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A REMARK ON GROUND STATE SOLUTIONS FOR LANE-EMDEN-FOWLER EQUATIONS WITH A CONVECTION TERM

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ABSTRACT. Via a sub-supersolution method and a perturbation argument, we study the Lane-Emden-Fowler equation

$$-\Delta u = p(x)[g(u) + f(u) + |\nabla u|^q]$$

in $\mathbb{R}^N$ ($N \geq 3$), where $0 < q < 1$, p is a positive weight such that $\int_0^\infty r\varphi(r)dr < \infty$, where $\varphi(r) = \max_{|x|=r} p(x)$, $r \geq 0$. Under the hypotheses that both g and f are sublinear, which include no monotonicity on the functions $g(u)$, $f(u)$, $g(u)/u$ and $f(u)/u$, we show the existence of ground state solutions.

1. Introduction

This paper concerns the Lane-Emden-Fowler type problem

$$\begin{gathered}
-\Delta u = p(x)[g(u) + f(u) + |\nabla u|^q], \quad \text{in } \mathbb{R}^N,\\
u > 0, \quad \text{in } \mathbb{R}^N,\\
u(x) \to 0, \quad \text{as } |x| \to \infty,
\end{gathered} \tag{1.1}$$

where $N \geq 3$, $0 < q < 1$, and $p : \mathbb{R}^N \to (0, +\infty)$ is a locally Hölder continuous function of exponent $0 < \alpha < 1$ satisfying

$$\int_0^\infty r\varphi(r)dr < \infty, \tag{1.2}$$

where $\varphi(r) = \max_{|x|=r} p(x)$, $r \geq 0$. We also assume that g satisfies

- (G1) $g \in C^1((0,\infty),(0,\infty))$;
- (G2) $\lim_{u\to 0^+} \frac{g(u)}{u} = +\infty$;
- (G3) $\lim_{u\to\infty} \frac{g(u)}{u} = 0$,

and $f : [0,\infty) \to [0,\infty)$ is a locally Hölder continuous function of exponent $0 < \alpha < 1$ satisfying

- (F1) $\lim_{u\to 0^+} \frac{f(u)}{u} = +\infty$;
- (F2) $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.

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The Lane-Emden-Fowler equation arises in the study of gaseous dynamics in astrophysics, fluid mechanics, relativistic mechanics, nuclear physics and chemical reaction systems. By far, it has been studied by many authors using various methods. But we note that in most works, monotonicity is necessary to some extent.

With regard to semilinear elliptic problems in bounded domains, we refer for details to [3, 4, 11, 14, 15, 21] and their references. Here, we mention the works of Ghergu and Rădulescu [7], Zhang [18, 20], where the influence of the convection term has been emphasized.

Concerning with ground state solutions for elliptic problems, that is, positive solutions defined in the whole space and decaying to zero at infinity, we refer the reader to the works of Cîrstea and Rădulescu [2], Dinu [5, 6], Goncalves and Santos [10], Sun and Li [16], Ye and Zhou [17], Zhang [19]. We mention here the work of Zhang [22]. In [22], it showed that if g satisfies (G1)-(G3) and condition (1.2), then the following boundary value problem

$$\begin{gathered} -\Delta u = p(x)g(u), \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.3}$$

has at least one solution $u \in C^{2+\alpha}_{\rm loc}(\mathbb{R}^N)$. For problem (1.3), we see that no monotonicity conditions are imposed on the functions $g(u)$ and $g(u)/u$. On the other hand, condition (1.2) is necessary to prove the existence (see also Lair and Shaker [13]).

Recently, in Ghergu and Rădulescu [8], the same problem (1.1) is considered, where $g \in C^1(0,\infty)$ is a positive decreasing function such that

$$\lim_{u\to 0^+} g(u) = +\infty,$$

and $f : [0,\infty) \to [0,\infty)$ is a Hölder continuous function of exponent $0 < \alpha < 1$ which is non-decreasing such that $f > 0$ on $(0,\infty)$ and satisfies (F1)-(F2) and

(F3) the mapping $(0,\infty) \ni u \longmapsto \frac{f(u)}{u}$ is non-increasing.

Finally, they showed that in addition to condition (1.2) , if the above assumptions are fulfilled, then problem (1.1) has at least one ground state solution.

In the present paper, we consider the existence of ground state solutions for problem (1.1) under more general conditions. Our main result is summarized in the following theorem.

Theorem 1.1. *Assume (G1)–(G3) and (F1)–(F2). Then problem* (1.1) *has at least one solution provided that condition* (1.2) *is fulfilled.*

Remark 1.2. Some basic examples of the function g satisfying (G1)–(G3) are:

(i) $u^{-\gamma} + u^p + \sin\psi(u) + 1$, where $\gamma > 0$, $p < 1$ and $\psi \in C^2(\mathbb{R})$;
(ii) $e^{1/u^\gamma} + u^p + \cos\psi(u) + 1$, where $\gamma > 0$, $p < 1$ and $\psi \in C^2(\mathbb{R})$;
(iii) $u^{-\gamma}\ln^{-q_1}(1+u) + \ln^{q_2}(1+u) + u^p + \sin\psi(u) + 2$ with $\psi \in C^2(\mathbb{R})$, $\gamma > 0$, $p < 1$, $q_2 > 0$ and $q_1 > 0$;
(iv) $u^{-\gamma} + \arctan\psi(u) + \pi$ with $\psi \in C^2(\mathbb{R})$ and $\gamma > 0$.

Remark 1.3. Some basic examples of the function f satisfying (F1)–(F2) are:

(i) $c_1(1+u)^{-\alpha} + c_2u^\gamma + c_3$, where $c_1, c_2, c_3 \ge 0$, $\alpha > 0$, $0 < \gamma < 1$;
(ii) $e^{1/(u+1)} + u^\gamma + \sin\psi(u) + 1$, where $0 < \gamma < 1$ and $\psi \in C^2(\mathbb{R})$;

(iii) $\ln^q(u+1) + (1+u)^{-\alpha}$, where $q > 0$, $\alpha > 0$;
(iv) $u^p \ln^q(u+1)$, where $p, q \in (0,1)$ and $p + q < 1$.

2. Proof of Theorem 1.1

In this section, we first show the existence of positive solutions for problem (1.1) in smooth bounded domains by a sub-supersolution method. Then, via the perturbation argument, we prove Theorem 1.1. First we recall the following auxiliary results.

Lemma 2.1 ([4, Lemma 3]). *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Suppose the boundary-value problem*

$$-\Delta u = p(x)[g(u) + f(u) + |\nabla u|^q], \quad u > 0,\ x \in \Omega,\ u|_{\partial\Omega} = 0, \tag{2.1}$$

has a super-solution $\bar{u}$ and a sub-solution $\underline{u}$ such that $\underline{u} \le \bar{u}$ in Ω, then problem (2.1) *has at least one solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ in the ordered interval $[\underline{u}, \bar{u}]$.*

Lemma 2.2 ([22, Lemma 2.3]). *Suppose (G1)–(G3) are satisfied. Then there exists a function $\overline{g}_0$ such that*

(i) $\overline{g}_0 \in C^1((0,\infty),(0,\infty))$;
(ii) $\frac{g(s)}{s} \le \overline{g}_0, \forall\ s > 0$;
(iii) $\overline{g}_0(s)$ *is non-increasing on* $(0,\infty)$;
(iv) $\lim_{s\to 0+} \overline{g}_0(s) = \infty$ *and* $\lim_{s\to\infty} \overline{g}_0(s) = 0$.

Note that if $g \in C((0,\infty),(0,\infty))$, the function $\overline{g}_0$ still exists in Lemma 2.2.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Assume (G1)-(G3) and (F1)-(F2) are fulfilled. Then problem* (2.1) *has at least one solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$.*

Proof. Let $\underline{u}$ be a solution of

$$-\Delta \underline{u} = p(x)g(\underline{u}), \quad \underline{u} > 0,\ x \in \Omega, \quad \underline{u}|_{\partial\Omega} = 0. \tag{2.2}$$

The existence of $\underline{u}$ follows from the results in Zhang [22]. Obviously, $\underline{u}$ is a sub-solution of (2.1). The main point is to find a super-solution $\overline{u}$ of (2.1) such that $\underline{u} \le \overline{u}$ in Ω. Then, by Lemma 2.1 we deduce that problem (2.1) has at least one solution.

Denote $\sigma(u) := g(u) + f(u)$. Then σ satisfies

- $\sigma \in C((0,\infty),(0,\infty))$;
- $\lim_{u\to 0^+} \frac{\sigma(u)}{u} = +\infty$;
- $\lim_{u\to\infty} \frac{\sigma(u)}{u} = 0$.

By Lemma 2.2, corresponding to σ, there exists a function $\overline{\sigma}_0$ satisfying

(i) $\overline{\sigma}_0 \in C^1((0,\infty),(0,\infty))$;
(ii) $\frac{\sigma(u)}{u} \le \overline{\sigma}_0$, for all $u > 0$;
(iii) $\overline{\sigma}_0(u)$ is non-increasing on $(0,\infty)$;
(iv) $\lim_{u\to 0+} \overline{\sigma}_0(u) = +\infty$ and $\lim_{u\to\infty} \overline{\sigma}_0(u) = 0$,

such that $G(u) := u(\overline{\sigma}_0(u) + \frac{1}{u})$ satisfying

(G1) $G \in C^1((0,\infty),(0,\infty))$;
(G2) $\frac{G(u)}{u}$ is decreasing on $(0,\infty)$;
(G3) $\lim_{u\to 0^+} \frac{G(u)}{u} = \infty$;

(G4) $\lim_{u\to\infty} \frac{G(u)}{u} = 0$.

Then, we consider the problem

$$-\Delta u = p(x)[G(u) + |\nabla u|^q], \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.3}$$

We claim that this problem has at least one classical solution, which is a super-solution of (2.1). Indeed, let $h : [0, \eta] \to [0, \infty)$ be such that

$$\begin{gathered} -h''(t) = \frac{G(h(t))}{h(t)}, \quad 0 < t < \eta < 1, \\ h > 0, \quad 0 < t \le \eta < 1, \\ h(0) = 0. \end{gathered} \tag{2.4}$$

The existence of h follows from the results in Agarwal and O'Regan [1, Theorem 2.1]. Since h is concave, there exists $h'(0+) \in (0, \infty]$, namely, h is increasing on $[0, \eta]$ for $\eta > 0$ small enough. Multiplying by $h'(t)$ in (2.4) and integrating on $[t, \eta]$, we combine (G2) and get

$$(h')^2(t) \le 2h(\eta)\frac{G(h(t))}{h(t)} + (h')^2(\eta), \quad 0 < t < \eta.$$

Since $s^q \le s^2 + 1$, for all $s \ge 0$. Combining the above inequality we have

$$(h')^q(t) \le C\frac{G(h(t))}{h(t)}, \tag{2.5}$$

for all $0 < t < \eta < 1$ and some $C > 0$.

Let ϕ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta$ in $H_0^1(\Omega)$. By Höpf's maximum principle, there exist $\delta > 0$ and $\omega \Subset \Omega$ such that

$$|\nabla\phi_1| > \delta, \quad \text{in } \Omega \setminus \omega. \tag{2.6}$$

For the rest of this paper we denote

$$\begin{gathered} |\phi_1|_\infty := \max_{x\in\bar\Omega} \phi_1(x), \quad |\phi_1|_0 := \min_{x\in\bar\omega} \phi_1(x), \\ |p|_\infty := \max_{x\in\bar\Omega} p(x), \quad |\nabla\phi_1|_\infty := \max_{x\in\bar\Omega} |\nabla\phi_1(x)|. \end{gathered}$$

And we fix $c > 0$ such that $c|\phi_1|_\infty < \eta$.

Using the monotonicity of h and h', it follows that

$$0 < h(c|\phi_1|_0) \le h(c\phi_1) \le h(\eta), \quad \text{in } \omega, \tag{2.7}$$

$$0 < h'(\eta) \le h'(c\phi_1) \le h'(c|\phi_1|_0), \quad \text{in } \omega. \tag{2.8}$$

Let $M > 1$ be such that

$$\lambda_1(Mch'(\eta))^{1-q}|\phi|_0 > 2|p|_\infty|\nabla\phi_1|_\infty^q, \tag{2.9}$$

$$M^{1-q}C^{-1}(c\delta)^{2-q} > 2|p|_\infty. \tag{2.10}$$

By (G4), we can choose $M > 1$ large enough such that

$$\frac{G(Mh(c|\phi_1|_0))}{Mh(c|\phi_1|_0)} \le \frac{\lambda_1 c|\phi_1|_0 h'(\eta)}{2|p|_\infty h(\eta)}. \tag{2.11}$$

Next, we show that $\overline{u}_0 = Mh(c\phi_1)$ is a super-solution of (2.3) provided that M satisfies (2.9)-(2.11). We have

$$-\Delta\overline{u}_0 = \lambda_1 Mc\phi_1 h'(c\phi_1) + Mc^2|\nabla\phi_1|^2\frac{G(h(c\phi_1))}{h(c\phi_1)}.$$

By (G2), (2.7)-(2.8) and (2.11), we have

$$\frac{G(Mh(c\phi_1))}{Mh(c\phi_1)} \le \frac{G(Mh(c|\phi_1|_0))}{Mh(c|\phi_1|_0)} \le \frac{\lambda_1 c|\phi_1|_0 h'(\eta)}{2|p|_\infty h(\eta)} \le \frac{\lambda_1 c\phi_1 h'(c\phi_1)}{2p(x)h(c\phi_1)}, \quad \text{in } \omega.$$

It follows that

$$\lambda_1 Mc\phi_1 h'(c\phi_1) \ge 2p(x)G(Mh(c\phi_1)), \quad \text{in } \omega. \tag{2.12}$$

From (2.8)-(2.9), we have

$$\lambda_1 Mc\phi_1 h'(c\phi_1) \ge 2p(x)|Mch'(c\phi_1)\nabla\phi_1|^q = 2p(x)|\nabla\overline{u}_0|^q, \text{ in } \quad \omega. \tag{2.13}$$

Since $h(0) = 0$, we get

$$\lim_{x\to\partial\Omega}\Big(\frac{(c\delta)^2}{h(c\phi_1)} - 2|p|_\infty\Big) = +\infty,$$

namely,

$$\frac{(c\delta)^2}{h(c\phi_1)} > 2|p|_\infty > 2p(x), \quad \text{in } \Omega\setminus\omega. \tag{2.14}$$

From (2.6), (G2) and (2.14), we have

$$Mc^2|\nabla\phi_1|^2\frac{G(h(c\phi_1))}{h(c\phi_1)} \ge Mc^2\delta^2\frac{G(Mh(c\phi_1))}{Mh(c\phi_1)} \ge 2p(x)G(Mh(c\phi_1)), \quad \text{in } \Omega\setminus\omega. \tag{2.15}$$

From (2.5)-(2.6) and (2.10), we have

$$Mc^2|\nabla\phi_1|^2\frac{G(h(c\phi_1))}{h(c\phi_1)} \ge 2p(x)|Mch'(c\phi_1)\nabla\phi_1|^q = 2p(x)|\nabla\bar{u}_0|^q, \quad \text{in } \Omega\setminus\omega. \tag{2.16}$$

From (2.12)-(2.13) and (2.15)-(2.16), we deduce that

$$-\Delta\overline{u}_0 \ge p(x)[G(\overline{u}_0) + |\nabla\overline{u}_0|^q], \quad \text{in } \Omega,$$

namely, $\overline{u}_0 = Mh(c\phi_1)$ is a super-solution of (2.3).

On the other hand, the unique solution $\underline{u}_0$ of the boundary-value problem

$$-\Delta\underline{u}_0 = p(x)G(\underline{u}_0), \quad \underline{u}_0 > 0,\ x\in\Omega,\ \underline{u}_0|_{\partial\Omega} = 0,$$

is a sub-solution of problem (2.3). Here the existence of $\underline{u}_0$ follows from the results in Goncalves and Santos [10].

Next, we prove that

$$\underline{u}_0 \le \bar{u}_0 \quad \text{in } \Omega.$$

Assume the contrary; i.e., there exists $x_0\in\Omega$ such that $\bar{u}_0(x_0) < \underline{u}_0(x_0)$. Then, $\sup_{x\in\Omega}(\ln(\underline{u}_0(x)) - \ln(\bar{u}_0(x)))$ exists and is positive in Ω. At the point, we have

$$\nabla(\ln(\underline{u}_0(x_0)) - \ln(\bar{u}_0(x_0))) = 0,$$
$$\Delta(\ln(\underline{u}_0(x_0)) - \ln(\bar{u}_0(x_0))) \le 0.$$

By (G2), we see that

$$\begin{aligned}
&\Delta\left(\ln(\underline{u}_0(x_0)) - \ln(\bar{u}_0(x_0))\right)\\
&= \frac{\Delta\underline{u}_0(x_0)}{\underline{u}_0(x_0)} - \frac{\Delta\bar{u}_0(x_0)}{\bar{u}_0(x_0)} - \frac{|\nabla\underline{u}_0(x_0)|^2}{(\underline{u}_0(x_0))^2} + \frac{|\nabla\bar{u}_0(x_0)|^2}{(\bar{u}_0(x_0))^2}\\
&= \frac{\Delta\underline{u}_0(x_0)}{\underline{u}_0(x_0)} - \frac{\Delta\bar{u}_0(x_0)}{\bar{u}_0(x_0)}\\
&\geq p(x_0)\Big(\big[\frac{G(\bar{u}_0(x_0))}{\bar{u}_0(x_0)} - \frac{G(\underline{u}_0(x_0))}{\underline{u}_0(x_0)}\big] + \frac{|\nabla\bar{u}_0(x_0)|^q}{\bar{u}_0(x_0)}\Big) > 0,
\end{aligned}$$

which is a contradiction. Therefore, $\overline{u}_0 \geq \underline{u}_0$ in Ω. Then by Lemma 2.1, problem (2.3) has at least one classical solution denoted by $\bar{u}$, which is a super-solution of problem (2.1).

Finally, we show that $\underline{u} \leq \bar{u}$ in Ω. Assume the contrary, i.e., there exists $x_1 \in \Omega$ such that $\bar{u}(x_1) < \underline{u}(x_1)$. Then, $\sup_{x\in\Omega}\left(\ln(\underline{u}(x)) - \ln(\bar{u}(x))\right)$ exists and is positive in Ω. At the point, we have

$$\nabla\left(\ln(\underline{u}(x_1)) - \ln(\bar{u}(x_1))\right) = 0 \quad \text{and} \quad \Delta\left(\ln(\underline{u}(x_1)) - \ln(\bar{u}(x_1))\right) \leq 0.$$

Since $\overline{\sigma}_0$ is non-increasing on $(0,\infty)$, we have

$$\overline{\sigma}_0(\bar{u}(x_1)) \geq \overline{\sigma}_0(\underline{u}(x_1)) \geq \frac{g(\underline{u}(x_1)) + f(\underline{u}(x_1))}{\underline{u}(x_1)}.$$

Then we obtain

$$\begin{aligned}
&\Delta\left(\ln(\underline{u}(x_1)) - \ln(\bar{u}(x_1))\right)\\
&= \frac{\Delta\underline{u}(x_1)}{\underline{u}(x_1)} - \frac{\Delta\bar{u}(x_1)}{\bar{u}(x_1)} - \frac{|\nabla\underline{u}(x_1)|^2}{(\underline{u}(x_1))^2} + \frac{|\nabla\bar{u}(x_1)|^2}{(\bar{u}(x_1))^2}\\
&= \frac{\Delta\underline{u}(x_1)}{\underline{u}(x_1)} - \frac{\Delta\bar{u}(x_1)}{\bar{u}(x_1)}\\
&= p(x_1)\Big(\big[\frac{G(\bar{u}(x_1))}{\bar{u}(x_1)} - \frac{g(\underline{u}(x_1))}{\underline{u}(x_1)}\big] + \frac{|\nabla\bar{u}(x_1)|^q}{\bar{u}(x_1)}\Big)\\
&= p(x_1)\Big(\big[\overline{\sigma}_0(\bar{u}(x_1)) - \frac{g(\underline{u}(x_1))}{\underline{u}(x_1)}\big] + \frac{1}{\bar{u}(x_1)} + \frac{|\nabla\bar{u}(x_1)|^q}{\bar{u}(x_1)}\Big) > 0,
\end{aligned}$$

which is a contradiction. Therefore, $\overline{u} \geq \underline{u}$ in Ω. By Lemma 2.1, the proof is complete. □

Lemma 2.4. *Assume condition* (1.2) *is fulfilled. Then there is a function w such that*

$$\begin{gathered}
-\Delta w \geq p(x)[g(w) + f(w) + |\nabla w|^q], \quad w > 0,\ x \in \mathbb{R}^N,\\
w(x) \to 0, \quad \text{as } |x| \to \infty.
\end{gathered} \tag{2.17}$$

Proof. Denote

$$\Psi(r) := r^{1-N}\int_0^r t^{N-1}\varphi(t)dt, \quad \forall r > 0.$$

By condition (1.2) and the L'Hôpital's rule, we have $\lim_{r\to 0}\Psi(r) = \lim_{r\to\infty}\Psi(r) = 0$. Thus, Ψ is bounded on $(0,\infty)$ and it can be extended in the origin by taking

$\Psi(0)=0$. On the other hand, by integration by parts and the L'Hôpital's rule (see details in [8]) , we get

$$\int_0^\infty \Psi(r)dr = \lim_{r\to\infty}\int_0^r \Psi(t)dt = \frac{1}{N-2}\int_0^\infty r\varphi(r)dr < \infty.$$

Let $\mu > 2$ be such that

$$\mu^{1-q} \ge 2\max_{r\ge 0}\Psi^q(r). \tag{2.18}$$

Define

$$\rho(x) := \mu\int_{|x|}^\infty \Psi(t)dt, \quad \text{for } x\in\mathbb{R}^N.$$

Then ρ is bounded and satisfies

$$\begin{gathered} -\Delta\rho = \mu\varphi(|x|), \quad \rho>0,\ x\in\mathbb{R}^N,\\ \rho(x)\to 0, \quad \text{as } |x|\to\infty. \end{gathered}$$

We claim that there are $R>0$ and a function $w\in C^2(\mathbb{R}^N)$ such that

$$\rho(x) = \frac{1}{R}\int_0^{w(x)} \frac{t}{G(t)+1}dt. \tag{2.19}$$

Indeed, since

$$\lim_{r\to+\infty}\frac{\int_0^r \frac{t}{G(t)+1}dt}{r} = \lim_{r\to+\infty}\frac{r}{G(r)+1} = +\infty,$$

we notice first for some $R>0$

$$|\rho|_\infty \le \frac{1}{R}\int_0^R \frac{t}{G(t)+1}dt,$$

and in particular

$$w(x)\le R,\ x\in\mathbb{R}^N. \tag{2.20}$$

From (2.19) we have

$$|\nabla w| = R\frac{G(w)+1}{w}|\nabla\rho| = \mu R\Psi(|x|)\frac{G(w)+1}{w},$$

and combining with (G2), we get

$$\frac{1}{R}\frac{w}{G(w)+1}\Delta w + \frac{1}{R}\frac{d}{dw}\Big(\frac{w}{G(w)+1}\Big)|\nabla w|^2 = \Delta\rho,$$

i.e.,

$$-\Delta w \ge \mu R\varphi(|x|)\frac{G(w)+1}{w}.$$

Then from (2.18), (2.20) and (G2), we obtain

$$\begin{aligned} -\Delta w &\ge \mu R\varphi(|x|)\frac{G(w)+1}{w}\\ &\ge Rp(x)\frac{G(w)+1}{w} + \frac{\mu}{2}Rp(x)\frac{G(w)+1}{w}\\ &\ge p(x)(G(w)+1) + p(x)|\mu R\frac{G(w)+1}{w}\Psi(|x|)|^q\\ &= p(x)(G(w)+1) + p(x)|\nabla w|^q. \end{aligned}$$

Hence,

$$-\Delta w \geq p(x)[G(w)+1+|\nabla w|^q],\ w>0,\ x\in\mathbb{R}^N,$$
$$w(x)\to 0,\ \text{as}\quad |x|\to\infty.$$

Since $G(w) > g(w)+f(w)$ on $(0,+\infty)$, it follows that w satisfies (2.17). The proof is complete. □

Proof of Theorem 1.1. Consider the perturbed problem

$$-\Delta u_n = p(x)[g(u_n)+f(u_n)+|\nabla u_n|^q],\quad u_n>0,\ x\in B_n,\ u_n|_{\partial B_n}=0, \tag{2.21}$$

where $B_n := \{x\in\mathbb{R}^N; |x|<n\}$, $n=1,2,3,\dots$. It follows by Lemma 2.3 that problem (2.21) has at least one solution $u_n\in C^2(B_n)\cap C(\bar{B_n})$. Put

$$u_n(x)=0,\quad \forall\ |x|>n.$$

Let w be as in Lemma 2.4, with the same proof above, we deduce that

$$u_n(x)\leq w(x),\ x\in\mathbb{R}^N,\ n=1,2,3,\dots. \tag{2.22}$$

Now, we need to estimate $\{u_n\}$. For any bounded $C^{2+\alpha}$-smooth domain $\Omega'\subset\mathbb{R}^N$, take Ω_1 and Ω_2 with $C^{2+\alpha}$-smooth boundaries, and K_1 large enough, such that

$$\Omega'\subset\subset\Omega_1\subset\subset\Omega_2\subset\subset B_n,\quad n\geq K_1.$$

Note that

$$u_n(x)\geq \underline{u}(x)>0,\quad \forall x\in B_{K_1}, \tag{2.23}$$

where B_{K_1} is the substitution for Ω in the proof of Lemma 2.3. Let

$$\Psi_n(x)=p(x)[g(u_n)+f(u_n)+|\nabla u_n|^q],\quad x\in\bar{B}_{K_1}.$$

Since $-\Delta u_n(x)=\Psi_n(x)$, $x\in B_{K_1}$, by the interior estimate theorem of Ladyzenskaja and Ural'tseva [12, Theorem 3.1, p. 266], we get a positive constant C_1 independent of n such that

$$\max_{x\in\bar{\Omega}_2}|\nabla u_n(x)|\leq C_1\max_{x\in\bar{B}_{K_1}}u_n(x)\leq C_1\max_{x\in\bar{B}_{K_1}}w(x),\quad \forall\ x\in B_{K_1},$$

i.e., $|\nabla u_n(x)|$ is uniformly bounded on $\bar{\Omega}_2$. It follows that $\{\Psi_n\}_{K_1}^\infty$ is uniformly bounded on $\bar{\Omega}_2$ and hence $\Psi_n\in L^p(\Omega_2)$ for any $p>1$. Since $-\Delta u_n(x)=\Psi_n(x)$, $x\in\Omega_2$, we see by [9, Theorem 9.11], that there exists a positive constant C_2 independent of n such that

$$\|u_n\|_{W^{2,p}(\Omega_1)}\leq C_2\left(\|\Psi_n\|_{L^p(\Omega_2)}+\|u_n\|_{L^p(\Omega_2)}\right),\quad \forall\ n\geq K_1.$$

Taking $p>N$ such that $\alpha<1-N/p$ and applying Sobolev's embedding inequality, we see that $\{\|u_n\|_{C^{1+\alpha}(\bar{\Omega}_1)}\}_{K_1}^\infty$ is uniformly bounded. Therefore $\Psi_n\in C^\alpha(\bar{\Omega}_1)$ and $\{\|\Psi_n\|_{C^\alpha(\bar{\Omega}_1)}\}_{K_1}^\infty$ is uniformly bounded. It follows by Schauder's interior estimate theorem (see [9, Chapter 1, p. 2]) that there exists a positive constant C_3 independent of n such that

$$\|u_n\|_{C^{2+\alpha}(\bar{\Omega}')}\leq C_3\left(\|\Psi_n\|_{C^\alpha(\bar{\Omega}_1)}+\|u_n\|_{C(\bar{\Omega}_1)}\right),\quad \forall\ n\geq K_1;$$

i.e., $\{\|u_n\|_{C^{2+\alpha}(\bar{\Omega}')}\}_{K_1}^\infty$ is uniformly bounded. Using Ascoli-Arzela's theorem and the diagonal sequential process, we see that $\{u_n\}_{K_1}^\infty$ has a subsequence that converges uniformly in the $C^2(\bar{\Omega}')$ norm to a function $u\in C^2(\bar{\Omega}')$ and u satisfies

$$-\Delta u=p(x)[g(u)+f(u)+|\nabla u|^q],\quad x\in\bar{\Omega}'.$$

By (2.23), we obtain that

$$u > 0, \quad \forall x \in \bar{\Omega}'.$$

Applying Schauder's regularity theorem we see that $u \in C^{2+\alpha}(\bar{\Omega}')$. Since Ω' is arbitrary, we also see that $u \in C^{2+\alpha}_{\rm loc}(\mathbb{R}^N)$. It follows by (2.22) that $\lim_{|x|\to\infty} u(x) = 0$. Thus, a standard bootstrap argument shows that u is a classical solution to problem (1.1). The proof is complete. $\square$

At last, it is worth pointing out that Ye and Zhou [17] proved that in many situations condition (1.2) can be replaced by the following more general condition

(P1) $-\Delta u = p(x)$ has a bounded ground state solution.

Obviously, condition (1.2) implies (P1) (see [17] for details about comparison between condition (1.2) and (P1)). Therefore, we have an unsolved problem as follows.

Remark 2.5. *We note that the existence of ground state solutions for problem* (1.1) *is left an open problem if p satisfies condition (P1) instead of* (1.2).

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