of all, given our bounded Lipschitz domain Ω we define $\Omega_i := \{X \in \mathbb{R}^2 : \phi(\mathbf{e}_i\cdot X) < \mathbf{e}_i^\perp\cdot X\}$ and introduce a partition of unity $1 = \sum_{i=1}^N \eta_i$ such that $\eta_i = 1$ on $\partial\Omega \cap B_{r_i}(Z_i)$ and $\operatorname{supp}(\eta_i) \subset \partial\Omega \cap B_{2r_i}(Z_i)$. Set $f_i = (\nu\cdot A\nabla u)\eta_i$ and $g_i = (\tau\cdot\nabla u)\eta_i$. Then, from (2.3),

$$\nabla u(X) = -\sum_{i=1}^N \int_{\partial\Omega_i} (\nabla_X\widetilde{\Gamma}_X)g_i + (\nabla_X\Gamma_X)f_i\,d\sigma.$$

Therefore, using Theorem 2.4 and the boundedness of $\mathcal{I}$ and $\mathcal{J}$, we have

$$\begin{aligned}\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} &\leq C\|\nabla u\|_{L^p(\partial\Omega)} + \sum_{i=1}^{N} C(\|\mathcal{I}(g_i)\|_{L^p(\partial\Omega_i)} + \|\mathcal{J}(f_i)\|_{L^p(\partial\Omega_i)}) \\ &\leq C\|\nabla u\|_{L^p(\partial\Omega)} + \sum_{i=1}^{N} C(\|g_i\|_{L^p(\partial\Omega_i)} + \|f_i\|_{L^p(\partial\Omega_i)}) \\ &\leq CN\|\nabla u\|_{L^p(\partial\Omega)}.\end{aligned}$$

This estimate when combined with Theorems 2.3 and 1.4 concludes the proof of Theorem 2.1.

References

[1] G. David, J.-L. Journé, S. Semmes; Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. *Rev. Mat. Iberoamericana*, 1(4):1–56, 1985.
[2] C. E. Kenig, H. Koch, J. Pipher, T. Toro; A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations. *Advances in Mathematics*, 153:231–298, 2000.
[3] C. E. Kenig, J. Pipher; The Neumann problem for elliptic equations with non-smooth coefficients. *Invent. Math.*, 113:447–509, 1993.
[4] C. E. Kenig, D. J. Rule; The regularity and Neumann problem for non-symmetric elliptic operators. to appear in *Transactions of the American Mathematical Society*.
[5] D. J. Rule; *The Regularity and Neumann Problem for Non-Symmetric Elliptic Operators*. PhD thesis, University of Chicago, 2007.

David J. Rule
School of Mathematics and Maxwell Institute for Mathematical Sciences, The University of Edinburgh, James Clerk Maxwell Building, The King's Buildings, Mayfield Road, Edinburgh, EH9 3JZ, UK
E-mail address: rule@uchicago.edu