Electronic Journal of Differential Equations, Vol. 2007(2007), No. 161, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ANTIPLANE FRICTIONAL CONTACT OF ELECTRO-VISCOELASTIC CYLINDERS

MOHAMED DALAH, MIRCEA SOFONEA

Abstract. We study a mathematical model that describes the antiplane shear deformation of a cylinder in frictional contact with a rigid foundation. The material is assumed to be electro-viscoelastic, the process is quasistatic, friction is modelled with Tresca's law and the foundation is assumed to be electrically conductive. We derive a variational formulation of the model which is in a form of a system coupling a first order evolutionary variational inequality for the displacement field with a time-dependent variational equation for the electric potential field. Then, we prove the existence of a unique weak solution to the model. The proof is based on arguments of evolutionary variational inequalities and fixed points of operators. Also, we investigate the behavior of the solution as the viscosity converges to zero and prove that it converges to the solution of the corresponding electro-elastic antiplane contact problem.

1. Introduction

Antiplane shear deformations are one of the simplest classes of deformations that solids can undergo: in antiplane shear of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. For this reason, the antiplane problems play a useful role as pilot problems, allowing for various aspects of solutions in Solid Mechanics to be examined in a particularly simple setting. Considerable attention has been paid to the modelling of such kind of problems, see for instance [9, 10, 11] and the references therein. In particular, the review article [9] deals with modern developments for the antiplane shear model involving linear and nonlinear solid materials, various constitutive settings and applications. Antiplane frictional contact problems were used in geophysics in order to describe pre-earthquake evolution of the regions of hight tectonic activity, see for instance [5, 6] and the references therein. The mathematical analysis of models for antiplane frictional contact problems can be found in [1, 8, 13, 14, 17].

Currently there is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, i.e. materials characterized by the coupling between the mechanical and electrical properties. This coupling, in a piezoelectric material, leads to the appearance of electric potential when mechanical stress is present, and conversely, mechanical stress is generated when electric potential is

2000 *Mathematics Subject Classification.* 74M10, 74F15, 74G25, 49J40.
Key words and phrases. Antiplane problem; electro-viscoelastic material; contact process; Tresca's friction law; evolutionary variational inequality; weak solution.

Submitted September 2, 2007. Published November 21, 2007.

applied. The first effect is used in mechanical sensors, and the reverse effect is used in actuators, in engineering control equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials and piezoelectric materials for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. General models for piezoelectric materials can be found in [2, 12, 19]. Static frictional contact problems for electro-elastic materials were studied in [3, 16, 18, 21], under the assumption that the foundation is insulated. Contact problems with normal compliance for electro-viscoelastic materials were investigated in [15, 22]. There, variational formulations of the problems were considered and their unique solvability was proved. Antiplane problems for piezoelectric materials were considered in [4, 23, 24]. We rarely actually load piezoelectric bodies so as to cause them to deform in antiplane shear; however, the governing equations and boundary conditions for antiplane shear problems involving piezoelectric materials are beautifully simple and the solution has many of the features of the more general case and may help us to solve the more complex problem too.

The present paper represents a continuation of [23]; there a model for the antiplane contact of an electro-elastic cylinder was considered under the assumption that the foundation is electrically conductive; the variational formulation of the model was derived and the existence of a unique solution to the model was proved by using arguments of evolutionary variational inequalities. Unlike [23], in the present paper we deal with an antiplane contact problem for an electro-viscoelastic cylinder, which leads to a new mathematical model, different to that presented in [23]. Our interest is to describe a simple physical process in which both frictional contact, viscosity and piezoelectric effects are involved, and to show that the resulting model leads to a well-posed mathematical problem. Taking into account the frictional contact between a viscous piezoelectric body and an electrically conductive foundation in the study of an antiplane problem leads to a new and interesting mathematical model which has the virtue of relative mathematical simplicity without loss of essential physical relevance.

Our paper is structured as follows. In Section 2 we present the model of the antiplane frictional contact of an electro-viscoelastic cylinder. In Section 3 we introduce the notation, list the assumption on problem's data, derive the variational formulation of the problem and state our main existence and uniqueness result, Theorem 3.1. The proof of the theorem is provided in Section 4; it is based on arguments of evolutionary variational inequalities and fixed point. Finally, in Section 5 we investigate the behavior of the solution as the viscosity converges to zero and prove that it converges to the solution of the corresponding electro-elastic antiplane contact problem studied in [23].

2. The model

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times (-\infty, +\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measure of Γ_1 and Γ_a, denoted $\operatorname{meas}\Gamma_1$ and $\operatorname{meas}\Gamma_a$, are positive. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty, +\infty)$ and a surface electrical charge of density q_b is prescribed on $\Gamma_b \times (-\infty, +\infty)$. The cylinder is in contact over $\Gamma_3 \times (-\infty, +\infty)$ with a conductive obstacle, the so called foundation. The contact is frictional and is modeled with Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0, T]$.

Below in this paper the indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable; also, a dot above represents the time derivative. We use $\mathcal{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and " $\cdot$ ", $\|\cdot\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and $\mathcal{S}^3$; we have:

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2} \quad \text{for all } \mathbf{u} = (u_i),\ \mathbf{v} = (v_i) \in \mathbb{R}^3,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{S}^3.$$

We assume that

$$\mathbf{f}_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.1}$$
$$\mathbf{f}_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0, T] \to \mathbb{R}, \tag{2.2}$$
$$q_0 = q_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.3}$$
$$q_2 = q_2(x_1, x_2, t) : \Gamma_b \times [0, T] \to \mathbb{R}. \tag{2.4}$$

The forces (2.1), (2.2) and the electric charges (2.3), (2.4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent on x_3 and have the form

$$\mathbf{u} = (0, 0, u) \quad \text{with } u = u(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.5}$$
$$\varphi = \varphi(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}. \tag{2.6}$$

Such kind of deformation, associated to a displacement field of the form (2.5), is called an *antiplane shear*, see for instance [9, 11] for details.

We denote by $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ the strain tensor and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the stress tensor; we also denote by $\mathbf{E}(\varphi) = (E_i(\varphi))$ the electric field and by $\mathbf{D} = (D_i)$ the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i}\,.$$

The material's behavior is modelled by an electro-viscoelastic constitutive law of the form

$$\boldsymbol{\sigma} = 2\theta\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \zeta\,\operatorname{tr}\boldsymbol{\varepsilon}(\dot{\mathbf{u}})\,\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda\,\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u})\,\mathbf{I} - \mathcal{E}^*\mathbf{E}(\varphi), \tag{2.7}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \beta\mathbf{E}(\varphi), \tag{2.8}$$

where ζ and θ are viscosity coefficients, λ and μ are the Lamé coefficients, tr $\boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, β is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ is its transpose. We assume that

$$\mathcal{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13}+\varepsilon_{31}) \\ e(\varepsilon_{23}+\varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathcal{S}^3, \tag{2.9}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, β and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\varepsilon}\cdot\mathbf{v} = \boldsymbol{\varepsilon}\cdot\mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon} \in \mathcal{S}^3$, $\mathbf{v} \in \mathbb{R}^3$, it follows from (2.9) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{2.10}$$

In the antiplane context (2.5), (2.6), using the constitutive equations (2.7), (2.8) and equalities (2.9), (2.10) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} \\ 0 & 0 & \theta\dot{u}_{,2} + \mu u_{,2} + e\varphi_{,2} \\ \theta\dot{u}_{,1} + \mu u_{,1} + e\varphi_{,1} & \theta\dot{u}_{,2}\, \mu u_{,2} + e\varphi_{,2} & 0 \end{pmatrix}, \tag{2.11}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix}. \tag{2.12}$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0}, \quad D_{i,i} - q_0 = 0 \quad \text{in } \mathcal{B}\times(0,T),$$

where $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$. Taking into account (2.11), (2.12), (2.5), (2.6), (2.1) and (2.3), the equilibrium equations above reduce to the following scalar equations

$$\operatorname{div}(\theta\nabla\dot{u} + \mu\nabla u + e\nabla\varphi) + f_0 = 0 \quad \text{in } \Omega\times(0,T), \tag{2.13}$$

$$\operatorname{div}(e\nabla u - \beta\nabla\varphi) = q_0 \quad \text{in } \Omega\times(0,T). \tag{2.14}$$

Here and below we use the notation

$$\operatorname{div}\boldsymbol{\tau} = \tau_{1,1} + \tau_{1,2} \quad \text{for } \boldsymbol{\tau} = (\tau_1(x_1,x_2,t), \tau_2(x_1,x_2,t)),$$
$$\nabla v = (v_{,1}, v_{,2}), \quad \partial_\nu v = v_{,1}\,\nu_1 + v_{,2}\,\nu_2 \quad \text{for } v = v(x_1,x_2,t).$$

We now describe the boundary conditions. During the process the cylinder is clamped on $\Gamma_1\times(-\infty,+\infty)$ and the electric potential vanish on $\Gamma_1\times(-\infty,+\infty)$; thus, (2.5) and (2.6) imply that

$$u = 0 \quad \text{on } \Gamma_1\times(0,T), \tag{2.15}$$

$$\varphi = 0 \quad \text{on } \Gamma_a\times(0,T). \tag{2.16}$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma\times(-\infty,+\infty)$. We have

$$\boldsymbol{\nu} = (\nu_1,\nu_2,0) \quad \text{with} \quad \nu_i = \nu_i(x_1,x_2) : \Gamma\to\mathbb{R}, \quad i = 1,2. \tag{2.17}$$

For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}. \tag{2.18}$$

For a given stress field σ we denote by σ_ν and σ_τ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}. \tag{2.19}$$

From (2.11), (2.12) and (2.17) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0,0,\theta\partial_\nu\dot u + \mu\partial_\nu u + e\partial_\nu\varphi), \quad \mathbf{D}\cdot\boldsymbol{\nu} = e\partial_\nu u - \beta\partial_\nu\varphi. \tag{2.20}$$

Taking into account (2.2), (2.4) and (2.20), the traction condition on $\Gamma_2 \times (-\infty,\infty)$ and the electric conditions on $\Gamma_b \times (-\infty,\infty)$ are given by

$$\theta\partial_\nu\dot u + \mu\partial_\nu u + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2\times(0,T), \tag{2.21}$$

$$e\partial_\nu u - \beta\partial_\nu\varphi = q_b \quad \text{on } \Gamma_b\times(0,T). \tag{2.22}$$

We now describe the frictional contact condition and the electric conditions on $\Gamma_3\times(-\infty,+\infty)$. First, from (2.5) and (2.17) we infer that the normal displacement vanishes, $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (2.5), (2.11), (2.17)–(2.19) we conclude that

$$\mathbf{u}_\tau = (0,0,u), \quad \boldsymbol{\sigma}_\tau = (0,0,\theta\partial_\nu\dot u + \mu\partial_\nu u + e\partial_\nu\varphi). \tag{2.23}$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is

$$\|\boldsymbol{\sigma}_\tau\| \le g, \quad \boldsymbol{\sigma}_\tau = -\,g\,\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \ne \mathbf{0} \quad \text{on } \Gamma_3\times(0,T). \tag{2.24}$$

Here $g:\Gamma_3\to\mathbb{R}_+$ is a given function, the friction bound, and $\dot{\mathbf{u}}_\tau$ represents the tangential velocity on the contact boundary, see [7, 20] for details. Using now (2.23) it is straightforward to see that the friction law (2.24) implies

$$\begin{gathered} |\theta\partial_\nu\dot u + \mu\partial_\nu u + e\partial_\nu\varphi| \le g,\\ \theta\partial_\nu\dot u + \mu\,\partial_\nu u + e\partial_\nu\varphi = -g\,\frac{\dot u}{|\dot u|} \quad \text{if } \dot u \ne 0 \end{gathered} \tag{2.25}$$

on $\Gamma_3\times(0,T)$.

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D}\cdot\boldsymbol{\nu} = k\,(\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3\times(0,T),$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. We use (2.20) and the previous equality to obtain

$$e\partial_\nu u - \beta\partial_\nu\varphi = k\,(\varphi - \varphi_F) \quad \text{on } \Gamma_3\times(0,T). \tag{2.26}$$

Finally, we prescribe the initial displacement,

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2.27}$$

where u_0 is a given function on Ω.

We collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation.

Problem $\mathcal{P}$. Find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}$ and the electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$ such that (2.13)–(2.16), (2.21), (2.22), (2.25)–(2.27) hold.

Note that once the displacement field u and the electric potential φ which solve Problem $\mathcal{P}$ are known, then the stress tensor $\boldsymbol{\sigma}$ and the electric displacement field $\mathbf{D}$ can be obtained by using the constitutive laws (2.11) and (2.12), respectively.

3. Variational formulation

We derive now the variational formulation of the Problem $\mathcal{P}$. To this end we introduce the function spaces

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}, \quad W = \{\, \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_a\}$$

where, here and below, we write w for the trace γw of a function $w \in H^1(\Omega)$ on Γ. Since *meas* $\Gamma_1 > 0$ and $\operatorname{meas} \Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V, \quad (\varphi, \psi)_W = \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx \quad \forall \varphi,\, \psi \in W.$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2} \quad \forall v \in V, \quad \|\psi\|_V = \|\nabla \psi\|_{L^2(\Omega)^2} \quad \forall \psi \in W \tag{3.1}$$

are equivalent on V and W, respectively, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist two positive constants $c_V > 0$ and $c_W > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \le c_V \|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \le c_W \|\psi\|_W \quad \forall \psi \in W. \tag{3.2}$$

For a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \le p \le \infty,\ k = 1, 2, \dots$; we also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the respective norms

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X,$$
$$\|x\|_{C^1([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X + \max_{t \in [0,T]} \|\dot{x}(t)\|_X.$$

In the study of the Problem $\mathcal{P}$ we assume that the viscosity coefficient and the electric permittivity coefficient satisfy

$$\theta \in L^\infty(\Omega) \text{ and there exists } \theta^* > 0 \text{ such that } \ \theta(\mathbf{x}) \ge \theta^* \text{ a.e. } \mathbf{x} \in \Omega, \tag{3.3}$$

$$\beta \in L^\infty(\Omega) \text{ and there exists } \beta^* > 0 \text{ such that } \ \beta(\mathbf{x}) \ge \beta^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{3.4}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad \text{and} \quad \mu(\mathbf{x}) > 0 \ \text{ a.e. } \mathbf{x} \in \Omega, \tag{3.5}$$

$$e \in L^\infty(\Omega). \tag{3.6}$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad f_2 \in W^{1,2}(0,T;L^2(\Gamma_2)), \tag{3.7}$$

$$q_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad q_2 \in W^{1,2}(0,T;L^2(\Gamma_b)). \tag{3.8}$$

The friction bound and the electric conductivity coefficient satisfy

$$g \in L^2(\Gamma_3) \quad \text{and} \quad g(\mathbf{x}) \geq 0 \ \text{ a.e. } \mathbf{x} \in \Gamma_3, \tag{3.9}$$

$$k \in L^\infty(\Gamma_3) \quad \text{and} \quad k(\mathbf{x}) \geq 0 \ \text{ a.e. } \mathbf{x} \in \Gamma_3. \tag{3.10}$$

Finally, we assume that the electric potential of the foundation and the initial displacement are such that

$$\varphi_F \in W^{1,2}(0,T;L^2(\Gamma_3)), \tag{3.11}$$

$$u_0 \in V. \tag{3.12}$$

Next, we define the bilinear forms $a_\theta : V \times V \to \mathbb{R}$, $a_\mu : V \times V \to \mathbb{R}$, $a_e : V \times W \to \mathbb{R}$, $a_e^* : W \times V \to \mathbb{R}$, and $a_\beta : W \times W \to \mathbb{R}$, by equalities

$$a_\theta(u,v) = \int_\Omega \theta \, \nabla u \cdot \nabla v \, dx, \tag{3.13}$$

$$a_\mu(u,v) = \int_\Omega \mu \, \nabla u \cdot \nabla v \, dx, \tag{3.14}$$

$$a_e(u,\varphi) = \int_\Omega e \, \nabla u \cdot \nabla \varphi \, dx = a_e^*(\varphi, u), \tag{3.15}$$

$$a_\beta(\varphi,\psi) = \int_\Omega \beta \, \nabla \varphi \cdot \nabla \psi \, dx + \int_{\Gamma_3} k \, \varphi \psi \, dx, \tag{3.16}$$

for all u, $v \in V$, $\varphi, \psi \in W$. Assumptions (3.3)–(3.6), (3.10) imply that the integrals above are well defined and, using (3.1) and (3.2), it follows that the forms a_θ, a_μ, a_e, a_e^* and a_β are continuous; moreover, the forms a_θ, a_μ and a_β are symmetric and, in addition, the form a_θ is V-elliptic and the form a_β is W-elliptic, since

$$a_\theta(v,v) \geq \theta^* \|v\|_V^2 \quad \forall v \in V, \tag{3.17}$$

$$a_\beta(\psi,\psi) \geq \beta^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{3.18}$$

We also define the mappings $f : [0,T] \to V$, $q : [0,T] \to W$ and $j : V \to \mathbb{R}$, respectively, by

$$(f(t), v)_V = \int_\Omega f_0(t) v \, dx + \int_{\Gamma_2} f_2(t) v \, da, \tag{3.19}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_2} q_b(t) \psi \, da + \int_{\Gamma_3} k \, \varphi_F(t) \psi \, da, \tag{3.20}$$

$$j(v) = \int_{\Gamma_3} g|v| \, da, \tag{3.21}$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. The definition of f and q are based on Riesz's representation theorem; moreover, it follows from assumptions by (3.7)–(3.10), that the integrals above are well-defined and

$$f \in W^{1,2}(0,T;V), \tag{3.22}$$

$$q \in W^{1,2}(0,T;W). \tag{3.23}$$

Performing integration by parts and using notation (3.13)–(3.16), (3.19)–(3.21) it is straightforward to derive the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. Find a displacement field $u:[0,T]\to V$ and an electric potential field $\varphi:[0,T]\to W$ such that

$$\begin{aligned}&a_\theta(\dot u(t),v-\dot u(t))+a_\mu(u(t),v-\dot u(t))+a_e^*(\varphi(t),v-\dot u(t))\\&+j(v)-j(\dot u(t))\ge (f(t),v-\dot u(t))_V\quad \forall v\in V,\ t\in[0,T],\end{aligned}\tag{3.24}$$

$$a_\beta(\varphi(t),\psi)-a_e(u(t),\psi)=(q(t),\psi)_W\quad \forall\psi\in W,\ t\in[0,T],\tag{3.25}$$

$$u(0)=u_0.\tag{3.26}$$

Our main existence and uniqueness result, which we state now and prove in the next section, is the following.

Theorem 3.1. *Assume that* (3.3)–(3.12) *hold. Then there exists a unique solution of problem $\mathcal{P}_V$. Moreover, the solution satisfies*

$$u\in W^{2,2}(0,T;V),\quad \varphi\in W^{1,2}(0,T;W).\tag{3.27}$$

A couple of functions (u,φ) which solves Problem $\mathcal{P}_V$ is called a *weak solution* of the electro-mechanical problem $\mathcal{P}$. We conclude by Theorem 3.1 that the antiplane contact problem $\mathcal{P}$ has a unique weak solution, provided that (3.3)–(3.12) hold.

4. Proof of Theorem 3.1

The proof is based on an abstract result for evolutionary variational inequalities that we present in what follows. Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$ and consider the problem of finding $u:[0,T]\to X$ such that

$$\begin{aligned}&a(\dot u(t),v-\dot u(t))_X+b(u(t),v-\dot u(t))_X+j(v)-j(\dot u(t))\\&\ge (h(t),v-\dot u(t))_X\quad \forall v\in X,\ t\in[0,T],\end{aligned}\tag{4.1}$$

$$u(0)=u_0.\tag{4.2}$$

In the study of the Cauchy problem (4.1)–(4.2) we assume that:

$a:X\times X\to\mathbb{R}$ is a bilinear symmetric form and
(a) there exists $M>0$ such that $|a(u,v)|\le M\|u\|_X\|v\|_X$ for all $u,\,v\in X$.
(b) there exists $m>0$ such that $a(v,v)\ge m\|v\|_X^2$ for all $v\in X$. (4.3)

$b:X\times X\to\mathbb{R}$ is a bilinear form and there exists $M'>0$ such that $|b(u,v)|\le M'\|u\|_X\|v\|_X$ for all $u,\,v\in X$. (4.4)

$$j:X\to\mathbb{R}\text{ is a convex lower semicontinuous functional.}\tag{4.5}$$

$$h\in C([0,T];X).\tag{4.6}$$

$$u_0\in X.\tag{4.7}$$

The following existence, uniqueness and regularity result represent a particular case of a more general result proved in [7, p. 230–234].

Theorem 4.1. *Let* (4.3)–(4.7) *hold. Then*
(1) *There exists a unique solution $u\in C^1([0,T];X)$ of problem* (4.1) *and* (4.2).
(2) *If u_1 and u_2 are two solutions of* (4.1) *and* (4.2) *corresponding to the data $h_1,\,h_2\in C([0,T];X)$, then there exists $c>0$ such that*

$$\|\dot u_1(t)-\dot u_2(t)\|_X\le c\,(\|h_1(t)-h_2(t)\|_X+\|u_1(t)-u_2(t)\|_X)\quad \forall\,t\in[0,T].\tag{4.8}$$

(3) *If, moreover, $h\in W^{1,p}(0,T;X)$, for some $p\in[1,\infty]$, then the solution satisfies $u\in W^{2,p}(0,T;X)$.*

We turn now to the proof of Theorem 3.1 which will be carried out in several steps. We assume in what follows that (3.3)–(3.12) hold and, everywhere below, we denote by c various positive constants which are independent of time and whose value may change from line to line. Let $\eta \in C([0,T],V)$ be given and, in the first step, consider the following intermediate variational problem.

Problem. $\mathcal{P}^1_\eta$. Find a displacement field $u_\eta : [0,T] \to V$ such that

$$\begin{aligned} &a_\theta(\dot u_\eta(t), v-\dot u_\eta(t)) + a_\mu(u_\eta(t), v-\dot u_\eta(t)) + (\eta(t), v-\dot u_\eta(t))_V \\ &+ j(v) - j(\dot u_\eta(t)) \geq (f(t), v-\dot u_\eta(t))_V \quad \forall\, v\in V,\ t\in[0,T], \end{aligned} \tag{4.9}$$

$$u_\eta(0) = u_0. \tag{4.10}$$

We have the following result for $\mathcal{P}^1_\eta$.

Lemma 4.2. (1) *There exists a unique solution* $u_\eta \in C^1([0,T];V)$ *to the problem* (4.9)–(4.10).
(2) *If* u_1 *and* u_2 *are two solutions of* (4.9)–(4.10) *corresponding to the data* η_1, $\eta_2 \in C([0,T];V)$*, then there exists* $c>0$ *such that*

$$\|\dot u_1(t) - \dot u_2(t)\|_V \leq c\,(\|\eta_1(t)-\eta_2(t)\|_V + \|u_1(t)-u_2(t)\|_V) \quad \forall\, t\in[0,T]. \tag{4.11}$$

(3) *If, moreover,* $\eta \in W^{1,2}(0,T;V)$*, then the solution satisfies* $u_\eta \in W^{2,2}(0,T;V)$.

Proof. We apply Theorem 4.1 on the space $X=V$ with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$, with the choice $a=a_\theta$, $b=a_\mu$, $h=f-\eta$. Clearly a_θ and a_μ satisfy conditions (4.3) and (4.4), respectively, and using (3.9) it follows from that the functional j satisfies condition (4.5). Moreover, using (3.22) and the regularity $\eta\in C([0,T],V)$ it is easy to see that $f-\eta \in C([0,T];V)$ i.e. h satisfies (4.6). Finally, we note that (4.7) is satisfied too and, if $\eta\in W^{1,2}(0,T;V)$ then $h=f-\eta\in W^{1,2}(0,T;V)$. Lemma 4.2 is a direct consequence of Theorem 4.1. □

In the next step we use the solution $u_\eta \in C^1([0,T],V)$, obtained in Lemma 4.2, to construct the following variational problem for the electrical potential.

Problem $\mathcal{P}^2_\eta$. Find an electrical potential $\varphi_\eta : [0,T]\to W$ such that

$$a_\beta(\varphi_\eta(t),\psi) - a_e(u_\eta(t),\psi) = (q(t),\psi)_W \quad \forall \psi\in W,\, t\in[0,T]. \tag{4.12}$$

The well-posedness of problem $\mathcal{P}^2_\eta$ follows.

Lemma 4.3. *There exists a unique solution* $\varphi_\eta \in W^{1,2}(0,T;W)$ *which satisfies* (4.12)*. Moreover, if* φ_{η_1} *and* φ_{η_2} *are the solutions of* (4.12) *corresponding to* η_1, $\eta_2 \in C([0,T];V)$ *then, there exists* $c>0$*, such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq c\,\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V \quad \forall\, t\in[0,T]. \tag{4.13}$$

Proof. Let $t\in[0,T]$. We use the properties of the bilinear form a_β and the Lax-Milgram lemma to see that there exists a unique element $\varphi_\eta(t)\in W$ which solves (4.12) at any moment $t\in[0,T]$. Consider now t_1, $t_2\in[0,T]$; using (4.12) and (3.18) we find that

$$\beta^*\,\|\varphi(t_1)-\varphi(t_2)\|_W^2 \leq \|e\|_{L^\infty(\Omega)}\|u(t_1)-u(t_2)\|_V + \|q(t_1)-q(t_2)\|_W\|\varphi(t_1)-\varphi(t_2)\|_W$$

which implies

$$\|\varphi(t_1)-\varphi(t_2)\|_W \leq c\,(\|u(t_1)-u(t_2)\|_V + \|q(t_1)-q(t_2)\|_W). \tag{4.14}$$

We note that regularity $u_\eta \in C^1([0,T];V)$ combined with (3.23) and (4.14) imply that $\varphi_\eta \in W^{1,2}(0,T;W)$. Also, arguments similar to those used in the proof of (4.14) lead to (4.13), which concludes the proof. □

We now use Riesz's representation theorem to define the element $\Lambda\eta(t) \in V$ by equality

$$(\Lambda\eta(t), v)_V = a_e^*(\varphi_\eta(t), v) \quad \forall v \in V,\ t \in [0,T]. \tag{4.15}$$

Clearly, for a given $\eta \in C([0,T];V)$ the function $t \mapsto \Lambda\eta(t)$ belongs to $C([0,T];V)$. In the newt step we show that the operator $\Lambda : C([0,T];V) \to C([0,T];V)$ a unique fixed point.

Lemma 4.4. *There exists a unique $\widetilde{\eta} \in W^{1,2}(0,T;V)$ such that $\Lambda\widetilde{\eta} = \widetilde{\eta}$.*

Proof. Let η_1, $\eta_2 \in C([0,T];V)$ and denote by u_i and φ_i the functions u_{η_i} and φ_{η_i} obtained in Lemmas 4.2 and 4.3, for $i = 1,2$. Let $t \in [0,T]$. Using (4.15) and (3.15) we obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \le c\,\|\varphi_1(t) - \varphi_2(t)\|_W,$$

and, keeping in mind (4.13), we find

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \le c\,\|u_1(t) - u_2(t)\|_V. \tag{4.16}$$

On the other hand, since $u_i(t) = u_0 + \int_0^t \dot{u}_i(s)\,ds$, we have

$$\|u_1(t) - u_2(t)\|_V \le \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V\,ds, \tag{4.17}$$

and using this inequality in (4.11) yields

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \le c\Big(\|\eta_1(t) - \eta_2(t)\|_V + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V\,ds\Big).$$

It follows now from a Gronwall-type argument that

$$\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V\,ds \le c\int_0^t \|\eta_1(t) - \eta_2(t)\|_V\,ds. \tag{4.18}$$

Combining (4.16)–(4.18) leads to

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \le c\int_0^t \|\eta_1(t) - \eta_2(t)\|_V\,ds$$

and, reiterating this inequality n times results in

$$\|\Lambda^n\eta_1(t) - \Lambda^n\eta_2(t)\|_V \le \frac{c^n}{n!}\|\eta_1(t) - \eta_2(t)\|_{C([0,T];V)}.$$

This last inequality shows that for a sufficiently large n the operator Λ^n is a contraction on the Banach space $C([0,T];V)$ and, therefore, there exists a unique element $\widetilde{\eta} \in C([0,T];V)$ such that $\Lambda\widetilde{\eta} = \widetilde{\eta}$. It follows from Lemma 4.3 that $\varphi_\eta \in W^{1,2}(0,T;W)$ and, therefore, the definition (4.15) of the operator Λ combined with the properties of the bilinear form a_e^* implies that $\Lambda\widetilde{\eta} \in W^{1,2}(0,T;V)$; this regularity combined with equality $\Lambda\widetilde{\eta} = \widetilde{\eta}$ shows that $\widetilde{\eta} \in W^{1,2}(0,T;V)$ which concludes the proof. □

We have now all the ingredients to provide the proof of the Theorem 3.1.

Existence. Let $\widetilde{\eta} \in W^{1,2}(0,T;V)$ be the fixed point of the operator Λ, and let $u_{\widetilde{\eta}}$, $\varphi_{\widetilde{\eta}}$ be the solutions of problems $\mathcal{P}^1_\eta$ and $\mathcal{P}^2_\eta$, respectively, for $\eta = \widetilde{\eta}$. It follows from (4.15) that

$$(\widetilde{\eta}(t), v)_V = a^*_e(\varphi_{\widetilde{\eta}}(t), v) \quad \forall v \in V,\ t \in [0,T]$$

and, therefore, (4.9), (4.10) and (4.12) imply that $(u_{\widetilde{\eta}}, \varphi_{\widetilde{\eta}})$ is a solution of problem $\mathcal{P}_V$. Regularity (3.27) of the solution follows from Lemmas 4.2 (3) and 4.3.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. It can also be obtained by using arguments similar as those used in [7, 20].

5. A convergence result

In this section we investigate the behavior of the weak solution of the antiplane frictional problem as the viscosity converges to zero. In order to outline the dependence on the viscosity coefficient θ, we reformulate Problem $\mathcal{P}_V$ as follows.

Problem $\mathcal{P}^\theta_V$. Find a displacement field $u_\theta : [0,T] \to V$ and an electric potential field $\varphi_\theta : [0,T] \to W$ such that

$$\begin{aligned} &a_\theta(\dot{u}_\theta(t), v - \dot{u}_\theta(t)) + a_\mu(u_\theta(t), v - \dot{u}_\theta(t)) + a^*_e(\varphi_\theta(t), v - \dot{u}_\theta(t)) \\ &+ j(v) - j(\dot{u}_\theta(t)) \ge (f(t), v - \dot{u}_\theta(t))_V \quad \forall v \in V,\ t \in [0,T], \end{aligned} \tag{5.1}$$

$$a_\beta(\varphi_\theta(t), \psi) - a_e(u_\theta(t), \psi) = (q(t), \psi)_W \quad \forall \psi \in W,\ t \in [0,T], \tag{5.2}$$

$$u_\theta(0) = u_0. \tag{5.3}$$

We also consider the inviscid problem associated to (5.1)–(5.3); i.e., the problem obtained for $\theta = 0$, which is formulated as follows.

Problem $\mathcal{P}^0_V$. Find a displacement field $u : [0,T] \to V$ and an electric potential field $\varphi : [0,T] \to W$ such that

$$\begin{aligned} &a_\mu(u(t), v - \dot{u}(t)) + a^*_e(\varphi(t), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \\ &\ge (f(t), v - \dot{u}(t))_V \quad \forall v \in V,\ \text{a.e. } t \in (0,T), \end{aligned} \tag{5.4}$$

$$a_\beta(\varphi(t), \psi) - a_e(u(t), \psi) = (q(t), \psi)_W \quad \forall \psi \in W,\ t \in [0,T], \tag{5.5}$$

$$u(0) = u_0. \tag{5.6}$$

Clearly, Problem $\mathcal{P}^0_V$ represents the variational formulation of the model in Section 2, in the case when the piezoelectric cylinder is assumed to be electro-elastic.

Assume in what follows that (3.3)–(3.12) hold. Then, it follows from Theorem 3.1 that Problem $\mathcal{P}^\theta_V$ has a unique solution $(u_\theta, \varphi_\theta)$ which satisfies $u_\theta \in W^{2,2}(0,T;V)$, $\varphi_\theta \in W^{1,2}(0,T;W)$. In order to state an existence and uniqueness result in the study of Problem $\mathcal{P}^0_V$ we need additional assumptions. First, we reinforce (3.5) with assumption

$$\mu \in L^\infty(\Omega) \text{ and there exists } \mu^* > 0 \text{ such that } \ \mu(\mathbf{x}) \ge \mu^* \text{ a.e. } \mathbf{x} \in \Omega \tag{5.7}$$

and note that in this case the bilinear form a_μ is V-elliptic, since it safisfies

$$a_\mu(v,v) \ge \mu^* \|v\|^2_V \quad \forall v \in V. \tag{5.8}$$

Next, we employ the W-ellipticity of the form a_β, (3.18), and the Lax-Milgram lemma to see that there exists a unique element $\varphi_0 \in W$ such that

$$a_\beta(\varphi_0, \psi) - a_e(u_0, \psi) = (q(0), \psi)_W \quad \forall \psi \in W. \tag{5.9}$$

We use the element φ_0 defined above to introduce the condition

$$a_\mu(u_0, v)_V + a_e^*(\varphi_0, v) + j(v) \geq (f(0), v)_V \quad \forall v \in V. \tag{5.10}$$

This inequality represents a compatibility condition on the initial data that is necessary in many quasistatic problems, see for instance [20]. Physically, it is needed so as to guarantee that initially the state is in equilibrium, since otherwise the inertial terms cannot be neglected and the problems become dynamic. It follows from Theorem 4.1 in [23] that, under assumptions (3.3)–(3.12), (5.7) and (5.10), the electro-elastic Problem $\mathcal{P}_V^0$ has a unique solution (u, φ) with regularity $u \in W^{1,2}(0,T;V)$, $\varphi \in W^{1,2}(0,T;W)$.

Consider now the assumption

$$\frac{1}{\theta^*}\, \|\theta\|^2_{L^\infty(\Omega)} \to 0. \tag{5.11}$$

We have the following convergence result.

Theorem 5.1. *Assume that* (3.3)–(3.12)*,* (5.7)*,* (5.10) *and* (5.11) *hold. Then the solution* $(u_\theta, \varphi_\theta)$ *of Problem* $\mathcal{P}_V^\theta$ *converges to the solution* u *of Problem* $\mathcal{P}_V^0$*, i.e.*

$$\|u_\theta - u\|_{C([0,T];V)} \to 0, \quad \|\varphi_\theta - \varphi\|_{C([0,T];W)} \to 0. \tag{5.12}$$

Proof. The equalities and inequalities below hold for almost any $t \in (0,T)$. We take $v = \dot u(t)$ in (5.1), $v = \dot u_\theta(t)$ in (5.4) and add the resulting inequalities to obtain

$$\begin{aligned} &a_\theta(\dot u_\theta(t), \dot u(t) - \dot u_\theta(t)) + a_\mu(u_\theta(t) - u(t), \dot u(t) - \dot u_\theta(t)) \\ &+ a_e^*(\varphi_\theta(t) - \varphi(t), \dot u(t) - \dot u_\theta(t)) \geq 0. \end{aligned}$$

This implies that

$$\begin{aligned} &a_\theta(\dot u_\theta(t) - \dot u(t), \dot u_\theta(t) - \dot u(t)) + a_\mu(u_\theta(t) - u(t), \dot u_\theta(t) - \dot u(t)) \\ &\leq a_\theta(\dot u(t), \dot u(t) - \dot u_\theta(t)) + a_e^*(\varphi_\theta(t) - \varphi(t), \dot u(t) - \dot u_\theta(t)). \end{aligned} \tag{5.13}$$

We use now assumption (3.3) to see that

$$\begin{aligned} &\theta^* \|\dot u_\theta(t) - \dot u(t)\|_V^2 + a_\mu(u_\theta(t) - u(t), \dot u_\theta(t) - \dot u(t)) \\ &\leq \|\theta\|_{L^\infty(\Omega)}\, \|\dot u(t)\|_V \|\dot u_\theta(t) - \dot u(t)\|_V + a_e^*(\varphi_\theta(t) - \varphi(t), \dot u(t) - \dot u_\theta(t)) \end{aligned}$$

and combine this inequality with the elementary inequality

$$\|\theta\|_{L^\infty(\Omega)}\, \|\dot u(t)\|\, \|\dot u_\theta(t) - \dot u(t)\|_V \leq \frac{\|\theta\|^2_{L^\infty(\Omega)}}{4\theta^*}\, \|\dot u(t)\|_V^2 + \theta^*\, \|\dot u_\theta(t) - \dot u(t)\|_V^2.$$

As a result we obtain

$$a_\mu(u_\theta(t) - u(t), \dot u_\theta(t) - \dot u(t)) \leq \frac{\|\theta\|^2_{L^\infty(\Omega)}}{4\theta^*}\, \|\dot u(t)\|_V^2 + a_e^*(\varphi_\theta(t) - \varphi(t), \dot u(t) - \dot u_\theta(t)). \tag{5.14}$$

On the other hand, we recall that a_β and a_e are bilinear continuous forms and the functions u_θ, u, φ_θ, φ and q have the regularity $W^{1,2}$. Therefore, the two sides of equalities (5.2) and (5.5) are derivable with respect to the time variable. We derive (5.2) and (5.5), subtract the resulting equalities and use the definition of the form a_e^* to obtain

$$a_\beta(\dot\varphi_\theta(t) - \dot\varphi(t), \psi) = a_e(\dot u_\theta(t) - \dot u(t), \psi) = a_e^*(\psi, \dot u_\theta(t) - \dot u(t)) \quad \forall \psi \in W.$$

We take now $\psi = \varphi(t) - \varphi_\theta(t)$ in the previous equality to find that

$$a_e^*(\varphi_\theta(t) - \varphi(t), \dot u(t) - \dot u_\theta(t)) = a_\beta(\dot\varphi_\theta(t) - \dot\varphi(t), \varphi(t) - \varphi_\theta(t)). \tag{5.15}$$

Next, we write (5.2) and (5.5) at $t = 0$, use the initial condition $u_\theta(0) = u(0) = u_0$ and the unique solvability of the variational equation (5.9) to see that

$$\varphi_\theta(0) = \varphi(0) = \varphi_0. \tag{5.16}$$

We now combine (5.14) and (5.15) to obtain

$$a_\mu(u_\theta(t) - u(t), \dot{u}_\theta(t) - \dot{u}(t)) \le \frac{\|\theta\|^2_{L^\infty(\Omega)}}{4\theta^*} \|\dot{u}(t)\|^2_V + a_\beta(\dot{\varphi}_\theta(t) - \dot{\varphi}(t), \varphi(t) - \varphi_\theta(t)).$$

Let $s \in [0, T]$. We integrate the previous inequality on $[0, s]$ with the initial conditions (5.3), (5.6) and (5.16) and use (3.18), (5.8) to obtain

$$\frac{\mu^*}{2} \|u_\theta(s) - u(s)\|^2_V \le \frac{\|\theta\|^2_{L^\infty(\Omega)}}{2\theta^*} \int_0^s \|\dot{u}(t)\|^2_V \, dt. \tag{5.17}$$

Next, we write (5.2) and (5.5) with $t = s$, $\psi = \varphi_\theta(s) - \varphi(s)$ and subtract the resulting equalities to obtain

$$a_\beta(\varphi_\theta(s) - \varphi(s), \varphi_\theta(s) - \varphi(s)) = a_e(u_\theta(s) - u(s), \varphi_\theta(s) - \varphi(s)).$$

Then, we use the coercivity of the form a_β, (3.18), and the continuity of the form a_e; as a result we find that

$$\|\varphi_\theta(s) - \varphi(s)\|_W \le \frac{\|e\|_{L^\infty(\Omega)}}{\beta^*} \|u_\theta(s) - u(s)\|_V. \tag{5.18}$$

Assume now that (5.11) hold. Then (5.17) and (5.18) yield the convergence result (5.12) which concludes the proof. □

Consider now the case of homogeneous viscosity, i.e. the case when assumption (5.11) is replaced by the assumption

$$\theta(x) = \theta \quad \text{a.e.} \quad \mathbf{x} \in \Omega,$$

where θ is given positive constant. In this case $\|\theta\|_{L^\infty(\Omega)} = \theta$, $\theta^* = \theta$ and the convergence (5.11) is equivalent to $\theta \to 0$. Therefore, by (5.12) we conclude that the weak solution to the antiplane electro-viscoelastic problem with Tresca's friction law may be approached by the weak solution to the antiplane electro-elastic problem with Tresca's friction law, as the viscosity is small enough. From mechanical point of view this convergence result shows that the electro-elasticity with friction may be considered as a limit case of electro-viscoelasticity with friction as the viscosity decreases.

References

[1] F. Andreu, J. M. Mazón and M. Sofonea, Entropy solutions in the study of antiplane shear deformations for elastic solids, *Math. Models Methods Appl. Sci.* **10** (2000), 96–126.

[2] R.C. Batra and J.S. Yang, Saint-Venant's principle in linear piezoelectricity, *Journal of Elasticity* **38** (1995), 209–218.

[3] P. Bisenga, F. Lebon and F. Maceri, The unilateral frictional contact of a piezoelectric body with a rigid support, in *Contact Mechanics*, J.A.C. Martins and Manuel D.P. Monteiro Marques (Eds.), Kluwer, Dordrecht, 2002, 347–354.

[4] A. Borrelli, C. O. Horgan and M. C. Patria, Saint-Venant's principle for antiplane shear deformations of linear piezoelectic materials, *SIAM J. Appl. Math.* **62** (2002), 2027–2044.

[5] M. Campillo, C. Dascalu and I.R. Ionescu, Instability of a periodic system of faults, *Geophysical International Journal* **159** (2004), 212–222.

[6] M. Campillo and I.R. Ionescu, Initiation of antiplane shear instability under slip dependent friction, *Journal of Geophysical Research* **102 B9** (1997) 363–371.

[7] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity* **30**, Studies in Advanced Mathematics, Americal Mathematical Society, Providence, RI–International Press, Somerville, MA, 2002.
[8] T.-V. Hoarau-Mantel and A. Matei, Analysis of a viscoelastic antiplane contact problem with slip-dependent friction, *Int. J. Appl. Math. Comput. Sci.* **12** (2002), 51–58.
[9] C. O. Horgan, Anti-plane shear deformation in linear and nonlinear solid mechanics, *SIAM Rev.* **37** (1995), 53–81.
[10] C.O. Horgan, Recent developments concerning Saint-Venant's principle: a second update, *Appl. Mech. Rev.* **49** (1996), 101–111.
[11] C. O. Horgan and K. L. Miller, Anti-plane shear deformation for homogeneous and inhomogeneous anisotropic linearly elastic solids, *J. Appl. Mech.* **61** (1994), 23-29.
[12] T. Ikeda, *Fundamentals of Piezoelectricity*, Oxford University Press, Oxford, 1990.
[13] I.R. Ionescu, C. Dascalu and M. Campillo, Slip-weakening friction on a periodic System of faults: spectral analysis, *Z. Angew. Math. Phys. (ZAMP)* **53** (2002), 980–995.
[14] I.R. Ionescu and S. Wolf, Interaction of faults under slip dependent friction. Nonlinear eingenvalue analysis, *Mathematical Methods in Applied Sciences (M^2AS)* **28** (2005), 77–100.
[15] Z. Lerguet, M. Shillor and M. Sofonea, A frictional contact problem for an electro-viscoelastic body, submitted to *Electronic Journal of Differential Equations*.
[16] F. Maceri and P. Bisegna, The unilateral frictionless contact of a piezoelectric body with a rigid support, *Math. Comp. Modelling* **28** (1998), 19–28.
[17] A. Matei, V. V. Motreanu and M. Sofonea, A Quasistatic Antiplane Contact Problem With Slip Dependent Friction, *Advances in Nonlinear Variational Inequalities* **4** (2001), 1–21.
[18] S. Migórski, Hemivariational inequality for a frictional contact problem in elasto-piezoelectricity, *Discrete and Continous Dynamical Systems - Series B* **6** (2006), 1339–1356.
[19] V. Z. Patron and B. A. Kudryavtsev, *Electromagnetoelasticity, Piezoelectrics and Electrically Conductive Solids*, Gordon & Breach, London, 1988.
[20] M. Shillor, M. Sofonea and J.J. Telega, *Models and Analysis of Quasistatic Contact*, Lect. Notes Phys. **655**, Springer, Berlin Heidelberg, 2004.
[21] M. Sofonea and El H. Essoufi, A piezoelectric contact problem with slip dependent coefficient of friction, *Mathematical Modelling and Analysis* **9** (2004), 229–242.
[22] M. Sofonea and El H. Essoufi, Quasistatic frictional contact of a viscoelastic piezoelectric body, *Adv. Math. Sci. Appl.* **14** (2004), 613–631.
[23] M. Sofonea, M. Dalah and A. Ayadi, Analysis of an antiplane electro-elastic contact problem, *Adv. Math. Sci. Appl.* **17** (2007), 385–400.
[24] Z.-G. Zhou, B. Wang and S.-Y. Du, Investigation of antiplane shear behavior of two collinear permeable cracks in a piezoelectric material by using the nonlocal theory, *Journal of Applied Mechanics* **69** (2002), 1–3.

Mohamed Dalah
Département de Mathématiques, Faculté des Sciences, Université de Mentouri - Constantine, 25 000 Constantine Algérie
E-mail address: `mdalah17@yahoo.fr`

Mircea Sofonea
Laboratoire de Mathématiques et Physique pour les Systèmes, Université de Perpignan, 52 Avenue Paul Alduy, 66 860 Perpignan, France
E-mail address: `sofonea@univ-perp.fr`