Electronic Journal of Differential Equations, Vol. 2007(2007), No. 56, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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BOUNDARY-VALUE PROBLEMS FOR SECOND-ORDER DIFFERENTIAL OPERATORS WITH NONLOCAL BOUNDARY CONDITIONS

MOHAMED DENCHE, ABDERRAHMANE MEZIANI

Abstract. In this paper, we study a second-order differential operator combining weighting integral boundary condition with another two-point boundary condition. Under certain conditions on the weighting functions, called regular and non regular cases, we prove that the resolvent decreases with respect to the spectral parameter in $L^p(0,1)$, but there is no maximal decrease at infinity for $p>1$. Furthermore, the studied operator generates in $L^p(0,1)$, an analytic semi group for $p=1$ in the regular case, and an analytic semi group with singularities for $p>1$, in both cases, and for $p=1$, in the non regular case only. The obtained results are then used to show the correct solvability of a mixed problem for parabolic partial differential equation with non regular boundary conditions.

1. Introduction

In space $L^p(0,1)$ we consider the boundary-value problem

$$
\begin{aligned}
L(u) &:= u'' = f(x),\\
B_i(u) &:= a_iu(0)+b_iu'(0)+c_iu(1)+d_iu'(1)\\
&\quad+\int_0^1 R_i(t)u(t)dt+\int_0^1 S_i(t)u'(t)dt=0,
\end{aligned}
\tag{1.1}
$$

where $i=\overline{1,2}$ and the functions R_i, S_i belong to $C([0,1],\mathbb{C})$. To problem (1.1) in $L^p(0,1)$ we associate the operator

$$L_p(u)=u'',$$

with domain $D(L_p)=\{u\in W^{2,p}(0,1): B_i(u)=0, i=\overline{1,2}\}$.

Many papers and books give the full spectral theory of Birkhoff regular differential operators with two point linearly independent boundary conditions, in terms of coefficients of boundary conditions. The reader should refer to [9, 13, 24, 25, 26, 33, 36, 37] and references therein. Few works were devoted to the study of a non regular situation. The case of separated non regular boundary conditions was studied by, Eberhard, Hopkins, Jakson, Keldysh, Khromov, Seifert, Stone, Ward (see Yakubov

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2000 *Mathematics Subject Classification.* 47E05, 35K20.
Key words and phrases. Green's function; regular and non regular boundary conditions; semi group with singularities; weighted mixed boundary conditions.

Submitted May 10, 2006. Published April 17, 2007.

and Yakubov [37] for exact references). A situation of non regular non-separated boundary conditions was considered by Benzinger [2], Denche [4, 5, 6], Eberhard and Freiling [10], Gasumov and Magerramov [16, 17], Khromov [22], Mamedov [23], Shkalikov [29], Silchenko [31], Tretter [34], Vagabov [35], Yakubov [38] and Yakubov [39].

A mathematical model with integral boundary conditions was derived by [11, 27] in the context of optical physics. The importance of this kind of problems has been also pointed out by Samarskii [28].

In this paper, we study a problem for second order ordinary differential equation with mixed nonlocal boundary conditions combined weighting integral boundary conditions with another two point boundary conditions. The regular case was studied in the space $L_1(0,1)$ by Gallardo [15]. The Particular case where $S_i(t)=0$, and nonregular boundary conditions is studied by Silchenko [31]. A situation of a variable coefficient of $u''(x)$ in the equation has been treated in [12, 32]. The integral boundary conditions are again non regular but they assume less restrictions on the functions $R_i(t)$ (here again $S_i(t)=0$, $a_i=b_i=c_i=d_i=0$). In particular the corresponding estimate in $L_2(0,1)$ has been established.

Following the technique in [5, 6, 14, 15, 24, 25, 26], we should bound the resolvent in the space $L^p(0,1)$ by means of a suitable formula for Green's function. Under certain conditions on the weighting functions and on the coefficients in the boundary conditions, called non regular boundary conditions, we prove that the resolvent decreases with respect to the spectral parameter in $L^p(0,1)$, but there is no maximal decreasing at infinity for $p\geq 1$. In contrast to the regular case this decreasing is maximal for $p=1$ as shown in [14, 15]. Furthermore, the studied operator generates in $L^p(0,1)$ an analytic semi group with singularities for $p\geq 1$. The obtained results are then used to show the correct solvability of a mixed problem for a parabolic partial differential equation with non regular non local boundary conditions.

2. Green's Function

Let $\lambda\in\mathbb{C}$, $u_1(x)=u_1(x,\lambda)$ and $u_2(x)=u_2(x,\lambda)$ be a fundamental system of solutions of equation

$$L(u)-\lambda u=0.$$

Following [24] , the Green's function of problem (1.1) is

$$G(x,s,\lambda)=\frac{N(x,s,\lambda)}{\Delta(\lambda)}, \tag{2.1}$$

where $\Delta(\lambda)$ is the characteristic determinant of the considered problem defined by

$$\Delta(\lambda)=\begin{vmatrix} B_1(u_1) & B_1(u_2)\\ B_2(u_1) & B_2(u_2)\end{vmatrix}, \tag{2.2}$$

and

$$N(x,s,\lambda)=\begin{vmatrix} u_1(x) & u_2(x) & g(x,s,\lambda)\\ B_1(u_1) & B_1(u_2) & B_1(g)_x\\ B_2(u_1) & B_2(u_2) & B_2(g)_x\end{vmatrix}, \tag{2.3}$$

for $x,s\in[0,1]$. The function $g(x,s,\lambda)$ is defined as follows

$$g(x,s,\lambda)=\pm\frac{1}{2}\,\frac{u_2(s)u_1(x)-u_2(x)u_1(s)}{u_1'(s)u_2(s)-u_1(s)u_2'(s)}, \tag{2.4}$$

where it takes the plus sign for $x > s$ and the minus for $x < s$. Given an arbitrary $\delta \in (0, \frac{\pi}{2})$, we consider the sector

$$\sum\nolimits_{\delta} = \{\lambda \in \mathbb{C}; |\arg(\lambda)| \le \frac{\pi}{2} + \delta, \lambda \neq 0\}.$$

For $\lambda \in \sum_{\delta}$, define ρ as the square root of λ with positive real part. For $\lambda \neq 0$, we can consider a fundamental system of solutions of equation $u'' = \lambda u = \rho^2 u$ given by $u_1(t) = e^{-\rho t}$ and $u_2(t) = e^{\rho t}$. In the following we are going to deduce an adequate formulae for $\Delta(\lambda)$ and $G(x, s, \lambda)$. First of all, for $i, j = \overline{1,2}$, we have

$$\begin{aligned} B_i(u_j) &= a_i + (-1)^j \rho b_i + c_i e^{(-1)^j \rho} + (-1)^j \rho d_i e^{(-1)^j \rho} \\ &\quad + \int_0^1 (R_i(t) + (-1)^j \rho S_i(t)) e^{(-1)^j \rho t} dt. \end{aligned}$$

So we obtain from (2.2),

$$\begin{aligned} \Delta(\lambda) &= (a_1 - \rho b_1 + c_1 e^{-\rho} - \rho d_1 e^{-\rho} + \int_0^1 (R_1(t) - \rho S_1(t)) e^{-\rho t} dt) \\ &\quad \times (a_2 + \rho b_2 + c_2 e^{\rho} + \rho d_2 e^{\rho} + \int_0^1 (R_2(t) + \rho S_2(t)) e^{\rho t} dt) \\ &\quad - (a_2 - \rho b_2 + c_2 e^{-\rho} - \rho d_2 e^{-\rho} + \int_0^1 (R_2(t) - \rho S_2(t)) e^{-\rho t} dt) \\ &\quad \times (a_1 + \rho b_1 + c_1 e^{\rho} + \rho d_1 e^{\rho} + \int_0^1 (R_1(t) + \rho S_1(t)) e^{\rho t} dt), \end{aligned} \tag{2.5}$$

and $g(x, s, \lambda)$ has the form

$$g(x, s, \lambda) = \begin{cases} \frac{1}{4\rho}(e^{\rho(x-s)} - e^{\rho(s-x)}) & \text{if } x > s, \\ \frac{1}{4\rho}(e^{\rho(s-x)} - e^{\rho(x-s)}) & \text{if } x < s. \end{cases}$$

Thus we have

$$\begin{aligned} B_i(g)_x &= \frac{e^{\rho s}}{4\rho}\Big[a_i - \rho b_i - c_i e^{-\rho} + \rho d_i e^{-\rho} \\ &\quad + \int_0^s (R_i(t) - \rho S_i(t)) e^{-\rho t} dt + \int_s^1 (-R_i(t) + \rho S_i(t)) e^{-\rho t} dt] \\ &\quad + \frac{e^{-\rho s}}{4\rho}[-a_i - \rho b_i + c_i e^{\rho} + \rho d_i e^{\rho} \\ &\quad - \int_0^s (R_i(t) + \rho S_i(t)) e^{\rho t} dt + \int_s^1 (R_i(t) + \rho S_i(t)) e^{\rho t} dt\Big], \end{aligned}$$

where $i = \overline{1,2}$. For $x, y \in \{a_i, b_i, c_i, d_i\}$ and $F, G \in \{R, S\}$, we introduce $\Delta_{xy} = x_1 y_2 - x_2 y_1, \Delta_{xF}(t) = x_1 F_2(t) - x_2 F_1(t), \Delta_F(t, \xi) = F_1(t) F_2(\xi) - F_2(t) F_1(\xi)$, and $\Delta_{FG}(t, \xi) = F_1(t) G_2(\xi) - F_2(t) G_1(\xi)$. After a long calculation, formula (2.3) can be written as

$$N(x, s, \lambda) = \varphi(x, s; \lambda) + \varphi_i(x, s, \lambda), \tag{2.6}$$

where

$$\begin{aligned} &\varphi(x, s, \lambda) \\ &= \frac{1}{2\rho}\Big[e^{\rho(x+s)}\{(\Delta_{ac} - \rho(\Delta_{ad} + \Delta_{bc}) + \rho^2 \Delta_{bd}) e^{-\rho} \end{aligned}$$

$$
\begin{aligned}
&- \int_0^s (\Delta_{cR}(t) - \rho(\Delta_{dR}(t) + \Delta_{cS}(t)) + \rho^2 \Delta_{dS}(t)) e^{-\rho(t+1)} dt \\
&+ \int_s^1 (\Delta_{aR}(t) - \rho(\Delta_{aS}(t) + \Delta_{bR}(t)) + \rho^2 \Delta_{bS}(t)) e^{-\rho t} dt \\
&+ \int_s^1 \int_0^s (\Delta_R(\xi,t) + \rho(\Delta_{RS}(t,\xi) - \Delta_{RS}(\xi,t)) + \rho^2 \Delta_S(\xi,t)) e^{-\rho(\xi+t)} d\xi dt\} \\
&+ e^{-\rho(x+s)} \{(\Delta_{ac} + \rho(\Delta_{ad} + \Delta_{bc}) + \rho^2 \Delta_{bd}) e^{\rho} \\
&- \int_0^s (\Delta_{cR}(t) + \rho(\Delta_{dR}(t) + \Delta_{cS}(t)) + \rho^2 \Delta_{dS}(t)) e^{\rho(t+1)} dt \\
&+ \int_s^1 (\Delta_{aR}(t) + \rho(\Delta_{bR}(t) + \Delta_{aS}(t)) + \rho^2 \Delta_{bS}(t)) e^{\rho t} dt \\
&+ \int_s^1 \int_0^s (\Delta_R(\xi,t) + \rho(\Delta_{RS}(\xi,t) - \Delta_{RS}(t,\xi)) + \rho^2 \Delta_S(\xi,t)) e^{\rho(\xi+t)} d\xi dt\} \Big],
\end{aligned}
\tag{2.7}
$$

and the function $\varphi_i(x,s,\lambda)$ is given by

$$
\varphi_i(x,s,\lambda) = \begin{cases} \varphi_1(x,s,\lambda) & \text{if } x > s, \\ \varphi_2(x,s,\lambda) & \text{if } x < s, \end{cases}
\tag{2.8}
$$

with

$$
\begin{aligned}
&\varphi_1(x,s,\lambda) \\
&= \frac{1}{2\rho} \Big[e^{\rho(x-s)} \{(-\Delta_{ac} + \rho(\Delta_{ad} - \Delta_{bc}) + \rho^2 \Delta_{bd}) e^{-\rho} \\
&+ 2\rho \Delta_{ab} + \int_0^s (\Delta_{aR}(t) + \rho(\Delta_{aS}(t) + \Delta_{bR}(t)) - \rho^2 \Delta_{bS}(t)) e^{\rho t} dt \\
&+ \int_0^s (\Delta_{cR}(t) + \rho(\Delta_{cS}(t) - \Delta_{dR}(t)) - \rho^2 \Delta_{dS}(t)) e^{\rho(t-1)} dt \\
&- \int_0^1 (\Delta_{aR}(t) + \rho(\Delta_{bR}(t) - \Delta_{aS}(t)) - \rho^2 \Delta_{bS}(t)) e^{-\rho t} dt \\
&+ \int_0^1 \int_0^s (\Delta_R(t,\xi) + \rho(\Delta_{RS}(\xi,t) - \Delta_{RS}(t,\xi)) - \rho^2 \Delta_S(t,\xi)) e^{\rho(\xi-t)} d\xi dt\} \\
&+ e^{\rho(s-x)} \{(-\Delta_{ac} + \rho(\Delta_{bc} - \Delta_{ad}) + \rho^2 \Delta_{bd}) e^{\rho} - 2\rho \Delta_{ab} \\
&+ \int_0^s (\Delta_{aR}(t) + \rho(\Delta_{bR}(t) - \Delta_{aS}(t)) - \rho^2 \Delta_{bS}(t)) e^{-\rho t} dt \\
&+ \int_0^s (\Delta_{cR}(t) + \rho(\Delta_{dR}(t) - \Delta_{cS}(t)) - \rho^2 \Delta_{dS}(t)) e^{\rho(1-t)} dt \\
&- \int_0^1 (\Delta_{aR}(t) + \rho(\Delta_{aS}(t) - \Delta_{bR}(t)) - \rho^2 \Delta_{bS}(t)) e^{\rho t} dt \\
&+ \int_0^1 \int_0^s (\Delta_R(t,\xi) - \rho(\Delta_{RS}(\xi,t) + \Delta_{RS}(t,\xi)) - \rho^2 \Delta_S(t,\xi)) e^{\rho(t-\xi)} d\xi dt\} \Big],
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
&\varphi_2(x,s,\lambda)\\
&= \frac{1}{2\rho}\Big[e^{\rho(x-s)}\{(-\Delta_{ac}+\rho(\Delta_{bc}-\Delta_{ad})+\rho^2\Delta_{bd})e^{\rho}-2\rho\Delta_{ad}\\
&\quad -\int_s^1(\Delta_{aR}(t)+\rho(\Delta_{aS}(t)-\Delta_{bR}(t))-\rho^2\Delta_{bS}(t))e^{\rho t}dt\\
&\quad -\int_s^1(\Delta_{cR}(t)+\rho(\Delta_{cS}(t)-\Delta_{dR}(t))-\rho^2\Delta_{dS}(t))e^{\rho(t-1)}dt\\
&\quad +\int_0^1(\Delta_{cR}(t)+\rho(\Delta_{dR}(t)-\Delta_{cS}(t))-\rho^2\Delta_{dS}(t))e^{\rho(1-t)}dt\\
&\quad +\int_0^1\int_s^1(\Delta_R(\xi,t)-\rho(\Delta_{RS}(t,\xi)+\Delta_{RS}(\xi,t))-\rho^2\Delta_S(\xi,t))e^{\rho(\xi-t)}d\xi dt\}\\
&\quad + e^{\rho(s-x)}\{(-\Delta_{ac}+\rho(\Delta_{ad}-\Delta_{bc})+\rho^2\Delta_{bd})e^{-\rho}+2\rho\Delta_{cd}\\
&\quad -\int_s^1(\Delta_{aR}(t)+\rho(\Delta_{bR}(t)-\Delta_{aS}(t))-\rho^2\Delta_{bS}(t))e^{-\rho t}dt\\
&\quad -\int_s^1(\Delta_{cR}(t)+\rho(\Delta_{dR}(t)-\Delta_{cS}(t))-\rho^2\Delta_{dS}(t))e^{\rho(1-t)}dt\\
&\quad +\int_0^1(\Delta_{cR}(t)+\rho(\Delta_{cS}(t)-\Delta_{dR}(t))-\rho^2\Delta_{dS}(t))e^{\rho(t-1)}dt\\
&\quad +\int_0^1\int_s^1(\Delta_R(\xi,t)+\rho(\Delta_{RS}(t,\xi)+\Delta_{RS}(\xi,t))-\rho^2\Delta_S(\xi,t))e^{\rho(t-\xi)}d\xi dt\}\Big]
\end{aligned}
\tag{2.10}
$$

3. Bounds on the Resolvent

Every $\lambda\in\mathbb{C}$ such that $\Delta(\lambda)\neq 0$ belongs to $\rho(L_p)$, and the associated resolvent operator $R(\lambda,L_p)$ can be expressed as a Hilbert Schmidt operator

$$
(\lambda I-L_p)^{-1}f=R(\lambda;L_p)f=-\int_0^1 G(.,s;\lambda)f(s)ds,\ \ f\in L^p(0,1). \tag{3.1}
$$

Then, for every $f\in L^p(0,1)$ we estimate (3.1),

$$
\|R(\lambda;L_p)f\|_{L_p(0,1)}\le\Big(\sup_{0\le s\le 1}\int_0^1|G(x,s;\lambda)|^p dx\Big)^{1/p}\|f\|_{L_p(0,1)},
$$

and so we need to bound

$$
(\sup_{0\le s\le 1}\int_0^1|G(x,s;\lambda)|^p dx)^{1/p}=\frac{1}{|\Delta(\lambda)|}(\sup_{0\le s\le 1}\int_0^1|N(x,s;\lambda)|^p dx)^{1/p}. \tag{3.2}
$$

3.1. **Estimation of $\mathbf{N}(x,s,\lambda)$.** We will denote by $\|\cdot\|$ the supremum norm for functions in one and two variables. Since

$$
N(x,s,\lambda)=\begin{cases}\varphi(x,s,\lambda)+\varphi_1(x,s,\lambda) & \text{if } x>s,\\ \varphi(x,s,\lambda)+\varphi_2(x,s,\lambda) & \text{if } x<s,\end{cases}
$$

it follows that

$$
|N(x,s,\lambda)|^p\le 2^{p-1}(|\varphi(x,s,\lambda)|^p+|\varphi_i(x,s,\lambda)|^p).
$$

Form (2.8), we have

$$\begin{aligned} &\int_0^1 |N(x,s,\lambda)|^p dx \\ &\leq 2^{p-1}\Big(\int_0^1 |\varphi(x,s,\lambda)|^p dx + \int_s^1 |\varphi_1(x,s,\lambda)|^p dx + \int_0^s |\varphi_1(x,s,\lambda)|^p dx\Big). \end{aligned} \tag{3.3}$$

From (2.7), we have

$$\begin{aligned} &\int_0^1 |\varphi(x,s,\lambda)|^p dx \\ &\leq \frac{2^{2p-2}}{2p|\rho|^p \operatorname{Re}(\rho)}\Big[(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ad}| + |\Delta_{bc}|) + |\Delta_{ac}|)^p \\ &\times \Big((e^{ps\operatorname{Re}(\rho)} - e^{p(s-1)\operatorname{Re}(\rho)}) + (e^{p(1-s)\operatorname{Re}(\rho)} - e^{-ps\operatorname{Re}(\rho)})\Big) \\ &+ \Big(\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{bS}\| + |\rho|(\|\Delta_{aS}\| + \|\Delta_{bR}\|) + \|\Delta_{aR}\|)\Big)^p \\ &\times ((e^{p\operatorname{Re}(\rho)} - 1) \times (1 - e^{(s-1)\operatorname{Re}(\rho)})^p + (1 - e^{-p\operatorname{Re}(\rho)}) \times (e^{(1-s)\operatorname{Re}(\rho)} - 1)^p) \\ &+ \Big(\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{dS}\| + |\rho|(\|\Delta_{dR}\| + \|\Delta_{cS}\|) + \|\Delta_{cR}\|)\Big)^p \times \big((e^{p\operatorname{Re}(\rho)} - 1) \\ &\times (e^{(s-1)\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})^p + (1 - e^{-p\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)} - e^{(1-s)\operatorname{Re}(\rho)})^p\big) \\ &+ \Big(\frac{1}{(\operatorname{Re}(\rho))^2}(\|\Delta_R\| + 2|\rho|\|\Delta_{RS}\| + |\rho|^2\|\Delta_S\|)\Big)^p \times ((e^{p\operatorname{Re}(\rho)} - 1) \\ &\times (e^{s\operatorname{Re}(\rho)} - 1)^p \times (e^{-s\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})^p + (1 - e^{-p\operatorname{Re}(\rho)}) \\ &\times (e^{\operatorname{Re}(\rho)} - e^{s\operatorname{Re}(\rho)})^p(1 - e^{-s\operatorname{Re}(\rho)})^p)\Big], \end{aligned}$$

from (2.9), we have

$$\begin{aligned} &\int_s^1 |\varphi_1(x,s,\lambda)|^p dx \\ &\leq \frac{5^{p-1}}{2p|\rho|^p \operatorname{Re}(\rho)}[(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ad}| + |\Delta_{bc}|) \\ &+ |\Delta_{ac}|)^p \times ((e^{p\operatorname{Re}(\rho)} - e^{-ps\operatorname{Re}(\rho)}) + (e^{-ps\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)})) + (2|\rho||\Delta_{ab}|)^p \\ &\times \big(e^{p(1-s)\operatorname{Re}(\rho)} - e^{p(s-1)\operatorname{Re}(\rho)}\big) + (\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{bS}\| + |\rho|(\|\Delta_{aS}\| + \|\Delta_{bR}\|) \\ &+ \|\Delta_{aR}\|))^p \times ((e^{-ps\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)}) \times (e^{(1+s)\operatorname{Re}(\rho)} - 1)^p + (e^{p\operatorname{Re}(\rho)} \\ &- e^{ps\operatorname{Re}(\rho)}) \times (1 - e^{-(1+s)\operatorname{Re}(\rho)})^p) + (\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{dS}\| + |\rho|(\|\Delta_{dR}\| \\ &+ \|\Delta_{cS}\|) + \|\Delta_{cR}\|))^p \times ((e^{-ps\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)}) \times (e^{p(s+1)\operatorname{Re}(\rho)} - e^{p\operatorname{Re}(\rho)})^p \\ &+ (e^{p\operatorname{Re}(\rho)} - e^{ps\operatorname{Re}(\rho)}) \times (e^{-\operatorname{Re}(\rho)} - e^{-(1+s)\operatorname{Re}(\rho)})^p) + (\frac{1}{(\operatorname{Re}(\rho))^2}(\|\Delta_R\| \\ &+ 2|\rho|\|\Delta_{RS}\| + |\rho|^2\|\Delta_S\|))^p \times ((e^{-ps\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)} - 1)^p \\ &\times (e^{s\operatorname{Re}(\rho)} - 1)^p + (e^{p\operatorname{Re}(\rho)} - e^{ps\operatorname{Re}(\rho)}) \times (1 - e^{-\operatorname{Re}(\rho)})^p \times (1 - e^{-s\operatorname{Re}(\rho)})^p)\Big]. \end{aligned}$$

From (2.10), we have

$$
\begin{aligned}
&\int_0^s |\varphi_2(x,s,\lambda)|^p dx \\
&\le \frac{5^{p-1}}{2p|\rho|^p \operatorname{Re}(\rho)} \Big[(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ad}| + |\Delta_{bc}|) \\
&\quad + |\Delta_{ac}|)^p \times ((e^{p\operatorname{Re}(\rho)} - e^{p(1-s)\operatorname{Re}(\rho)}) + (e^{p(s-1)\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)})) \\
&\quad + (2|\rho||\Delta_{cd}|)^p \times (e^{ps\operatorname{Re}(\rho)} - e^{-ps\operatorname{Re}(\rho)}) + (\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{bS}\| + |\rho|(\|\Delta_{aS}\| \\
&\quad + \|\Delta_{bR}\|) + \|\Delta_{aR}\|))^p \times ((e^{ps\operatorname{Re}(\rho)} - 1) \times (e^{(1-s)\operatorname{Re}(\rho)} - 1)^p \\
&\quad + (1 - e^{-ps\operatorname{Re}(\rho)}) \times (1 - e^{(s-1)\operatorname{Re}(\rho)})^p) + (\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{dS}\| + |\rho|(\|\Delta_{dR}\| \\
&\quad + \|\Delta_{cS}\|) + \|\Delta_{cR}\|))^p \times ((e^{ps\operatorname{Re}(\rho)} - 1) \times (e^{(1-s)\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})^p \\
&\quad + (1 - e^{-ps\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)} - e^{(s-1)\operatorname{Re}(\rho)})^p) + (\frac{1}{(\operatorname{Re}(\rho))^2}(\|\Delta_R\| + 2|\rho|\|\Delta_{RS}\| \\
&\quad + |\rho|^2\|\Delta_S\|))^p \times ((e^{ps\operatorname{Re}(\rho)} - 1) \times (e^{(1-s)\operatorname{Re}(\rho)} - 1)^p \times (1 - e^{-\operatorname{Re}(\rho)})^p \\
&\quad + (1 - e^{-ps\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)} - 1)^p \times (1 - e^{(s-1)\operatorname{Re}(\rho)})^p)\Big].
\end{aligned}
$$

So that

$$
\begin{aligned}
&\int_0^1 |N(x,s,\lambda)|^p dx \\
&\le \frac{2^{p-1} \times 5^{p-1}}{2p|\rho|^p \operatorname{Re}(\rho)}[(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ad}| + |\Delta_{bc}|) + |\Delta_{ac}|)^p \\
&\quad \times 2(e^{p\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)}) + (2|\rho||\Delta_{ab}|)^p \times (e^{p(1-s)\operatorname{Re}(\rho)} - e^{p(s-1)\operatorname{Re}(\rho)}) \\
&\quad + (2|\rho||\Delta_{cd}|)^p \times (e^{ps\operatorname{Re}(\rho)} - e^{-ps\operatorname{Re}(\rho)}) + (\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{bS}\| + |\rho|(\|\Delta_{aS}\| \\
&\quad + \|\Delta_{bR}\|) + \|\Delta_{aR}\|))^p \times ((e^{p\operatorname{Re}(\rho)} - 1) \times (1 - e^{(s-1)\operatorname{Re}(\rho)})^p \\
&\quad + (1 - e^{-p\operatorname{Re}(\rho)})(e^{(1-s)\operatorname{Re}(\rho)} - 1)^p + (e^{-ps\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)}) \times (e^{(1+s)\operatorname{Re}(\rho)} - 1)^p \\
&\quad + (e^{p\operatorname{Re}(\rho)} - e^{ps\operatorname{Re}(\rho)})(1 - e^{-(1+s)\operatorname{Re}(\rho)})^p + (e^{ps\operatorname{Re}(\rho)} - 1) \times (e^{(1-s)\operatorname{Re}(\rho)} - 1)^p \\
&\quad + (1 - e^{-ps\operatorname{Re}(\rho)})(1 - e^{(s-1)\operatorname{Re}(\rho)})^p) + (\frac{1}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{dS}\| + |\rho|(\|\Delta_{dR}\| \\
&\quad + \|\Delta_{cS}\|) + \|\Delta_{cR}\|))^p((e^{p\operatorname{Re}(\rho)} - 1) \times (e^{(s-1)\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})^p + (1 - e^{-p\operatorname{Re}(\rho)}) \\
&\quad \times (e^{\operatorname{Re}(\rho)} - e^{(1-s)\operatorname{Re}(\rho)})^p + (e^{-ps\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)}) \times (e^{p(s+1)\operatorname{Re}(\rho)} - e^{p\operatorname{Re}(\rho)})^p \\
&\quad + (e^{p\operatorname{Re}(\rho)} - e^{ps\operatorname{Re}(\rho)}) \times (e^{-\operatorname{Re}(\rho)} - e^{-(1+s)\operatorname{Re}(\rho)})^p + (e^{ps\operatorname{Re}(\rho)} - 1) \\
&\quad \times (e^{(1-s)\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})^p + (1 - e^{-ps\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)} - e^{(s-1)\operatorname{Re}(\rho)})^p) \\
&\quad + (\frac{1}{(\operatorname{Re}(\rho))^2}(\|\Delta_R\| + 2|\rho|\|\Delta_{RS}\| + |\rho|^2\|\Delta_S\|))^p \times ((e^{p\operatorname{Re}(\rho)} - 1) \\
&\quad \times (e^{s\operatorname{Re}(\rho)} - 1)^p \times (e^{-s\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})^p + (1 - e^{-p\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)} - e^{s\operatorname{Re}(\rho)})^p \\
&\quad \times (1 - e^{-s\operatorname{Re}(\rho)})^p + (e^{-ps\operatorname{Re}(\rho)} - e^{-p\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)} - 1)^p \times (e^{s\operatorname{Re}(\rho)} - 1)^p \\
&\quad + (e^{p\operatorname{Re}(\rho)} - e^{ps\operatorname{Re}(\rho)}) \times (1 - e^{-\operatorname{Re}(\rho)})^p \times (1 - e^{-s\operatorname{Re}(\rho)})^p + (e^{ps\operatorname{Re}(\rho)} - 1)
\end{aligned}
$$

$$\times (e^{(1-s)\operatorname{Re}(\rho)}-1)^p \times (1-e^{-\operatorname{Re}(\rho)})^p + (1-e^{-ps\operatorname{Re}(\rho)}) \times (e^{\operatorname{Re}(\rho)}-1)^p \times (1-e^{(s-1)\operatorname{Re}(\rho)})^p)\Big].$$

Since $\operatorname{Re}(\rho) > 0$, we obtain

$$\begin{aligned}
&\sup_{0\le s\le 1}\Big(\int_0^1 |N(x,s,\lambda)|^p dx\Big)^{1/p} \\
&\le \frac{2^{(2-\frac{2}{p})}\times 5^{(1-\frac{1}{p})}}{p^{\frac{1}{p}}|\rho|(\operatorname{Re}(\rho))^{1/p}} e^{\operatorname{Re}(\rho)}\Big[|\rho|^2|\Delta_{bd}| \\
&\quad + |\rho|(|\Delta_{ad}|+|\Delta_{bc}|+|\Delta_{cd}|) + |\Delta_{ac}| + \frac{3}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{bS}\| + |\rho|(\|\Delta_{aS}\| \\
&\quad + \|\Delta_{bR}\|) + \|\Delta_{aR}\|) + \frac{3}{\operatorname{Re}(\rho)}(|\rho|^2\|\Delta_{dS}\| + |\rho|(\|\Delta_{dR}\| + \|\Delta_{cS}\|) \\
&\quad + \|\Delta_{cR}\|) + \frac{3}{(\operatorname{Re}(\rho))^2}(\|\Delta_R\| + 2|\rho|\|\Delta_{RS}\| + |\rho|^2\|\Delta_S\|)\Big]. \qquad (3.4)
\end{aligned}$$

From the above inequality, (3.2) and (3.3), we obtain

$$\begin{aligned}
\|R(\lambda, L_p)\| &\le \frac{2^{(2-\frac{2}{p})}\times 5^{(1-\frac{1}{p})}}{|\Delta(\rho^2)||\rho|(\operatorname{Re}(\rho))^{1/p}p^{1/p}} e^{\operatorname{Re}(\rho)}\Big[(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ad}| + |\Delta_{bc}| \\
&\quad + |\Delta_{ab}| + |\Delta_{cd}|) + |\Delta_{ac}| + \frac{3}{\operatorname{Re}(\rho)}(|\rho|^2(\|\Delta_{bS}\| + \|\Delta_{dS}\|) \\
&\quad + |\rho|(\|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_{aS}\| + \|\Delta_{bR}\|) + \|\Delta_{cR}\| + \|\Delta_{aR}\|) \\
&\quad + \frac{3}{(\operatorname{Re}(\rho))^2}(\|\Delta_R\| + 2|\rho|\|\Delta_{RS}\| + |\rho|^2\|\Delta_S\|)\Big],
\end{aligned}$$

for $\rho \in \sum_{\frac{\delta}{2}} = \{\rho \in \mathbb{C} : |\arg\rho| \le \frac{\pi}{4} + \frac{\delta}{2}, \rho \neq 0\}$, we have $(\operatorname{Re}(\rho))^{-1} < \frac{1}{|\rho|\cos(\frac{\pi}{4}+\frac{\delta}{2})}$. Then

$$\begin{aligned}
&\|R(\lambda, L_p)\| \\
&\le \frac{2^{(2-\frac{2}{p})}\times 5^{(1-\frac{1}{p})}}{|\Delta(\rho^2)||\rho|^{1+\frac{1}{p}}(\cos(\frac{\pi}{4}+\frac{\delta}{2}))^{1/p}p^{\frac{1}{p}}} e^{\operatorname{Re}(\rho)}\Big[(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ad}| \\
&\quad + |\Delta_{bc}| + |\Delta_{bc}| + |\Delta_{ab}| + |\Delta_{cd}|) + |\Delta_{ac}| + \frac{3}{|\rho|\cos(\frac{\pi}{4}+\frac{\delta}{2})}(|\rho|^2(\|\Delta_{bS}\| + \|\Delta_{dS}\|) \\
&\quad + |\rho|(\|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_{aS}\| + \|\Delta_{bR}\|) + \|\Delta_{cR}\| + \|\Delta_{aR}\|) \\
&\quad + \frac{3}{(|\rho|\cos(\frac{\pi}{4}+\frac{\delta}{2}))^2}(\|\Delta_R\| + 2|\rho|\|\Delta_{RS}\| + |\rho|^2\|\Delta_S\|)\Big].
\end{aligned}$$

Finally, we obtain, for $\lambda = \rho^2 \in \sum_\delta$,

$$\begin{aligned}
&\|R(\lambda, L_p)\| \\
&\le \frac{c}{|\Delta(\rho^2)||\rho|^{1+\frac{1}{p}}} e^{\operatorname{Re}(\rho)}\Big[(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ab}| + |\Delta_{ad}| \\
&\quad + |\Delta_{bc}| + |\Delta_{cd}| + \|\Delta_{bS}\| + \|\Delta_{dS}\|) + |\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| \\
&\quad + \|\Delta_{cS}\| + \|\Delta_S\| + \frac{1}{|\rho|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^2}\Big]
\end{aligned}$$

$$\leq c\frac{H(\rho)}{|\rho|^{1+\frac{1}{p}}}, \tag{3.5}$$

where

$$\begin{aligned} H(\rho) = \frac{e^{\mathrm{Re}(\rho)}}{|\Delta(\rho^2)|}\Big[&(|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ab}| + |\Delta_{ad}| \\ &+ |\Delta_{bc}| + |\Delta_{cd}| + \|\Delta_{bS}\| + \|\Delta_{dS}\|) + |\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| \\ &+ \|\Delta_{cS}\| + \|\Delta_{S}\| + \frac{1}{|\rho|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^2}\Big], \end{aligned} \tag{3.6}$$

and

$$c = \frac{2^{(2-\frac{2}{p})}\times 5^{(1-\frac{1}{p})}}{p^{1/p}}\max\Big(\frac{1}{(\cos(\frac{\delta}{2}))^{1/p}}, \frac{3}{(\cos(\frac{\delta}{2}))^{1+\frac{1}{p}}}, \frac{3}{(\cos(\frac{\delta}{2}))^{2+\frac{1}{p}}}\Big).$$

The following step is to analyze the function $H(\rho)$ in order to determine the cases for which it is bounded in the sector $\sum_{\delta/2}$.

3.2. **Estimation of the characteristic determinant, regular case.** The next step is to determine the cases for which $|\Delta(\lambda)|$ remains bounded below. It will then be necessary to bound $|\Delta(\lambda)|$ appropriately. However, formula (2.5) is not useful for this purpose, it will be then necessary to make some additional regularity hypotheses on the functions R_i and S_i, and so we assume that the functions R_i and S_i, $i = \overline{1,2}$, are in $C^1([0,1],\mathbb{C})$. Integrating twice by parts in (2.5), we obtain

$$\begin{aligned} \Delta(\lambda) = e^{\rho}\Big[&-\rho^2\Delta_{bd} + \rho(\Delta_{ad} - \Delta_{bc} + \Delta_{dS}(0) - \Delta_{bS}(1)) + \Delta_{ac} \\ &+ \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_{S}(1,0) + \frac{1}{\rho}(\Delta_{aR}(1) \\ &- \Delta_{cR}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0)) + \frac{1}{\rho^2}(\Delta_R(0,1)) + \Phi(\rho)\Big], \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \Phi(\rho) = 2e^{-\rho}[&\rho(\Delta_{ab} + \Delta_{cd} + \Delta_{dS}(0) - \Delta_{bS}(1)) + \frac{1}{\rho}(\Delta_{cR}(1) \\ &- \Delta_{aR}(0) - \Delta_{RS}(0,0) - \Delta_{RS}(1,1))] \\ &\times e^{-2\rho}[\rho^2\Delta_{bd} + \rho(\Delta_{ad} - \Delta_{bc} + \Delta_{dS}(0) - \Delta_{bS}(1)) - \Delta_{ac} - \Delta_{aS}(1) \\ &- \Delta_{cS}(0) + \Delta_{bR}(1) + \Delta_{dR}(0) - \Delta_{S}(1,0) + \frac{1}{\rho}(\Delta_{aR}(1) - \Delta_{cR}(0) \\ &+ \Delta_{RS}(0,1) + \Delta_{RS}(1,0)) + \frac{1}{\rho^2}(\Delta_R(1,0))] \\ &- \frac{1}{\rho}[\int_0^1 (\Delta_{aR'}(t) + \rho(\Delta_{aS'}(t) - \Delta_{bR'}(t)) - \rho^2\Delta_{bS'}(t))e^{\rho(t-1)}dt \\ &+ \int_0^1 (\Delta_{aR'}(t) + \rho(\Delta_{bR'}(t) - \Delta_{aS'}(t)) - \rho^2\Delta_{bS'}(t))e^{-\rho(t+1)}dt \\ &+ \int_0^1 (\Delta_{cR'}(t) + \rho(\Delta_{cS'}(t) - \Delta_{dR'}(t)) - \rho^2\Delta_{dS'}(t))e^{\rho(t-2)}dt \\ &+ \int_0^1 (\Delta_{cR'}(t) + \rho(\Delta_{dR'}(t) - \Delta_{cS'}(t)) - \rho^2\Delta_{dS'}(t))e^{-\rho t}dt] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\rho^2}[\int_0^1 (\Delta_{R'R}(t,0)-\rho(\Delta_{R'S}(t,0)+\Delta_{RS'}(0,t))-\rho^2\Delta_{S'S}(t,0))e^{\rho(t-1)}dt\\
&+\int_0^1 (\Delta_{R'R}(0,t)-\rho(\Delta_{R'S}(0,t)+\Delta_{RS'}(t,0))+\rho^2\Delta_{S'S}(1,t))e^{-\rho(t+1)}dt\\
&+\int_0^1 (\Delta_{R'R}(1,t)+\rho(\Delta_{R'S}(1,t)+\Delta_{RS'}(t,1))+\rho^2\Delta_{S'S}(t,1))e^{\rho(t-2)}dt\\
&+\int_0^1 (\Delta_{R'R}(t,1)+\rho(\Delta_{R'S}(t,1)+\Delta_{RS'}(1,t))+\rho^2\Delta_{S'S}(1,t))e^{-\rho t}dt\\
&+\int_0^1\int_0^1 (\Delta_{R'}(\xi,t)+\rho(\Delta_{R'S'}(t,\xi)+\Delta_{R'S'}(\xi,t))\\
&-\rho^2\Delta_{S'}(\xi t))e^{\rho(\xi-t-1)}d\xi dt].
\end{aligned}
$$

Suppose first that $\Delta_{bd}\neq 0$. From (3.7) we can write the characteristic determinant in the form

$$\Delta(\lambda)=\rho^2\Delta_{bd}e^{\rho}(-1+F(\rho))$$

for a certain function $F(\rho)$, It is not difficult to see that a constant r_0 can be chosen in order that $|F(\rho)|\leq \frac{1}{2}$ for $|\rho|>r_0$. Thus, for $\rho\in r_0+\sum_{\frac{\delta}{2}}$, we have

$$|\Delta(\lambda)|\geq|\rho|^2|\Delta_{bd}|e^{\mathrm{Re}(\rho)}(1-|F(\rho)|)\geq\frac{1}{2}|\rho|^2|\Delta_{bd}|e^{\mathrm{Re}(\rho)}.$$

Finally, from (3.6) we obtain

$$
\begin{aligned}
H(\rho)\leq \frac{2}{|\Delta_{bd}|}\Big[&|\Delta_{bd}|+\frac{1}{r_0}(|\Delta_{ab}|+|\Delta_{ad}|+|\Delta_{bc}|+|\Delta_{cd}|+\|\Delta_{bS}\|+\|\Delta_{dS}\|)\\
&+\frac{1}{r_0^2}(|\Delta_{ac}|+\|\Delta_{aS}\|+\|\Delta_{bR}\|+\|\Delta_{dR}\|+\|\Delta_{cS}\|+\|\Delta_{S}\|)\\
&+\frac{1}{r_0^3}(\|\Delta_{aR}\|+\|\Delta_{cR}\|+2\|\Delta_{RS}\|)+\frac{\|\Delta_R\|}{r_0^4}\Big]\equiv H_0,
\end{aligned}
$$

which proves that $H(\rho)$ is bounded by a constant H_0 in the sector $r_0+\sum_{\frac{\delta}{2}}$. The other cases can be treated in a similar way [5, 6, 13, 15], and we do not include them here for lack of space. After doing the complete analysis of cases, we obtain that $H(\rho)$ is bounded by a constant $H_0>0$ in a sector of the form $r_0+\sum_{\frac{\delta}{2}}$, only in the following five cases

(1) $\Delta_{bd}\neq 0$
(2) $\Delta_{bd}=0$ and $\Delta_{ad}-\Delta_{bc}-\Delta_{bS}(1)+\Delta_{dS}(0)\neq 0$
(3) $\Delta_{ab}=\Delta_{ad}=\Delta_{bc}=\Delta_{bd}=\Delta_{cd}=0$, $\Delta_{bS}\equiv 0$, $\Delta_{dS}\equiv 0$ and $\Delta_{ac}+\Delta_{aS}(1)+\Delta_{cS}(0)-\Delta_{bR}(1)-\Delta_{dR}(0)+\Delta_{S}(1,0)\neq 0$
(4) $\Delta_{ab}=\Delta_{ac}=\Delta_{ad}=\Delta_{bc}=\Delta_{bd}=\Delta_{cd}=0$, $\Delta_{bR}\equiv 0, \Delta_{dR}\equiv 0, \Delta_{aS}\equiv 0, \Delta_{bS}\equiv 0$, $\Delta_{cS}\equiv 0$, $\Delta_{dS}\equiv 0$, $\Delta_S\equiv 0$ and $\Delta_{aR}(1)-\Delta_{cR}(0)+\Delta_{RS}(0,1)+\Delta_{RS}(1,0)\neq 0$
(5) $a_i=b_i=c_i=d_i=0, S_i\equiv 0$, $i=\overline{1,2}$ and $\Delta_R(0,1)\neq 0$.

Definition 3.1. Suppose that $R_i,S_i\in C^1([0,1],\mathbb{C})$, $i=\overline{1,2}$. The boundary conditions in (1.1) are called regular if they verify one of the conditions above.

The above arguments prove the following theorem.

Theorem 3.2. *If the boundary value conditions in* (1.1) *are regular, then* $\sum_\delta \subset \rho(L_p)$ *for sufficiently large* $|\lambda|$ *and there exists* $c > 0$ *such that*

$$\|R(\lambda, L_p)\| \le \frac{c}{|\lambda|^{\frac{1}{2}+\frac{1}{2p}}}, |\lambda| \to +\infty.$$

Remark 3.3. From theorem 3.2 results that the operator L_p for $p \neq \infty$, generates an analytic semi group with singularities [30] of type $A(\frac{p-1}{p+1}, \frac{3p-1}{p+1})$.

Remark 3.4. For $p = 1$, the decrease of the resolvent is maximal and the operator L_1 generates an analytic semi group [15].

3.3. **Estimation of the characteristic determinant, non regular case.** As in the regular case Formula (3.7) is not useful for the estimation of the characteristic determinant, it will be then necessary to make some additional hypotheses on the functions R_i and S_i, and so we assume that the functions R_i and S_i are in $C^2([0,1],\mathbb{C})$, $i = \overline{1,2}$. Integrating twice by parts in (3.7), we obtain

$$\begin{aligned}
\Delta(\lambda) = e^{\rho}\Big[&-\rho^2\Delta_{bd} + \rho(\Delta_{ad} - \Delta_{bc} + \Delta_{dS}(0) - \Delta_{bS}(1)) + \Delta_{ac} \\
&+ \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_{S}(1,0) + \Delta_{bS'}(1) \\
&+ \Delta_{dS'}(0) + \frac{1}{\rho}(\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) + \Delta_{cS'}(0) \\
&- \Delta_{dR'}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) + \Delta_{SS'}(1,0) + \Delta_{SS'}(0,1)) \\
&+ \frac{1}{\rho^2}(\Delta_{R}(0,1) - \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1) \\
&- \Delta_{dR'}(1) - \Delta_{cR'}(0)) + \frac{1}{\rho^3}(\Delta_{R'R}(1,0) + \Delta_{R'R}(0,1)) + \Phi(\rho)\Big],
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\Phi(\rho) = 2e^{-\rho}\Big[&\rho(\Delta_{ab} + \Delta_{cd} + \Delta_{bS}(0) - \Delta_{dS}(1)) + \frac{1}{\rho}(\Delta_{cR}(1) \\
&- \Delta_{aR}(0) + \Delta_{dR'}(1) - \Delta_{cS'}(1) + \Delta_{aS'}(0) - \Delta_{bR'}(0) - \Delta_{RS}(0,0) \\
&- \Delta_{RS}(1,1) + \Delta_{S'S}(1,1) + \Delta_{S'S}(0,0)) + \frac{1}{\rho^3}(\Delta_{RR'}(1,1) + \Delta_{RR'}(0,0))\Big] \\
&+ e^{-2\rho}[\rho^2\Delta_{bd} + \rho(\Delta_{ad} - \Delta_{bc} + \Delta_{dS}(0) - \Delta_{bS}(1)) - \Delta_{ac} - \Delta_{aS}(1) \\
&- \Delta_{cS}(0) + \Delta_{bR}(1) + \Delta_{dR}(0) - \Delta_{S}(1,0) - \Delta_{bS'}(1) - \Delta_{dS'}(0) \\
&+ \frac{1}{\rho}(\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) + \Delta_{cS'}(0) - \Delta_{dR'}(0) \\
&+ \Delta_{RS}(0,1) + \Delta_{RS}(1,0) + \Delta_{SS'}(1,0) + \Delta_{SS'}(0,1)) \\
&+ \frac{1}{\rho^2}(\Delta_{R}(0,1) - \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1) \\
&- \Delta_{dR'}(1) - \Delta_{cR'}(0)) + \frac{1}{\rho^3}(\Delta_{R'R}(1,0) + \Delta_{R'R}(0,1))] \\
&+ \frac{1}{\rho^2}[\int_0^1 (\Delta_{aR''}(t) + \rho(\Delta_{aS''}(t) - \Delta_{bR''}(t)) - \rho^2\Delta_{bS''}(t))e^{\rho(t-1)}dt \\
&+ \int_0^1 (\Delta_{cR''}(t) + \rho(\Delta_{cS''}(t) - \Delta_{dR''}(t)) - \rho^2\Delta_{dS''}(t))e^{\rho(t-2)}dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^1 (\Delta_{aR''}(t) + \rho(\Delta_{bR''}(t) - \Delta_{aS''}(t)) - \rho^2 \Delta_{bS''}(t)) e^{-\rho(t+1)} dt \\
&- \int_0^1 (\Delta_{cR''}(t) + \rho(\Delta_{dR''}(t) - \Delta_{cS''}(t)) - \rho^2 \Delta_{dS''}(t)) e^{-\rho t} dt \\
&+ \{ (\int_0^1 (R_2'(t) - \rho S_2'(t)) e^{-\rho t} dt) \times (\int_0^1 (R_1'(t) + \rho S_1'(t)) e^{\rho(t-1)} dt) \\
&- (\int_0^1 (R_1'(t) - \rho S_1'(t)) e^{-\rho t} dt) \times (\int_0^1 (R_2'(t) + \rho S_2'(t)) e^{\rho(t-1)} dt) \} \Big] \\
&+ \frac{1}{\rho^3} \Big[- \int_0^1 (\Delta_{RR''}(1,t) + \rho(\Delta_{RS''}(1,t) + \Delta_{R''S}(t,1)) \\
&+ \rho^2 \Delta_{S''S}(t,1)) e^{\rho(t-2)} dt \\
&+ \int_0^1 (\Delta_{RR''}(0,t) + \rho(\Delta_{S''R}(t,0) + \Delta_{SR''}(0,t)) + \rho^2 \Delta_{S''S}(t,0)) e^{-\rho(t+1)} dt \\
&+ \int_0^1 (\Delta_{R''R}(t,1) + \rho(\Delta_{R''S}(1,t) + \Delta_{RS''}(1,t)) + \rho^2 \Delta_{SS''}(1,t)) e^{-\rho t} dt \\
&- \int_0^1 (\Delta_{R''R}(t,0) + \rho(\Delta_{R''S}(t,0) + \Delta_{RS''}(0,t)) + \rho^2 \Delta_{SS''}(0,t)) e^{\rho(t-1)} dt \Big].
\end{aligned}
$$

After a straightforward calculation, we obtain the following inequality valid for $\rho \in \sum_{\frac{\delta}{2}}$, with $|\rho|$ sufficiently large,

$$
\begin{aligned}
&|\Phi(\rho)| \\
&\leq 2e^{-\operatorname{Re}(\rho)} \Big[|\rho| (|\Delta_{ab}| + |\Delta_{cd}| + \|\Delta_{bS}\| + \|\Delta_{dS}\|) \\
&\quad + \frac{1}{|\rho|} (\|\Delta_{cR}\| + \|\Delta_{aR}\| + \|\Delta_{dR'}\| + \|\Delta_{cS'}\| + \|\Delta_{aS'}\| + \|\Delta_{bR'}\| \\
&\quad + 2\|\Delta_{SR}\| + 2\|\Delta_{S'S}\|) \frac{1}{|\rho|^3} (\|\Delta_{RR'}\| + \|\Delta_{RR'}\|) \\
&\quad + e^{-\operatorname{Re}(\rho)} [|\rho|^2 |\Delta_{bd}| + |\rho| (|\Delta_{ad}| + |\Delta_{bc}| + \|\Delta_{dS}\| + \|\Delta_{bS}\|) \\
&\quad + |\Delta_{ac}| + \|\Delta_{bR}\| + \|\Delta_{aS}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_{dS'}\| + \|\Delta_{bS'}\| \\
&\quad + \|\Delta_S\| + \frac{1}{|\rho|} (\|\Delta_{aR}\| + \|\Delta_{cR}\| + \|\Delta_{bR'}\| + \|\Delta_{aS'}\| + \|\Delta_{cS'}\| \\
&\quad + \|\Delta_{dR'}\| + 2\|\Delta_{RS'}\| + 2\|\Delta_{S'S}\|) + \frac{1}{|\rho|^2} (\|\Delta_{aR'}\| + \|\Delta_{cR'}\| \\
&\quad + \|\Delta_R\| + 2\|\Delta_{RS'}\| + 2\|\Delta_{R'S}\| + \frac{2}{|\rho|^3} \|\Delta_{R'R}\| \Big] \\
&\quad + \frac{1}{|\rho|^2 \operatorname{Re}(\rho)} [(\|\Delta|_{aR''}| + |\rho| (\|\Delta_{aS''}\| + \|\Delta_{bR''}\|) + |\rho|^2 \|\Delta_{bS''}\|) \times (1 - e^{-\operatorname{Re}(\rho)}) \\
&\quad + (\|\Delta_{cR''}\| + |\rho| (\|\Delta_{cS''}\| + \|\Delta_{dR''}\|) + |\rho|^2 \|\Delta_{dS''}\|) \times (e^{-\operatorname{Re}(\rho)} - e^{-2\operatorname{Re}(\rho)}) \\
&\quad + (\|\Delta_{aR''}\| + |\rho| (\|\Delta_{aS''}\| + \|\Delta_{bR''}\|) + |\rho|^2 \|\Delta_{bS''}\|) \times (e^{-\operatorname{Re}(\rho)} - e^{-2\operatorname{Re}(\rho)}) \\
&\quad + (\|\Delta_{cR''}\| + |\rho| (\|\Delta_{cS''}\| + \|\Delta_{dR''}\|) + |\rho|^2 \|\Delta_{dS''}\|) \times (1 - e^{-\operatorname{Re}(\rho)}) \\
&\quad + \frac{1}{|\rho|^2 (\operatorname{Re}(\rho))^2} (\|\Delta_{R'}\| + 2|\rho| \|\Delta_{R'S'}\| + |\rho|^2 \|\Delta_{S'}\|) \times (1 - e^{-\operatorname{Re}(\rho)})^2 + \frac{2}{|\rho|^3 \operatorname{Re}(\rho)}
\end{aligned}
$$

$$\times (\|\Delta_{R''R}\| + |\rho|(\|\Delta_{R''S}\| + \|\Delta_{RS''}\|) + |\rho|^2\|\Delta_{SS''}\|) \times (e^{-\operatorname{Re}(\rho)} - e^{-2\operatorname{Re}(\rho)})$$
$$+ \frac{2}{|\rho|^3 \operatorname{Re}(\rho)}(\|\Delta_{R''R}\| + |\rho|(\|\Delta_{R''S}\| + \|\Delta_{RS''}\|) + |\rho|^2\|\Delta_{SS''}\|) \times (1 - e^{-\operatorname{Re}(\rho)}).$$

Then

$$\begin{aligned}
|\Phi(\rho)| \le \frac{4}{|\rho|^2(\cos(\frac{\pi}{4}+\frac{\delta}{2}))^2}\Big[&|\rho|(|\Delta_{ab}| + |\Delta_{cd}| + \|\Delta_{bS}\| + \|\Delta_{dS}\|) \\
&+ \frac{1}{|\rho|}(\|\Delta_{cR}\| + \|\Delta_{aR}\| + \|\Delta_{dR'}\| + \|\Delta_{cS'}\| + \|\Delta_{aS'}\| + \|\Delta_{bR'}\| \\
&+ 2\|\Delta_{SR}\| + 2\|\Delta_{S'S}\|)\frac{1}{|\rho|^3}(\|\Delta_{RR'}\| + \|\Delta_{RR'}\|) \\
&+ \frac{1}{2|\rho|^2(\cos(\frac{\pi}{4}+\frac{\delta}{2}))^2}[|\rho|^2|\Delta_{bd}| + |\rho|(|\Delta_{ad}| + |\Delta_{bc}| + \|\Delta_{dS}\| + \|\Delta_{bS}\|) \\
&+ |\Delta_{ac}| + \|\Delta_{bR}\| + \|\Delta_{aS}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_{dS'}\| + \|\Delta_{bS'}\| \\
&+ \|\Delta_{S}\| + \frac{1}{|\rho|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + \|\Delta_{bR'}\| + \|\Delta_{aS'}\| + \|\Delta_{cS'}\| \\
&+ \|\Delta_{dR'}\| + 2\|\Delta_{RS'}\| + 2\|\Delta_{S'S}\|) + \frac{1}{|\rho|^2}(\|\Delta_{aR'}\| + \|\Delta_{cR'}\| \\
&+ \|\Delta_{R}\| + 2\|\Delta_{RS'}\| + 2\|\Delta_{R'S}\| + \frac{2}{|\rho|^3}\|\Delta_{R'R}\|] \\
&+ \frac{2}{|\rho|^3\cos(\frac{\pi}{4}+\frac{\delta}{2})}[(\|\Delta_{aR''}\| + |\rho|(\|\Delta_{aS''}\| + \|\Delta_{bR''}\|) + |\rho|^2\|\Delta_{bS''}\|) \\
&+ (\|\Delta_{cR''}\| + |\rho|(\|\Delta_{cS''}\| + \|\Delta_{dR''}\|) + |\rho|^2\|\Delta_{dS''}\|)\Big] \\
&+ \frac{1}{|\rho|^4(\cos(\frac{\pi}{4}+\frac{\delta}{2}))^2}(\|\Delta_{R'}\| + 2|\rho|\|\Delta_{R'S'}\| + |\rho|^2\|\Delta_{S'}\|) \\
&+ \frac{4}{|\rho|^4\cos(\frac{\pi}{4}+\frac{\delta}{2})}(\|\Delta_{R''R}\| + |\rho|(\|\Delta_{R''S}\| + \|\Delta_{RS''}\|) + |\rho|^2\|\Delta_{SS''}\|).
\end{aligned} \tag{3.9}$$

Where we have used that $\operatorname{Re}(\rho) > |\rho|\cos(\frac{\pi}{4}+\frac{\delta}{2})$, $1-e^{-\operatorname{Re}(\rho)} < 1$, $1-e^{-2\operatorname{Re}(\rho)} < 1$, $|\rho|^2 e^{-\operatorname{Re}(\rho)} \le 2(\cos(\frac{\pi}{4}+\frac{\delta}{2}))^{-2}$ and $2|\rho|^2 e^{-2\operatorname{Re}(\rho)} \le (\cos(\frac{\pi}{4}+\frac{\delta}{2}))^{-2}$. There are several cases to analyze depending on the functions R_i and S_i, $i = \overline{1,2}$.

Case 1. Suppose that $\Delta_{bd} = 0$, $\Delta_{ad} - \Delta_{bc} + \Delta_{dS}(0) - \Delta_{bS}(1) = 0$,

$$\max(|\Delta_{ab}|, |\Delta_{ad}|, |\Delta_{bc}|, |\Delta_{cd}|, \|\Delta_{bS}\|, \|\Delta_{dS}\|) \ne 0$$

and $k_1 = \Delta_{ac}+\Delta_{aS}(1)+\Delta_{cS}(0)-\Delta_{bR}(1)-\Delta_{dR}(0)+\Delta_{S}(1,0)+\Delta_{bS'}(1)+\Delta_{dS'}(0) \ne 0$ From (3.3), we have for $|\rho|$ sufficiently large

$$\begin{aligned}
|\Delta(\lambda)| \ge e^{\operatorname{Re}(\rho)}\Big[&|\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_{S}(1,0) \\
&+ \Delta_{bS'}(1) + \Delta_{dS'}(0)| - \frac{1}{|\rho|}|\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) \\
&+ \Delta_{cS'}(0) - \Delta_{dR'}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) + \Delta_{SS'}(1,0) + \Delta_{SS'}(0,1)| \\
&- \frac{1}{|\rho|^2}|\Delta_{R}(0,1) - \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1)
\end{aligned}$$

$$- \Delta_{dR'}(1) - \Delta_{cR'}(0)| - \frac{1}{|\rho|^3}|\Delta_{R'R}(1,0) + \Delta_{R'R}(0,1)| - \Phi(\rho)\Big].$$

From (3.9) we deduce for $\rho \in \sum_{\frac{\delta}{2}}, |\rho| \geq r_0 > 0$.

$$|\Phi(\rho)| \leq \frac{c(r_0)}{|\rho|}.$$

Then, we have

$$\begin{aligned}
|\Delta(\lambda)| \geq e^{\operatorname{Re}(\rho)}[&|\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_S(1,0) \\
&+ \Delta_{bS'}(1) + \Delta_{dS'}(0)| - \frac{1}{|\rho|}|\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) \\
&+ \Delta_{cS'}(0) - \Delta_{dR'}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) + \Delta_{SS'}(1,0) + \Delta_{SS'}(0,1)| \\
&- \frac{1}{|\rho|^2}|\Delta_R(0,1) - \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1) \\
&- \Delta_{dR'}(1) - \Delta_{cR'}(0)| - \frac{1}{|\rho|^3}|\Delta_{R'R}(1,0) + \Delta_{R'R}(0,1)| - \frac{c(r_0)}{|\rho|}\Big],
\end{aligned}$$

we can now choose $r_0 > 0$, such that

$$\begin{aligned}
&\frac{1}{r_0}|\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) + \Delta_{cS'}(0) - \Delta_{dR'}(0) \\
&+ \Delta_{RS}(0,1) + \Delta_{RS}(1,0) + \Delta_{SS'}(1,0) + \Delta_{SS'}(0,1)| + \frac{1}{r_{0^2}}|\Delta_R(0,1) \\
&- \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1) - \Delta_{dR'}(1) - \Delta_{cR'}(0)| \\
&+ \frac{1}{r_0^3}|\Delta_{R'R}(1,0) + \Delta_{R'R}(0,1)| + \frac{c(r_0)}{|\rho|} \\
&\leq \frac{1}{2}|\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) \\
&\quad - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_S(1,0) + \Delta_{bS'}(1) + \Delta_{dS'}(0)|,
\end{aligned}$$

then, for $\rho \in \sum_{\frac{\delta}{2}}, |\rho| \geq r_0 > 0$, we get

$$|\Delta(\rho)| \geq \frac{e^{\operatorname{Re}(\rho)}}{2}|k_1|.$$

From (3.5), we deduce the following bound, valid for every $|\arg\rho| \leq \frac{\pi}{4} + \frac{\delta}{2}$, $\rho \neq 0$,

$$\|R(\lambda, L_p)\| \leq \frac{cH(\rho)}{|\rho|^{1+\frac{1}{p}}} \leq \frac{1}{|\rho|^{1/p}} \times \frac{cH(\rho)}{|\rho|},$$

where

$$\begin{aligned}
\frac{H(\rho)}{|\rho|} &= \frac{e^{\operatorname{Re}(\rho)}}{|\Delta(\rho^2)||\rho|}[|\rho|(|\Delta_{ab}| + |\Delta_{ad}| + |\Delta_{bc}| + |\Delta_{cd}| + \|\Delta_{bS}\| \\
&\quad + \|\Delta_{dS}\|) + (|\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| \\
&\quad + \|\Delta_S\|) + \frac{1}{|\rho|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^2}] \\
&= \frac{e^{\operatorname{Re}(\rho)}}{|\Delta(\rho^2)|}[(|\Delta_{ab}| + |\Delta_{ad}| + |\Delta_{bc}| + |\Delta_{cd}| + \|\Delta_{bS}\| + \|\Delta_{dS}\|) \\
&\quad + \frac{1}{|\rho|}(|\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_S\|)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{|\rho|^2}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^3}] \\
&\le \frac{2}{|k_1|}[(|\Delta_{ab}| + |\Delta_{ad}| + |\Delta_{bc}| + |\Delta_{cd}| + \|\Delta_{bS}\| + \|\Delta_{dS}\|) \\
&+ \frac{1}{|\rho|}(|\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_S\|) \\
&+ \frac{1}{|\rho|^2}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^3}],
\end{aligned}
$$

as $|\lambda| \to +\infty$, where

$$
\frac{cH(\rho)}{|\rho|} \le \frac{2c}{|k_1|}(|\Delta_{ab}| + |\Delta_{ad}| + |\Delta_{bc}| + |\Delta_{cd}| + \|\Delta_{bS}\| + \|\Delta_{dS}\|) = c_1
$$

then

$$
\|R(\lambda, L_p)\| \le \frac{c_1}{|\lambda|^{\frac{1}{2p}}}.
$$

Case 2. If $\Delta_{ab} = \Delta_{ad} = \Delta_{bc} = \Delta_{bd} = \Delta_{cd} = 0$, $\Delta_{bS} \equiv 0$, $\Delta_{dS} \equiv 0$, $\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_S(1, 0) = 0$,

$$
\max(|\Delta_{ac}|, \|\Delta_{aS}\|, \|\Delta_{cS}\|, \|\Delta_{bR}\|, \|\Delta_{dR}\|, \|\Delta_S\|) \ne 0
$$

and $k_2 = \Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) + \Delta_{cS'}(0) - \Delta_{dR'}(0) + \Delta_{RS}(0, 1) + \Delta_{RS}(1, 0) + \Delta_{SS'}(1, 0) + \Delta_{SS'}(0, 1) \ne 0$, we have the following bound, valid for $\lambda \in \sum_\delta$ and $|\lambda|$ big enough,

$$
\|R(\lambda, L_p)\| \le \frac{cH(\rho)}{|\rho|^{1+\frac{1}{p}}} \le \frac{1}{|\rho|^{1/p}} \times \frac{cH(\rho)}{|\rho|},
$$

where

$$
\begin{aligned}
\frac{H(\rho)}{|\rho|} &= \frac{e^{\operatorname{Re}(\rho)}}{|\rho||\Delta(\rho^2)|}[(|\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| \\
&\quad + \|\Delta_S\|) + \frac{1}{|\rho|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^2}] \\
&\le \frac{2}{|k_2|}[(|\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_S\|) \\
&\quad + \frac{1}{|\rho|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^2}],
\end{aligned}
$$

as $|\rho| \to +\infty$, we have

$$
\frac{cH(\rho)}{|\rho|} \le \frac{2c}{|k_2|}(|\Delta_{ac}| + \|\Delta_{aS}\| + \|\Delta_{bR}\| + \|\Delta_{dR}\| + \|\Delta_{cS}\| + \|\Delta_S\|) = c_2
$$

then

$$
\|R(\lambda, L_p)\| \le \frac{c_2}{|\lambda|^{\frac{1}{2p}}}.
$$

Case 3. If $\Delta_{ab} = \Delta_{ac} = \Delta_{ad} = \Delta_{bc} = \Delta_{db} = \Delta_{cd} = 0$, $\Delta_{bR} \equiv 0, \Delta_{dR} \equiv 0$, $\Delta_{aS} \equiv 0$, $\Delta_{bS} \equiv 0$, $\Delta_{cS} \equiv 0$, $\Delta_{dS} \equiv 0$, $\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) = 0$, $\max(\|\Delta_{aR}\|, \|\Delta_{cR}\|, \|\Delta_{RS}\|) \neq 0$ and $k_3 = \Delta_R(0,1) - \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1) - \Delta_{dR'}(1) - \Delta_{cR'}(0) \neq 0$. Similarly, we get

$$\|R(\lambda, L_p)\| \leq \frac{cH(\rho)}{|\rho|^{1+\frac{1}{p}}} \leq \frac{1}{|\rho|^{1/p}} \times \frac{cH(\rho)}{|\rho|},$$

where

$$\begin{aligned}
\frac{H(\rho)}{|\rho|} &= \frac{e^{\mathrm{Re}(\rho)}}{|\rho||\Delta(\rho^2)|}\Big[\frac{1}{|\rho|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|^2}\Big] \\
&= \frac{e^{\mathrm{Re}(\rho)}}{|\rho|^2|\Delta(\rho^2)|}\Big[(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|}\Big] \\
&\leq \frac{2}{|k_3|}\big[(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) + \frac{\|\Delta_R\|}{|\rho|}\big],
\end{aligned}$$

as $|\rho| \to +\infty$, we have

$$\frac{cH(\rho)}{|\rho|} \leq \frac{2c}{|k_3|}(\|\Delta_{aR}\| + \|\Delta_{cR}\| + 2\|\Delta_{RS}\|) = c_3$$

then, we have

$$\|R(\lambda, L_p)\| \leq \frac{c_3}{|\lambda|^{\frac{1}{2p}}}.$$

Case 4. If $a_i = b_i = c_i = d_i = 0$, $S_i \equiv 0$ where $i = \overline{1,2}$, $\Delta_R(0,1) = 0$, $\|\Delta_R\| \neq 0$ and $k_4 = \Delta_{R'R}(1,0) + \Delta_{R'R}(0,1) \neq 0$, again in this case, we have

$$\|R(\lambda, L_p)\| \leq \frac{cH(\rho)}{|\rho|^{1+\frac{1}{p}}} \leq \frac{1}{|\rho|^{1/p}} \times \frac{cH(\rho)}{|\rho|},$$

where

$$\frac{H(\rho)}{|\rho|} = \frac{e^{\mathrm{Re}(\rho)}}{|\rho||\Delta(\rho^2)|} \times \frac{\|\Delta_R\|}{|\rho|^2} = \frac{e^{\mathrm{Re}(\rho)}}{|\rho|^2|\Delta(\rho^2)|} \times \frac{\|\Delta_R\|}{|\rho|} \leq \frac{2}{k_4}\|\Delta_R\|,$$

as $|\rho| \to +\infty$, we have

$$\frac{cH(\rho)}{|\rho|} \leq \frac{2c}{|k_4|}\|\Delta_R\| = c_4$$

then

$$\|R(\lambda, L_p)\| \leq \frac{c_4}{|\lambda|^{\frac{1}{2p}}}.$$

Definition 3.5. The boundary conditions in (1.1) are called non regular if the functions R_i, $S_i \in C^2([0,1], \mathbb{C})$, $i = \overline{1,2}$ and if and only if one of the following conditions holds.

(1) $\Delta_{bd} = 0$, $\Delta_{ad} - \Delta_{bc} + \Delta_{dS}(0) - \Delta_{bS}(1) = 0$,

$$\max(|\Delta_{ab}|, |\Delta_{ad}|, |\Delta_{bc}|, |\Delta_{cd}|, \|\Delta_{bS}\|, \|\Delta_{dS}\|) \neq 0$$

and $\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_S(1,0) + \Delta_{bS'}(1) + \Delta_{dS'}(0) \neq 0$

(2) $\Delta_{ab} = \Delta_{ad} = \Delta_{bc} = \Delta_{bd} = \Delta_{cd} = 0$, $\Delta_{bS} \equiv 0, \Delta_{dS} \equiv 0$, $\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_S(1,0) = 0$,
$$\max(|\Delta_{ac}|, \|\Delta_{aS}\|, \|\Delta_{cS}\|, \|\Delta_{bR}\|, \|\Delta_{dR}\|, \|\Delta_S\|) \neq 0$$
and $\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) + \Delta_{cS'}(0) - \Delta_{dR'}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) + \Delta_{SS'}(1,0) + \Delta_{SS'}(0,1) \neq 0$

(3) $\Delta_{ab} = \Delta_{ac} = \Delta_{ad} = \Delta_{bc} = \Delta_{db} = \Delta_{cd} = 0$, $\Delta_{bR} \equiv 0$, $\Delta_{dR} \equiv 0$, $\Delta_{aS} \equiv 0$, $\Delta_{bS} \equiv 0$, $\Delta_{cS} \equiv 0$, $\Delta_{dS} \equiv 0$, $\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) = 0$, $\max(\|\Delta_{aR}\|, \|\Delta_{cR}\|, \|\Delta_{RS}\|) \neq 0$ and $\Delta_R(0,1) - \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1) - \Delta_{dR'}(1) - \Delta_{cR'}(0) \neq 0$

(4) If $a_i = b_i = c_i = d_i = 0$, $S_i \equiv 0$ where $i = \overline{1,2}$, $\Delta_R(0,1) = 0$, $\|\Delta_R\| \neq 0$ and $\Delta_{R'R}(1,0) + \Delta_{R'R}(0,1) \neq 0$.

The above arguments prove the following theorem.

Theorem 3.6. *If the boundary value conditions in* (1.1) *are non regular, then* $\sum_\delta \subset \rho(L_p)$ *for sufficiently large* $|\lambda|$ *and there exists* $c > 0$ *such that*
$$\|R(\lambda, L_p)\| \leq \frac{c}{|\lambda|^{\frac{1}{2p}}}.$$

Remark 3.7. From theorem 3.6 it follows that the operator L_p, for $p \neq \infty$, generates an analytic semi group with singularities [30] of type $A(2p-1, 4p-1)$.

Remark 3.8. As particular cases, we obtain the results of [5, 6].

Remark 3.9. Contrarily to the regular case, we have a loss of $\frac{1}{2}$ in the resolvent estimate.

Remark 3.10. It is not difficult to see that the above definitions of regularity and non regularity of boundary conditions do not depend on possible elementary simplifications on the boundary conditions or integrations by parts.

3.4. **Applications.** In the following, we apply the results obtained to study of a class of a mixed problem for a parabolic equation with weighted integral boundary condition combined with anther two point boundary condition of the form
$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} - a\frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x) \\
\begin{aligned}
L_1(u) := {} & a_1 u(0,t) + b_1 u'(0,t) + c_1 u(1,t) + d_1 u'(1,t) \\
& + \int_0^1 R_1(\xi) u(t,\xi) d\xi + \int_0^1 S_1(\xi) u'(t,\xi) d\xi = 0, \\
L_2(u) := {} & a_2 u(0,t) + b_2 u'(0,t) + c_2 u(1,t) + d_2 u'(1,t) \\
& + \int_0^1 R_2(\xi) u(t,\xi) d\xi + \int_0^1 S_2(\xi) u'(t,\xi) d\xi = 0,
\end{aligned} \\
u(0,x) = u_0(x),
\end{gathered} \tag{3.10}$$
where $(t,\xi) \in [0,T] \times [0,1]$. Boundary-value problems for parabolic equations with integral boundary conditions are studied by [1, 3, 7, 18, 19, 21, 30] using various methods. For instance, the potential method in [3] and [21], Fourier method in [1, 18, 19, 20] and the energy inequalities method has been used in [7, 8, 40]. In our case, we apply the method of operator differential equation. The study of the problem is then reduced to a Cauchy problem for a parabolic abstract differential

equation, where the operator coefficients has been previously studied. For this purpose, let E, E_1 and E_2 be Banach spaces. Introduce two Banach spaces

$$C_\mu((0,T],E) = \big\{ f \in C((0,T],E) : \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| < +\infty \big\}, \quad \mu \geq 0,$$

$$\begin{aligned} C_\mu^\gamma((0,T],E) = \Big\{ f \in C((0,T],E) : \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| \\ + \sup_{0<t<t+h\leq T} \|f(t+h) - f(t)\| h^{-\gamma} t^\mu < +\infty \Big\}, \quad \mu \geq 0,\ \gamma \in (0,1], \end{aligned}$$

and a linear space

$$C^1((0,T],E_1,E_2) = \big\{ f \in C((0,T],E_1) \cap C^1((0,T],E_2) \big\}, \quad E_1 \subset E_2,$$

where $C((0,T],E)$ and $C^1((0,T],E)$ are spaces of continuous and continuously differentiable, respectively, vector-functions from $(0,T]$ into E. We denote, for a linear operator A in a Banach space E, by

$$\begin{gathered} E(A) = \{ u \in D(A) : \|u\|_{E(A)} = (\|Au\|^2 + \|u\|^2)^{\frac{1}{2}} \}, \\ C^1((0,T],E(A),E) = \{ f \in C((0,T],E(A)) : f' \in C((0,T],E) \}. \end{gathered}$$

Let us derive a theorem which was proved by various methods in [30] and [37]. Consider, in a Banach space E, the Cauchy problem

$$\begin{gathered} u'(t) = Au(t) + f(t), \quad t \in [0,T], \\ u(0) = u_0, \end{gathered} \tag{3.11}$$

where A is, generally speaking, unbounded linear operator in E, u_0 is a given element of E, $f(t)$ is a given vector-function and $u(t)$ is an unknown vector-function in E.

Theorem 3.11. *Let the following conditions be satisfied:*

(1) *A is a closed linear operator in a Banach space E and for some $\beta \in (0,1], \alpha > 0$*
$$\|R(\lambda, A)\| \leq C|\lambda|^{-\beta}, \quad |\arg\lambda| \leq \frac{\pi}{2} + \alpha, |\lambda| \to +\infty;$$
(2) *$f \in C_\mu^\gamma((0,T],E)$ for some $\gamma \in (1-\beta, 1], \mu \in [0,\beta)$;*
(3) *$u_0 \in D(A)$.*

Then the Cauchy problem (3.11) *has a unique solution*

$$u \in C((0,T],E) \cap C^1((0,T],E(A),E),$$

and for the solution u,

$$\begin{gathered} \|u(t)\| \leq C(\|Au_0\| + \|u_0\| + \|f\|_{C_\mu^\gamma((0,T],E)}), t \in (0,T], \\ \|u'(t)\| + \|Au(t)\| \leq C(t^{\beta-1}(\|Au_0\| + \|u_0\|) + t^{\beta-\mu-1}\|f\|_{C_\mu^\gamma((0,t],E)}), t \in (0,T]. \end{gathered}$$

As a result of this we get the following theorem.

Theorem 3.12. *Let the following conditions be satisfied:*

(1) *$a \neq 0$, $|\arg a| < \pi/2$,*

(2) *The functions* $R(t), S(t) \in C^2([0,1],\mathbb{C})$, $i = \overline{1,2}$, *and one of the following conditions is satisfied* $\Delta_{bd} = 0$, $\Delta_{ad} - \Delta_{bc} + \Delta_{dS}(0) - \Delta_{bS}(1) = 0$, $\max(|\Delta_{ab}|, |\Delta_{ad}|, |\Delta_{bc}|, |\Delta_{cd}|, \|\Delta_{bS}\|, \|\Delta_{dS}\|) \neq 0$ *and* $\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_S(1,0) + \Delta_{bS'}(1) + \Delta_{dS'}(0) \neq 0$ *or* $\Delta_{ab} = \Delta_{ad} = \Delta_{bc} = \Delta_{bd} = \Delta_{cd} = 0$, $\Delta_{bS} \equiv 0$, $\Delta_{dS} \equiv 0$, $\Delta_{ac} + \Delta_{aS}(1) + \Delta_{cS}(0) - \Delta_{bR}(1) - \Delta_{dR}(0) + \Delta_S(1,0) = 0$,
$$\max(|\Delta_{ac}|, \|\Delta_{aS}\|, \|\Delta_{cS}\|, \|\Delta_{bR}\|, \|\Delta_{dR}\|, \|\Delta_S\|) \neq 0$$
and $\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{bR'}(1) - \Delta_{aS'}(1) + \Delta_{cS'}(0) - \Delta_{dR'}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) + \Delta_{SS'}(1,0) + \Delta_{SS'}(0,1) \neq 0$ *or* $\Delta_{ab} = \Delta_{ac} = \Delta_{ad} = \Delta_{bc} = \Delta_{db} = \Delta_{cd} = 0$, $\Delta_{bR} \equiv 0$, $\Delta_{dR} \equiv 0, \Delta_{aS} \equiv 0$, $\Delta_{bS} \equiv 0$, $\Delta_{cS} \equiv 0$, $\Delta_{dS} \equiv 0$, $\Delta_{aR}(1) - \Delta_{cR}(0) + \Delta_{RS}(0,1) + \Delta_{RS}(1,0) = 0$,
$$\max(\|\Delta_{aR}\|, \|\Delta_{cR}\|, \|\Delta_{RS}\|) \neq 0$$
and $\Delta_R(0,1) - \Delta_{RS'}(0,1) + \Delta_{RS'}(1,0) - \Delta_{R'S}(1,0) + \Delta_{R'S}(0,1) - \Delta_{dR'}(1) - \Delta_{cR'}(0) \neq 0$ *or If:* $a_i = b_i = c_i = d_i = 0, S_i \equiv 0$ *where* $i = \overline{1,2}$, $\Delta_R(0,1) = 0$, $\|\Delta_R\| \neq 0$ *and* $\Delta_{R'R}(1,0) + \Delta_{R'R}(0,1) \neq 0$

(3) $f \in C^\gamma_\mu((0,T], L^q(0,1))$ *for some* $\gamma \in (1 - \frac{1}{2q}, 1]$ *and some* $\mu \in [0, \frac{1}{2q})$,

(4) $u_0 \in W^{2,q}((0,1), L_i u = 0, i = \overline{1,2})$.

Then problem (3.10) has a unique solution
$$u \in C((0,T], L^q(0,1)) \cap C^1((0,T], W^{2,q}(0,1), L^q(0,1))$$
and for this solution we have the estimates:
$$\|u(t,.)\|_{L^q(0,1)} \leq c(\|u_0\|_{W^{2,q}(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))}), t \in (0,T], \tag{3.12}$$
$$\begin{aligned} &\|u''(t,.)\|_{L^q(0,1)} + \|u'(t,.)\|_{L^q(0,1)} \\ &\leq c(t^{\frac{1}{2q}-1}\|u_0\|_{W^{2,q}(0,1)} + t^{\frac{1}{2q}-\mu-1}\|f\|_{C^\gamma_\mu((0,t],L^q(0,1))}), t \in (0,T]. \end{aligned} \tag{3.13}$$

Proof. In the space $L^q(0,1)$, $1 \leq q < +\infty$, we consider the operator A defined by
$$A(u) = au''(x), \quad D(A) = \{u \in W^{2,q}(0,1),\ L_i(u) = 0, i = \overline{1,2}\}.$$
Then problem (3.10) can be written as
$$\begin{gathered} u'(t) = Au(t) + f(t), \\ u(0) = u_0, \end{gathered}$$
where $u(t) = u(t,.), f(t) = f(t,.)$, and $u_0 = u_0(.)$ are functions with values in the Banach space $L^q(0,1)$. From Theorem 3.6 we conclude that $\|R(\lambda, A)\| \leq c|\lambda|^{-\frac{1}{2q}}$, for $|\arg\lambda| \leq \frac{\pi}{2} + \alpha$, as $|\lambda| \to +\infty$. Then , from Theorem 3.11 the problem (3.10) has a unique solution
$$u \in C((0,T], L^q(0,1)) \cap C^1((0,T], W^{2,q}(0,1), L^q(0,1))$$
and we have the following estimates
$$\|u(t,.)\|_{L^q(0,1)} \leq c(\|Au_0\|_{L^q(0,1)} + \|u_0\|_{L^q(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))}), \tag{3.14}$$
$$\begin{aligned} &\|u'(t,.)\|_{L^q(0,1)} + \|Au(t,.)\|_{L^q(0,1)} \\ &\leq c(t^{\frac{1}{2q}-1}(\|Au_0\|_{L^q(0,1)} + \|u_0\|_{L^q(0,1)}) + t^{\frac{1}{2q}-\mu-1}\|f\|_{C^\gamma_\mu((0,t],L^q(0,1))}), \end{aligned} \tag{3.15}$$
where $t \in [0,T]$, from (3.14) we get
$$\|u(t,.)\|_{L^q(0,1)} \leq c(\|u_0''\|_{L^q(0,1)} + \|u_0\|_{L^q(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))})$$

$$\leq c(\|u_0\|_{W^{2,q}(0,1)} + \|f\|_{C_\mu((0,t],L^q(0,1))}), \quad t \in [0,T].$$

And from (3.15) we get

$$\begin{aligned}
&\|u'(t,.)\|_{L^q(0,1)} + \|u''(t,.)\|_{L^q(0,1)} \\
&\leq c\Big(t^{\frac{1}{2q}-1}(\|u_0\|_{L^q(0,1)} + \|u_0''\|_{L^q(0,1)}) + t^{\frac{1}{2q}-\mu-1}\|f\|_{C_\mu^\gamma((0,t],L^q(0,1))}\Big) \\
&\leq c\Big(t^{\frac{1}{2q}-1}\|u_0\|_{W^{2,q}(0,1)} + t^{\frac{1}{2q}-\mu-1}\|f\|_{C_\mu^\gamma((0,t],L^q(0,1))}\Big), \quad t \in [0,T].
\end{aligned}$$

□

Acknowledgements. The authors gratefully acknowledge the useful suggestions of the anonymous referee.

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Mohamed Denche
Laboratoire Equations Differentielles, Departement de Mathematiques, Faculte des Sciences, Universite Mentouri, 25000 Constantine, Algeria
E-mail address: denech@wissal.dz

Abderrahmane Meziani
Laboratoire Equations Differentielles, Departement de Mathematiques, Faculte des Sciences, Universite Mentouri, 25000 Constantine, Algeria
E-mail address: mezianiar@yahoo.fr