Electronic Journal of Differential Equations, Vol. 2007(2007), No. 172, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SIMILARITIES OF DISCRETE AND CONTINUOUS STURM-LIOUVILLE PROBLEMS

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Abstract. In this paper we present a study on the analogous properties of discrete and continuous Sturm-Liouville problems arising in matrix analysis and differential equations, respectively. Green's functions in both cases have analogous expressions in terms of the spectral data. Most of the results associated to inverse problems in both cases are identical. In particular, in both cases Weyl's m-function determines the Sturm-Liouville operators uniquely. Moreover, the well known Rayleigh-Ritz Theorem in linear algebra can be proved by using the concept of Green's function in discrete case.

1. Introduction

First we present a brief description of the discrete Green's functions, which are discussed in more detail in [3]. It is well known that a Sturm-Liouville problem is an initial value problem of the form

$$\begin{gathered} Ly(x) = \lambda\rho(x)y(x), \quad 0 \le x \le \ell \\ p(0)y'(0) - hy(0) = 0 \\ p(\ell)y'(\ell) + Hy(\ell) = 0 \end{gathered} \tag{1.1}$$

where, $p(x), q(x), \rho(x)$ are given, $\rho(x) > 0$, and L is a second order differential operator of the form

$$Ly(x) = -(p(x)y'(x))' + q(x)y.$$

It is well known that for every Sturm-Liouville problem of the form (1.1) there is a corresponding *Green's function* $G(x, s, \lambda)$. If λ is not an eigenvalue of (1.1) then it is well known (see [2], chapter 7) that $G(x, y, \lambda)$ can be constructed as follows. Let $\varphi(x)$ and $\psi(x)$ be solutions of the Sturm-Liouville equation satisfying the first and second condition of (1.1), respectively. Then

$$[\varphi\psi](x) = p(x)\{\varphi(x)\psi'(x) - \varphi'(x)\psi(x)\} = \text{constant}. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 15A18, 15A24, 34B25, 34B27.
Key words and phrases. Green's function; Jacobi matrix; Sturm-Liouville equation; eigenvalue; eigenvector.

Submitted November 7, 2006. Published December 6, 2007.

This constant is zero if λ is an eigenvalue of (1.1) and nonzero otherwise. In the latter case we can choose the constant to be -1, and then

$$G(x,s,\lambda)=\begin{cases}\varphi(x)\psi(s), & 0\le x\le s\\ \varphi(s)\psi(x), & s\le x\le \ell.\end{cases} \tag{1.3}$$

The functions φ,ψ are functions of x,λ. If $\lambda=0$ is not an eigenvalue of (1.1); i.e. h,H are not both zero, then

$$G(x,s,\lambda)=\begin{cases}\varphi_0(x)\psi_0(s), & 0\le x\le s\\ \varphi_0(s)\psi_0(x), & s\le x\le \ell.\end{cases} \tag{1.4}$$

where φ_0,ψ_0 denote φ,ψ, respectively, for $\lambda=0$. Now let $a\le b$ be two fixed points in the interval $(0,\ell)$. We define the normalized Green's function as follows:

$$\Phi(a,b,\lambda)=\frac{G(a,b,\lambda)}{G(a,b,0)}=\frac{\varphi(a)\psi(b)}{\varphi_0(a)\psi_0(b)}. \tag{1.5}$$

Now if we consider a discrete form of the Sturm-Liouville problem, we may define a similar terminology. Using the concept of discrete derivative; i.e., $y'=y_{n+1}-y_n$, the discrete form of the Sturm-Liouville equation is a system of three-term recurrence relations that can be written in a compact form

$$Ax=\lambda Bx \tag{1.6}$$

where A is the tridiagonal matrix

$$A=\begin{pmatrix} a_1 & c_1 & 0 & . & . & .\\ c_1 & a_2 & c_2 & 0 & . & .\\ 0 & c_2 & a_3 & c_3 & 0 & .\\ . & . & . & . & . & .\\ . & . & . & . & . & c_{n-1}\\ . & . & . & . & c_{n-1} & a_n\end{pmatrix}$$

and $B=\operatorname{diag}(b_1,b_2,\dots,b_n)$. For more details see Atkinson [1]. The corresponding *Green's function* for discrete case is defined by

$$G(i,j,\lambda)=(\lambda)=e_i^T(A-\lambda B)^{-1}e_j. \tag{1.7}$$

The corresponding normalized Green's function is defined by

$$\Phi_{ij}(\lambda)=\frac{G(i,j,\lambda)}{G(i,j,0)}. \tag{1.8}$$

Now denote the leading principal submatrix of order k of the matrix A by A_k and denote the trailing principal submatrix of order k of the matrix A by A_k^R. By $\sigma(A,B)$ we denote the set of generalized eigenvalues corresponding to the generalized eigenvalue problem (1.6). A special case of Green's functions in both discrete and continuous cases are m-functions. In the discrete case m-functions corresponding to (1.6) are defined as $m_i(\lambda)=G(i,i,\lambda)$. In [5] we proved that for a given basic m-function $G(1,1,\lambda)$ and a diagonal matrix B we can construct a unique tridiagonal positive definite matrix A such that $m_i(\lambda)$, $1\le i\le n$, are the corresponding m-functions for the pair (A,B), (inverse problem). There are some interesting spectral similarities in discrete and continuous Sturm Liouville problems in the following theorems that have been proved in [3].

Theorem 1.1. *Suppose B is a positive definite diagonal matrix and A is a tridiagonal matrix. Let $\sigma(A,B)=\{\lambda_k\}_1^n$ and let $\{x^{(k)}\}$ be the corresponding eigenvectors. Then*

$$G(i,j,\lambda)=\sum_{k=1}^{n}\frac{x_i^{(k)}x_j^{(k)}}{\lambda_k-\lambda} \tag{1.9}$$

where $x_i^{(k)}$ is the i^{th} entry of $x^{(k)}$.

Remark 1.2. This result is the similar result to the expansion of the Green's function in terms of eigenfunctions, i.e.,

$$G(t,\tau,\lambda)=\sum_{k=1}^{\infty}\frac{\chi_k(t)\chi_k(\tau)}{\lambda_k-\lambda}.$$

Theorem 1.3. *Under the assumptions of Theorem 1.1, let*

$$\sigma(A_{i-1},B_{i-1})=\{\lambda_m^L\}_1^{i-1},\ \sigma(A_{n-j}^R,B_{n-j}^R)=\{\lambda_m^R\}_1^{i-1}.$$

Then

$$\Phi_{ij}(\lambda)=\frac{\prod_{m=1}^{i-1}(1-\lambda/\lambda_m^L)\prod_{m=1}^{n-j}(1-\lambda/\lambda_m^R)}{\prod_{m=1}^{n}(1-\lambda/\lambda_m)}. \tag{1.10}$$

Theorem 1.4.

$$\frac{x_i^{(k)}x_j^{(k)}}{G(i,j,0)}=\frac{\lambda_k\prod_{m=1}^{i-1}(1-\lambda_k/\lambda_m^L)\prod_{m=1}^{n-j}(1-\lambda_k/\lambda_m^R)}{\prod_{m\neq k}^{n}(1-\lambda_k/\lambda_m)}. \tag{1.11}$$

Therefore, the Green's function has been constructed using a pair of spectra. There is a similar construction procedure to recover the Green's function in continuous Sturm-Liouville as follows; consider a pair of Sturm-Liouville problems with boundary conditions at left $\{\lambda_n^L\}_1^\infty$,

$$\begin{gathered} Ly(x)=\lambda\rho(x)y(x),\quad 0\le x\le\ell\\ p(0)y'(0)-hy(0)=0\\ y(a)=0,\quad a\in(0,\ell) \end{gathered} \tag{1.12}$$

and right $\{\lambda_n^R\}_1^\infty$,

$$\begin{gathered} Ly(x)=\lambda\rho(x)y(x),\quad 0\le x\le\ell\\ p(\ell)y'(\ell)+Hy(\ell)=0\\ y(b)=0,\quad b\in(0,\ell)\,. \end{gathered} \tag{1.13}$$

Let $\{\lambda_n\}_1^\infty$ be the eigenvalues of the original Sturm-Liouville problem

$$\begin{gathered} Ly(x)=\lambda\rho(x)y(x),\quad 0\le x\le\ell\\ p(0)y'(0)-hy(0)=0\\ p(\ell)y'(\ell)+Hy(\ell)=0\,. \end{gathered} \tag{1.14}$$

Then we have the following analogous theorems [3].

Theorem 1.5.

$$\Phi_{ij}(\lambda)=\frac{\prod_{n=1}^{\infty}(1-\lambda/\lambda_n^L)\prod_{n=1}^{\infty}(1-\lambda/\lambda_n^R)}{\prod_{n=1}^{\infty}(1-\lambda/\lambda_n)}. \tag{1.15}$$

Theorem 1.6. *Let $\{u_r(x)\}_1^\infty$ be the corresponding eigenfunctions to the original Sturm-Liouville problem, then*
$$\frac{u_r(a)u_r(b)}{G(a,b,0)} = \frac{\lambda_k \prod_{m=1}^\infty (1-\lambda_k/\lambda_m^L)\prod_{m=1}^\infty(1-\lambda_k/\lambda_m^R)}{\prod_{m\neq k}^\infty (1-\lambda_k/\lambda_m)}. \tag{1.16}$$

2. Weyl m-functions

Another similar property of discrete and continuous Sturm-Liouville problems is the uniqueness of the solution of inverse problem in both cases by given weyl m-function. Weyl m-function for discrete case is defined by $G(1,1,\lambda)$. If we have two given spectra $\sigma(A,B) = \{\lambda_i\}_1^N$, and $\sigma(A_{[2,N]}, B_{[2,N]}) = \{\mu_i\}_1^{N-1}$, where $A_{[2,N]}, B_{[2,N]}$ are submatrices obtained from A and B, respectively, by deleting the first row and first column of A and B with interlacing property
$$0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \mu_{N-1} < \lambda_N,$$
then it is well known that we can construct the first entries of eigenvectors corresponding to $\{\lambda_i\}_1^N$ by Lanczos algorithm. In this case if B is a nonsingular diagonal matrix we can construct the m-function of the pair (A,B) as follows.

Theorem 2.1. *The Weyl function $m(\lambda)$ has the following representations in the discrete case*
$$m(\lambda) = \sum_{k=1}^n \frac{[x_1^{(k)}]^2}{\lambda_k - \lambda}, \tag{2.1}$$
$$m(\lambda) = \frac{1}{b_1}\prod_{j=1}^{N-1}(\mu_j - \lambda)\prod_{j=1}^N (\lambda_j - \lambda)^{-1}, \tag{2.2}$$
where
$$[x_1^{(k)}]^2 = \frac{1}{b_1}\prod_{j=1}^{N-1}(\mu_j - \lambda_k)\prod_{j\neq k}^N (\lambda_j - \lambda_k)^{-1}. \tag{2.3}$$

Proof. The first part is an immediate consequence of Theorem 1.1. For the second part consider the equation
$$(A - \lambda B)y = e_1 \tag{2.4}$$
If $\lambda \neq \lambda_i$, then
$$y_1 = e_1^T(A-\lambda B)^{-1}e_1 = m(\lambda). \tag{2.5}$$
By applying Cramer's rule to (2.5) we find
$$y_1 = \frac{\det(A_1 - \lambda B_1)}{\det(A - \lambda B)} = \frac{P_{N-1}(\lambda)}{P_N(\lambda)}.$$
Since $\{\mu_i\}_1^{N-1}$ and $\{\lambda_i\}_1^N$ are zeros of $P_{N-1}(\lambda)$ and $P_N(\lambda)$, respectively, we obtain
$$\begin{aligned} m(\lambda) = y_1 &= \frac{\det(B_1)}{\det(B)}\prod_{j=1}^{N-1}(\mu_j - \lambda)\prod_{j=1}^N(\lambda_j - \lambda)^{-1} \\ &= \frac{1}{b_1}\prod_{j=1}^{N-1}(\mu_j-\lambda)\prod_{j=1}^N(\lambda_j-\lambda)^{-1}. \end{aligned} \tag{2.6}$$
Multiplication of the two representations of $m(\lambda)$ by $\lambda_k - \lambda$ and taking the limit as $\lambda \to \lambda_k$ completes the proof. □

Theorem 2.2. *For a given m-function there is a unique positive definite Jacobi matrix with the prescribed m-function,* [5].

In the continuous case, let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be two solutions of Sturm-Liouville equation (1) with the boundary conditions $\varphi(0,\lambda) = 1,\ p(0)\varphi'(0,\lambda) = h$ and $\psi(\ell,\lambda) = 1,\ p(\ell)\psi'(\ell,\lambda) = -H$, respectively. Suppose $\Delta(\lambda)$ be the Wronskian of $\varphi,\ \psi$, that is

$$\Delta(\lambda) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda).$$

It is well known [4] that $\Delta(\lambda)$ is an analytic function in λ whose zeros are the eigenvalues of Sturm-Liouville problem (1). In this case the Weyl m-function is defined by

$$m(\lambda) = -\frac{\varphi(0,\lambda)}{\Delta(\lambda)}.$$

In the continuous case the following results are well known, for details see [4].

Theorem 2.3. *The m-function has the representation*

$$m(\lambda) = \sum_{k=1}^{\infty} \frac{1}{\alpha_k(\lambda_k - \lambda)}, \tag{2.7}$$

where $\frac{1}{\alpha_k} = Res_{\lambda=\lambda_k} m(\lambda)$.

Theorem 2.4. *The function $m(\lambda)$ uniquely determines the Sturm-Liouville operator.*

3. Eigenvalues of Hermitian matrices

For a general matrix $A \in M_n$ there is no way to characterize the eigenvalues of A except that they are the roots of the characteristic polynomial of A. For Hermitian matrices, however, the eigenvalues can be characterized as the solutions of a sequence of optimization problems. In this regard we have the well known Theorem of Rayleigh-Ritz, see [6] for more details. Now we want to prove a similar theorem by using the concept of the Green's function. That is to characterize the eigenvalues and corresponding orthonormal basis of eigenvectors for a Hermitian matrix. It is well known that every Hermitian matrix can be reduced to a tridiagonal matrix with the same eigenvalues. Some of the well-known methods in this regards are Householder, Givens and Lanczos transformations, [6]. For this reason we adopt the following method for a general Hermitian matrix instead of a Jacobi matix. Let A and B be two given Hermitian matrices. As we defined the Green's function by (1.7); i.e.,

$$G(i,j,\lambda) = (A - \lambda B)^{-1}(i,j) = e_i^T (A - \lambda B)^{-1} e_j.$$

Now consider the nonhomogeneous system

$$Ax = \lambda Bx + f, \quad f \in \mathbb{C}^n. \tag{3.1}$$

If $\lambda \notin \sigma(A,B)$, then the unique solution of this system is

$$x = (A - \lambda B)^{-1} f\,;$$

that is,

$$x_i = \sum_{j=1}^{n} G(i,j,\lambda) f_j, \quad i = 1,\dots,n. \tag{3.2}$$

For simplicity we assume $\lambda = 0$ is not an eigenvalue and let $G(i,j) = G(i,j,0)$. Note that the same algorithm can be applied for any $\lambda \notin \sigma(A,B)$. Thus we can define a linear operator $\mathbb{G}$ on $\mathbb{C}^n$ as follows:

$$\mathbb{G} f_i = \sum_{j=1}^{n} G(i,j) f_j, \quad i = 1, \dots, n. \tag{3.3}$$

Clearly in this case indeed $\mathbb{G} = A^{-1}$. Therefore $\mathbb{G}$ is selfadjoint; i.e.,

$$\langle f, \mathbb{G} g \rangle = \langle \mathbb{G} f, g \rangle, \quad \forall f, g \in \mathbb{C}^n. \tag{3.4}$$

First we prove the following results then we use them to characterize the eigenvalues.

Lemma 3.1. *The norm of $\mathbb{G}$ satisfies*

$$\|\mathbb{G}\| = max|\langle \mathbb{G} u, u\rangle|, \quad u \in \mathbb{C}^n, \ \|u\| = 1, \tag{3.5}$$

where $\langle u, u\rangle = u^* u$ is the standard inner product on $\mathbb{C}^n$.

Proof. By (3.4) $\langle \mathbb{G} u, u\rangle$ is real. If $\|u\| = 1$, then it follows from the properties of inner product that

$$|\langle \mathbb{G} u, u\rangle| \le \|\mathbb{G} u\| \|u\| \le \|\mathbb{G}\|,$$

and hence $\eta = \max|\langle \mathbb{G} u, u\rangle| \le \|\mathbb{G}\|$. In order to prove the inverse inequality, we have

$$\langle \mathbb{G}(u+v), u+v\rangle = \langle \mathbb{G} u, u\rangle + \langle \mathbb{G} v, v\rangle + 2\Re\langle \mathbb{G} u, v\rangle \le \eta \|u+v\|^2$$

and similarly,

$$\langle \mathbb{G}(u-v), u-v\rangle = \langle \mathbb{G} u, u\rangle + \langle \mathbb{G} v, v\rangle - 2\Re\langle \mathbb{G} u, v\rangle \ge -\eta \|u-v\|^2$$

Subtracting these equations we obtain

$$4\Re\langle \mathbb{G} u, v\rangle \le \ 2\eta(\|u\|^2 + \|v\|^2). \tag{3.6}$$

Note that $\mathbb{G} u \neq 0$ for $u \neq 0$ since $\mathbb{G}$ is invertible. Put $v = \mathbb{G} u / \|\mathbb{G} u\|$ in (3.6) to obtain $\|\mathbb{G} u\| \le \eta$ which completes the proof. $\square$

Theorem 3.2. *Either $\|\mathbb{G}\|$ or $-\|\mathbb{G}\|$ is an eigenvalue of $\mathbb{G}$.*

Proof. Using Lemma 3.1, suppose $\|\mathbb{G}\| = \max\langle \mathbb{G} u, u\rangle$ for $\|u\| = 1, \ u \in \ \mathbb{C}^n$. Since $u \longmapsto \langle \mathbb{G} u, u\rangle$ is continuous on the compact sphere $\|u\| = 1$, thus it attains its maximum on the sphere, i.e., there exists an x_0 with $\|x_0\| = 1$ such that $\max\langle \mathbb{G} u, u\rangle = \langle \mathbb{G} x_0, x_0\rangle$, for $\|u\| = 1$. Hence there is a sequence $\{u_m\}$ in $\mathbb{C}^n$ such that $\|u_m\| = 1$ and $\langle \mathbb{G} u_m, u_m\rangle \to \|\mathbb{G}\|$, as $m \to \infty$. Therefore, using Ascoli theorem there is a subsequence of $\{\mathbb{G} u_m\}$, call it $\{\mathbb{G} u_m\}$ also, which is convergent to a vector v_0 in $\mathbb{C}^n$; that is,

$$\|\mathbb{G} u_m - v_0\| \to 0, \quad (m \to \infty). \tag{3.7}$$

Let $\mu_0 = \|\mathbb{G}\|$. Now we prove that v_0 is eigenvector of $\mathbb{G}$ corresponding to eigenvalue of μ_0. Clearly $\|\mathbb{G} u_m\| \to \|v_0\|$ and we have

$$\|\mathbb{G} u_m - \mu_0 u_m\|^2 = \|\mathbb{G} u_m\|^2 + \mu_0^2 \|u_m\|^2 - 2\mu_0 \langle \mathbb{G} u_m, u_m\rangle \tag{3.8}$$

and the right side tends to $\|v_0\|^2 - \mu_0^2$. It follows that $\|v_0\|^2 \ge \mu_0^2 > 0$, hence v_0 is not zero. From (3.8) it follows that since $\|\mathbb{G} u_m\|^2 \le \mu_0^2$,

$$0 \le \|\mathbb{G} u_m - \mu_0 u_m\|^2 \le 2\mu_0^2 - 2\mu_0 \langle \mathbb{G} u_m, u_m\rangle$$

which tends to zero as $m \to \infty$. Thus

$$\|\mathbb{G}u_m - \mu_0 u_m\| \to 0 \tag{3.9}$$

On the other hand by using the Triangle inequality we have

$$0 \le \|\mathbb{G}v_0 - \mu_0 v_0\| \le \|\mathbb{G}v_0 - \mathbb{G}(\mathbb{G}u_m)\| + \|\mathbb{G}(\mathbb{G}u_m) - \mu_0\mathbb{G}u_m\| + \|\mu_0\mathbb{G}u_m - \mu_0 v_0\|$$

Combining (3.7), (3.9), and the inequality $\|\mathbb{G}u\| \le \|\mathbb{G}\|\|u\|$, the last inequality shows that $\|\mathbb{G}v_0 - \mu_0 v_0\| = 0$, which proves that $\mathbb{G}v_0 = \mu_0 v_0$. If the other case of Lemma 3.1 holds; i.e., $-\|\mathbb{G}\| = \min\langle\mathbb{G}u, u\rangle$, the proof is similar. □

Conclusion. In both discrete and continuous Sturm-Liouville problems Green's functions and Weyl m-functions have similar expressions in terms of spectral data. In both cases Weyl m-function uniquely determines the Sturm-Liouville operators. Rayleigh-Ritz Theorem may be proved by using the concept of Green's function in discrete case to characterize the eigenvalues and eigenvectors of a given Hermitian matrix as follows. Let $\mu_0 = \|\mathbb{G}\|$ and let v_0 be the corresponding eigenvector. Define $\chi_0 = v_0/\|v_0\|$. The eigenvector χ_0 is said to be *normalized*. Let

$$G_1(i,j) = G(i,j) - \mu_0\chi_0(i)\bar{\chi}_0(j), \tag{3.10}$$

and similar to $\mathbb{G}$ the operator $\mathbb{G}_1$ is defined by

$$\mathbb{G}_1 u_i = \sum_{j=1}^{n} G_1(i,j)u_j, \quad i = 1,\dots,n. \tag{3.11}$$

Then $\mathbb{G}_1$ has the same properties as $\mathbb{G}$, in particular $\mu_1 = \|\mathbb{G}_1\|$ is an eigenvalue of $\mathbb{G}_1$ and there is a corresponding eigenfunction φ_1. Let $\chi_1 = v_1/\|v_1\|$. We can easily verify that $\langle\mathbb{G}_1 u, \chi_0\rangle = 0$ for all $u \in \mathbb{C}^n$. This follows that χ_1 is orthogonal to χ_0. Therefore,

$$\mathbb{G}\chi_1 = \mathbb{G}_1\chi_1 = \mu_1\chi_1. \tag{3.12}$$

Moreover, we have

$$|\mu_0| \ge |\langle\mathbb{G}\chi_1, \chi_1\rangle| = |\mu_1|\|\chi_1\|^2 = |\mu_1|. \tag{3.13}$$

Letting

$$G_2(i,j) = G_1(i,j) - \mu_1\chi_1(i)\bar{\chi}_1(j), \tag{3.14}$$

and proceeding as above, the existence of χ_2 and μ_2 is established with $|\mu_2| \le |\mu_1|$, and χ_2 orthogonal to χ_1 and χ_0. In this way establish an orthonormal basis of eigenvectors and all eigenvalues of A^{-1}, hence the eigenvalues of A.

Acknowledgments. The author would like to thank the anonymous referee for careful reading of the original manuscript and for his/her valuable comments. The author also thanks the Sahand University of Technology for the financial support.

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