Electronic Journal of Differential Equations, Vol. 2007(2007), No. 61, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

BLOW-UP SOLUTIONS FOR N COUPLED SCHRÖDINGER EQUATIONS

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Abstract. It is proved that blow-up solutions to N coupled Schrödinger equations

$$i\varphi_{jt} + \varphi_{jxx} + \mu_j|\varphi_j|^{p-2}\varphi_j + \sum_{k\neq j,\ k=1}^{N} \beta_{kj}|\varphi_k|^{p_k}|\varphi_j|^{p_j-2}\varphi_j = 0$$

exist only under the condition that the initial data have strictly negative energy.

1. Introduction

In this paper, we consider the existence of blow-up solutions of the N coupled Schrödinger equations

$$\begin{gathered} i\varphi_{jt} + \varphi_{jxx} + \mu_j|\varphi_j|^{p-2}\varphi_j + \sum_{k\neq j,\ k=1}^{N} \beta_{kj}|\varphi_k|^{p_k}|\varphi_j|^{p_j-2}\varphi_j = 0, \\ \varphi_j(x,t)\big|_{t=0} = \psi_j(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $i = \sqrt{-1}$, $\varphi_j = \varphi_j(x,t) : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{C}$, $j,k \in \{1,\dots,N\}$ and μ_j, $\beta_{kj} \in \mathbb{R}$. System of this kind appears in several branches of physics, such as in the study of interactions of waves with different polarizations [3] or in the description of nonlinear modulations of two monochromatic waves [9].

When $p = 4$, $p_j = 2$, and $p_k = 2$, the solution φ_j of (1.1) denotes the jth component of the beam in Kerr-like photo refractive media [1]. The constants β_{kj} is the interaction between the kth and the jth component of the beam. As $\beta_{kj} > 0$, the interaction is attractive while the interaction is repulsive if $\beta_{kj} < 0$. Moreover, the system (1.1) is integrable and there are various analytical and numerical results on solitary wave solutions of the general N coupled Schrödinger equations [6, 8].

When $2 < p < 6$, $2 \le p_k + p_j < 6$ and $N = 2$, the existence and stability of standing wave, which is a trivial global solution, of (1.1) have been studied by Cipolatti et al [5]. Also when $2 < p < 6$ and $2 \le p_k + p_j < 6$, for any

2000 *Mathematics Subject Classification.* 35Q55, 35B35.
Key words and phrases. Blow-up solutions; coupled Schrödinger equations.

Submitted May 29, 2006. Published April 22, 2007.
J. Chen was supported by grant 10501006 from the Youth Foundation of NSFC, by the China Post-Doc Science Foundation, and by the Program NCETFJ.

$j, k \in \{1, \dots, N\}$, we know from [4] that for any

$$\overrightarrow{\varphi}(x,0) = (\varphi_1(x,0), \dots, \varphi_N(x,0)) = \overrightarrow{\psi} = (\psi_1(x), \dots, \psi_N(x)) \in (H^1(\mathbb{R}))^N,$$

Equation (1.1) admits a unique global solution $\overrightarrow{\varphi} \in C(\mathbb{R}_+, (H^1(\mathbb{R}))^N)$.

The main purpose here is to prove the existence of blow-up solutions of (1.1) only under the condition of the initial data with strictly negative energy. The main result is the following theorem.

Theorem 1.1. *Let $p = 6$, $p_k + p_j = 6$ and $\mu_j \geq 0$, $\beta_{kj} > 0$ with $\theta_{kj} := \frac{\beta_{kj}}{p_j} = \frac{\beta_{jk}}{p_k} := \theta_{jk}$, $p_k, p_j \geq 2$. If $E(\overrightarrow{\psi}) < 0$(for the definition of E, see Proposition 2.1), then the solution of* (1.1) *with initial data $\overrightarrow{\psi}$ must blow up in finite time.*

We emphasize that when $N = 1$, i.e. no coupling terms, the blow up problem has been studied extensively, see e.g. [10, 7, 4]. But as far as we know, there is no blow-up result to the N coupled Schrödinger equations. The main contribution here is to overcome the additional difficulties created by the coupling terms and then prove Theorem 1.1.

This paper is organized as follows. In Section 2, we give some preliminaries and derive a variant of virial identity which generalizes some previous works for the single equation. Section 3 is devoted to the proof of Theorem 1.1.

Notation. As above and henceforth, the integral $\int_{\mathbb{R}} \dots dx$ is simply denoted by $\int \dots$. For any t, the function $x \mapsto \varphi_j(x,t)$ is simply denoted by $\varphi_j(t)$. $\overline{f}$ denotes the complex conjugate of f. f_x and f_t denote the derivative of f with respect to x and t, respectively. By $f^{(m)}$ we denote the mth order derivatives of f. $\|\cdot\|_{L^q}$ denotes the norm in $L^q(\mathbb{R})$ or $(L^q(\mathbb{R}))^N$ which will be understood from the context. Re denotes the real part and Im the imaginary part.

2. Preliminaries

Throughout this paper, we always assume that the conditions of Theorem 1.1 hold. The following proposition is useful in what follows.

Proposition 2.1. *For any $\overrightarrow{\psi} = (\psi_1(x), \dots, \psi_N(x)) \in (H^1(\mathbb{R}))^N$, there is $T > 0$ and a unique solution $\overrightarrow{\varphi} \in C([0,T), (H^1(\mathbb{R}))^N)$ satisfying* (1.1). *Moreover, there holds the following conservation laws:*

$$\int |\varphi_j(t)|^2 \equiv \int |\psi_j|^2, \tag{2.1}$$

$$E(\overrightarrow{\varphi}(t)) = \sum_{j=1}^N \int \Big(|\varphi_{jx}|^2 - \frac{2}{p}\mu_j|\varphi_j|^p\Big) - 2\sum_{k<j} \theta_{kj} \int |\varphi_k|^{p_k}|\varphi_j|^{p_j} \equiv E(\overrightarrow{\psi}). \tag{2.2}$$

Proof. The existence of the local solution $\overrightarrow{\varphi}$ follows from [4]. We only sketch the proof on the conservative laws. Firstly, multiplying (1.1) by $\overline{\varphi}_j$, integrating over $\mathbb{R}$ and taking imaginary part, we obtain (2.1). Secondly, it is deduced from multiplying (1.1) by $\overline{\varphi}_{jt}$, integrating over $\mathbb{R}$ and taking real part that

$$\int \Big(-\frac{1}{2}|\varphi_{jx}|^2 + \frac{\mu_j}{p}|\varphi_j|^p\Big)_t + \sum_{k\neq j} \frac{\beta_{kj}}{p_j} \int |\varphi_k|^{p_k}(|\varphi_j|^{p_j})_t = 0. \tag{2.3}$$

Similarly, for (1.1) with k instead of j, we have

$$\int\Big(-\frac{1}{2}|\varphi_{kx}|^2+\frac{\mu_k}{p}|\varphi_k|^p\Big)_t+\sum_{j\neq k}\frac{\beta_{jk}}{p_k}\int|\varphi_j|^{p_j}(|\varphi_k|^{p_k})_t=0. \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$\sum_{j=1}^N\int\Big(-\frac{1}{2}|\varphi_{jx}|^2+\frac{\mu_j}{p}|\varphi_j|^p\Big)_t+\sum_{k<j}\theta_{kj}\int\Big(|\varphi_k|^{p_k}|\varphi_j|^{p_j}\Big)_t=0. \tag{2.5}$$

Then (2.2) holds. □

Next we derive a variant of virial identity.

Lemma 2.2. *Let φ_j be a local smooth solution of* (1.1) *with $\varphi_j(x,0)=\psi_j(x)$. For real function $\phi\in W^{3,\infty}(\mathbb{R})$, define $\Phi(x)=\int_0^x\phi(y)dy$. Then*

$$\begin{aligned}
&\sum_{j=1}^N\operatorname{Im}\int\phi\psi_j\overline{\psi}_{jx}-\sum_{j=1}^N\operatorname{Im}\int\phi\varphi_j(t)\overline{\varphi}_{jx}(t)\\
&=\int_0^t\Big\{2\sum_{j=1}^N\int|\varphi_{jx}|^2\phi_x-\frac{1}{2}\sum_{j=1}^N\int|\varphi_j|^2\phi^{(3)}+\frac{2-p}{p}\sum_{j=1}^N\mu_j\int|\varphi_j|^p\phi_x\\
&\quad-(p-2)\sum_{k<j}\theta_{kj}\int|\varphi_k|^{p_k}|\varphi_j|^{p_j}\phi_x\Big\}d\tau,
\end{aligned} \tag{2.6}$$

and

$$\int\Phi|\varphi_j|^2=\int\Phi|\psi_j|^2-2\int_0^t\int\operatorname{Im}\phi\varphi_j\overline{\varphi}_{jx}\,dx\,d\tau. \tag{2.7}$$

Proof. Let φ_j be a smooth solution of (1.1). Firstly, multiplying (1.1) by $\phi\overline{\varphi}_{jx}$, integrating over $\mathbb{R}$ and taking the real part, we obtain

$$-\operatorname{Im}\int\phi\varphi_{jt}\overline{\varphi}_{jx}+\int\Big(\frac{1}{2}\phi(|\varphi_{jx}|^2)_x+\frac{\mu_j}{p}\phi(|\varphi_j|^p)_x+\sum_{k\neq j}\theta_{kj}|\varphi_k|^{p_k}(|\varphi_j|^{p_j})_x\phi\Big)=0. \tag{2.8}$$

From

$$\begin{gathered}
-\operatorname{Im}\int\phi\varphi_{jt}\overline{\varphi}_{jx}=-\frac{d}{dt}\operatorname{Im}\int\phi\varphi_j\overline{\varphi}_{jx}+\operatorname{Im}\int\phi\varphi_j\overline{\varphi}_{jxt},\\
\operatorname{Im}\int\phi\varphi_j\overline{\varphi}_{jxt}=-\operatorname{Im}\int\phi_x\overline{\varphi}_{jt}\varphi_j+\operatorname{Im}\int\phi\varphi_{jt}\overline{\varphi}_{jx},
\end{gathered}$$

we obtain

$$-\operatorname{Im}\int\phi\varphi_{jt}\overline{\varphi}_{jx}=-\frac{1}{2}\frac{d}{dt}\operatorname{Im}\int\phi\varphi_j\overline{\varphi}_{jx}-\frac{1}{2}\operatorname{Im}\int\phi_x\overline{\varphi}_{jt}\varphi_j. \tag{2.9}$$

It is deduced from (2.8) and (2.9) that

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\operatorname{Im}\int\phi\varphi_j\overline{\varphi}_{jx}-\frac{1}{2}\operatorname{Im}\int\phi_x\overline{\varphi}_{jt}\varphi_j+\int\Big(\frac{1}{2}\phi(|\varphi_{jx}|^2)_x\\
&+\frac{\mu_j}{p}\phi(|\varphi_j|^p)_x+\sum_{k\neq j}\theta_{kj}|\varphi_k|^{p_k}(|\varphi_j|^{p_j})_x\phi\Big)=0.
\end{aligned} \tag{2.10}$$

For (1.1) with k instead of j, we obtain by a similar argument that

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\operatorname{Im}\int \phi\varphi_k\overline{\varphi}_{kx} - \frac{1}{2}\operatorname{Im}\int \phi_x\overline{\varphi}_{kt}\varphi_k + \int\Big(\frac{1}{2}\phi(|\varphi_{kx}|^2)_x \\ &+ \frac{\mu_k}{p}\phi(|\varphi_k|^p)_x + \sum_{j\neq k}\theta_{jk}|\varphi_j|^{p_j}(|\varphi_k|^{p_k})_x\phi\Big) = 0. \end{aligned} \tag{2.11}$$

Secondly, multiplying the complex conjugate of (1.1) by $\varphi_j\phi_x$, integrating by parts and taking the real part, we get that

$$-\operatorname{Im}\int \phi_x\overline{\varphi}_{jt}\varphi_j = \int\Big(-\phi_x|\varphi_{jx}|^2+\frac{1}{2}|\varphi_j|^2\phi^{(3)}+\mu_j\phi_x|\varphi_j|^p+\sum_{k\neq j}\beta_{kj}|\varphi_k|^{p_k}|\varphi_j|^{p_j}\phi_x\Big). \tag{2.12}$$

Similarly,

$$-\operatorname{Im}\int \phi_x\overline{\varphi}_{kt}\varphi_k = \int\Big(-\phi_x|\varphi_{kx}|^2+\frac{1}{2}|\varphi_k|^2\phi^{(3)}+\mu_k\phi_x|\varphi_k|^p+\sum_{j\neq k}\beta_{jk}|\varphi_j|^{p_j}|\varphi_k|^{p_k}\phi_x\Big). \tag{2.13}$$

We now obtain from (2.10)–(2.13) that

$$\begin{aligned} &-\frac{d}{dt}\operatorname{Im}\int \phi\varphi_j\overline{\varphi}_{jx} - 2\int\phi_x|\varphi_{jx}|^2 + \frac{1}{2}\int|\varphi_j|^2\phi^{(3)} + \frac{p-2}{p}\mu_j\int\phi_x|\varphi_j|^p \\ &+ \sum_{k\neq j}\beta_{kj}\int|\varphi_k|^{p_k}|\varphi_j|^{p_j}\phi_x + \sum_{k\neq j}\frac{2\beta_{kj}}{p_j}\int|\varphi_k|^{p_k}(|\varphi_j|^{p_j})_x\phi = 0 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &-\frac{d}{dt}\operatorname{Im}\int \phi\varphi_k\overline{\varphi}_{kx} - 2\int\phi_x|\varphi_{kx}|^2 + \frac{1}{2}\int|\varphi_k|^2\phi^{(3)} + \frac{p-2}{p}\mu_k\int\phi_x|\varphi_k|^p \\ &+ \sum_{j\neq k}\beta_{jk}\int|\varphi_j|^{p_j}|\varphi_k|^{p_k}\phi_x + \sum_{j\neq k}\frac{2\beta_{jk}}{p_k}\int|\varphi_j|^{p_j}(|\varphi_k|^{p_k})_x\phi = 0. \end{aligned} \tag{2.15}$$

It follows that

$$\begin{aligned} &-\frac{d}{dt}\sum_{j=1}^N\operatorname{Im}\int\phi\varphi_j\overline{\varphi}_{jx} - 2\sum_{j=1}^N\int\phi_x|\varphi_{jx}|^2 + \frac{1}{2}\sum_{j=1}^N\int|\varphi_j|^2\phi^{(3)} \\ &+ \frac{p-2}{p}\sum_{j=1}^N\mu_j\int\phi_x|\varphi_j|^p + (p-2)\sum_{k<j}\theta_{kj}\int|\varphi_k|^{p_k}|\varphi_j|^{p_j}\phi_x = 0. \end{aligned} \tag{2.16}$$

Hence (2.6) holds. Finally, multiplying the complex conjugate of (1.1) by $\Phi\varphi_j$, integrating by parts and taking the imaginary part, we obtain

$$-\operatorname{Re}\int\Phi\varphi_j\overline{\varphi}_{jt} + \operatorname{Im}\int\Phi\varphi_j\overline{\varphi}_{jxx} = 0,$$

which implies

$$\frac{d}{dt}\int\Phi|\varphi_j|^2 = -2\operatorname{Im}\int\phi\varphi_j\overline{\varphi}_{jx}. \tag{2.17}$$

So (2.7) easily follows. The proof is complete. $\square$

3. Proof of Theorem 1.1

In this section, we will borrow an idea from [7, 10] to prove Theorem 1.1. Firstly we introduce two lemmas from [10].

Lemma 3.1 ([10, Lemma 2.1]). *Let $u \in H^1(\mathbb{R})$ and ρ be a real valued function in $W^{1,\infty}(\mathbb{R})$. Then for any $r > 0$, we have*

$$\|\rho u\|_{L^\infty(|x|>r)} \le \|u\|^{1/2}_{L^2(|x|>r)} \Big(2\|\rho^2 u_x\|_{L^2(|x|>r)} + \|u(\rho^2)_x\|_{L^2(|x|>r)}\Big)^{1/2}. \tag{3.1}$$

Lemma 3.2 ([10, Lemma 2.3]). *Let $v(x)$ be in L^2. We define $R(x)$ such that $R(x) = |x|$ for $|x| < 1$ and $R(x) = 1$ for $|x| > 1$. Put $v_\varepsilon(x) = \varepsilon^{-1/2} v(x/\varepsilon)$ for $\varepsilon > 0$. Then for any $\delta > 0$, there exists an $\varepsilon_0 > 0$ such that $\|Rv_\varepsilon\|_{L^2} \le \delta$ for $0 < \varepsilon < \varepsilon_0$.*

We are now in a position to prove the theorem. Observe that $p = 6$, $p_j + p_k = 6$ for $j, k \in \{1, \dots, N\}$ and the solution $\varphi_j(x,t)$ of (1.1) has the following scaling invariance. More precisely, if we put

$$\varphi_{\varepsilon j}(x,t) = \varepsilon^{-1/2}\varphi_j(x/\varepsilon, t/\varepsilon^2), \quad \varphi_{\varepsilon k}(x,t) = \varepsilon^{-1/2}\varphi_k(x/\varepsilon, t/\varepsilon^2) \tag{3.2}$$

for $\varepsilon > 0$, then $\varphi_{\varepsilon j}$ and $\varphi_{\varepsilon k}$ also satisfy (1.1) and (1.1) with k instead of j and with initial data $\varphi_{\varepsilon j}(x,0) = \psi_{\varepsilon j} = \varepsilon^{-1/2}\psi_j(x/\varepsilon)$ and $\varphi_{\varepsilon k}(x,0) = \psi_{\varepsilon k} = \varepsilon^{-1/2}\psi_k(x/\varepsilon)$, respectively. The proof is divided into two steps. In the first step, we show that if $-E(\overrightarrow{\psi})$ is large and $\|\overrightarrow{\psi}\|_{L^2(|x|>1)}$ is small (but $\|\overrightarrow{\psi}\|_{L^2(|x|<1)}$ may be large), then $\|\overrightarrow{\varphi}(t)\|_{L^2(|x|>1)}$ is small for all $t > 0$.

In the second step, for any initial data $\overrightarrow{\psi}$ with negative energy, we use the scaling transform (3.2) to choose $\varepsilon > 0$ so small that $-E(\overrightarrow{\psi}_\varepsilon)$ ($\overrightarrow{\psi}_\varepsilon = (\psi_{\varepsilon 1}, \dots, \psi_{\varepsilon N})$) is sufficiently large and $\|\overrightarrow{\psi}_\varepsilon\|_{L^2(|x|>1)}$ is small enough. Then the proof of the second step is reduced to the first step and we complete the proof.

Let $\phi : [0,\infty) \to \mathbb{R}_+$ be a function with bounded third order derivatives and be such that

$$\phi(s) = \begin{cases} s, & 0 \le |s| < 1, \\ s - (s-1)^3, & 1 < s < 1 + \frac{\sqrt{3}}{3}, \\ s - (s+1)^3, & -(1 + \frac{\sqrt{3}}{3}) < s < -1, \\ \text{smooth}, \phi' < 0, & 1 + \frac{\sqrt{3}}{3} \le |s| < 2, \\ 0, & 2 \le |s|. \end{cases}$$

Putting $\Phi(x) = \int_0^x \phi(y)dy$ and $E_0 = E(\overrightarrow{\psi})$, we have the following proposition.

Proposition 3.3. *Let $\varphi_j(t)$ be a solution of (1.1) in $C([0,T), H^1(\mathbb{R}))$ with $\varphi_j(0) = \psi_j$. Put $a_0 = 3/(16M)$. If $\varphi_j(t)$ satisfies*

$$\sum_{j=1}^N \|\varphi_j(t)\|^4_{L^2(|x|>1)} \le 2a_0, \quad 0 \le t < T, \tag{3.3}$$

then we have

$$
\begin{aligned}
&-\sum_{j=1}^{N}\operatorname{Im}\int \phi\varphi_j(t)\overline{\varphi}_{jx}(t)+\sum_{j=1}^{N}\operatorname{Im}\int \phi\psi_j\overline{\psi}_{jx}\\
&\le \Big(2E_0+4M(1+M)^2\sum_{j=1}^{N}\|\psi_j\|_{L^2}^6+\frac{M}{2}\sum_{j=1}^{N}\|\psi_j\|_{L^2}^2\Big)t,
\end{aligned}
\tag{3.4}
$$

where $M=\|\phi_{xx}\|_{L^\infty}+\|\phi^{(3)}\|_{L^\infty}+\sum_{k,j=1}^{N}\beta_{kj}+\sum_{j=1}^{N}\mu_j$.

Proof. From the energy conserved identity

$$
\begin{aligned}
-\sum_{j=1}^{N}\int_{|x|<1}|\varphi_{jx}|^2 &= E(\overrightarrow{\varphi}(t))-\sum_{j=1}^{N}\int_{|x|>1}|\varphi_{jx}|^2\\
&\quad+\frac{1}{3}\sum_{j=1}^{N}\mu_j\int|\varphi_j|^6+2\sum_{k<j}\theta_{kj}\int|\varphi_k|^{p_k}|\varphi_j|^{p_j},
\end{aligned}
$$

we obtain by (2.6) that

$$
\begin{aligned}
&-\sum_{j=1}^{N}\operatorname{Im}\int \phi\varphi_j(t)\overline{\varphi}_{jx}(t)+\sum_{j=1}^{N}\operatorname{Im}\int \phi\psi_j\overline{\psi}_{jx}\\
&=\int_0^t\Big\{2E_0-\sum_{j=1}^{N}\int_{|x|>1}2\big(1-\phi_x\big)|\varphi_{jx}|^2+\frac{2}{3}\sum_{j=1}^{N}\mu_j\int\big(1-\phi_x\big)|\varphi_j|^6\\
&\quad-\frac{1}{2}\sum_{j=1}^{N}\int|\varphi_j|^2\phi^{(3)}+4\sum_{k<j}\theta_{kj}\int\big(1-\phi_x\big)|\varphi_k|^{p_k}|\varphi_j|^{p_j}\Big\}d\tau.
\end{aligned}
$$

By Lemma 3.1 with $\rho(x)=(1-\phi_x)^{1/4}$ and Hölder inequality, we obtain

$$
\begin{aligned}
&\int_{|x|>1}(1-\phi_x)|\varphi_j|^6\le\|\varphi_j\|_{L^2(|x|>1)}^2\|\rho\varphi_j\|_{L^\infty(|x|>1)}^4\\
&\le\|\varphi_j\|_{L^2(|x|>1)}^4\Big(2\|\rho^2\varphi_{jx}\|_{L^2(|x|>1)}+\|\varphi_j(\rho^2)_x\|_{L^2(|x|>1)}\Big)^2\\
&\le 8\|\varphi_j\|_{L^2(|x|>1)}^4\|\rho^2\varphi_{jx}\|_{L^2(|x|>1)}^2+2\|\varphi_j\|_{L^2(|x|>1)}^6\|(\rho^2)_x\|_{L^\infty(|x|>1)}^2.
\end{aligned}
\tag{3.5}
$$

On the other hand, we have from the definition of ϕ and ρ that $|(\rho^2)_x|\le\sqrt{3}$ for $1<|x|<1+1/\sqrt{3}$. For $|x|>1+1/\sqrt{3}$, we also have $|(\rho^2)_x|\le\frac{1}{2}\|\phi_{xx}\|_{L^\infty}$. It follows that $|(\rho^2)_x|\le\sqrt{3}(1+\frac{1}{2}\|\phi_{xx}\|_{L^\infty})$. So

$$
\begin{aligned}
&\int_{|x|>1}(1-\phi_x)|\varphi_j|^6\\
&\le 8\|\varphi_j\|_{L^2(|x|>1)}^4\|\rho^2\varphi_{jx}\|_{L^2(|x|>1)}^2+6(1+\frac{1}{2}\|\phi_{xx}\|_{L^\infty})^2\|\varphi_j\|_{L^2(|x|>1)}^6.
\end{aligned}
\tag{3.6}
$$

It is deduced from

$$
\int\big(1-\phi_x\big)|\varphi_k|^{p_k}|\varphi_j|^{p_j}\le\frac{p_k}{6}\int\big(1-\phi_x\big)|\varphi_k|^6+\frac{p_j}{6}\int\big(1-\phi_x\big)|\varphi_j|^6
$$

that

$$
\begin{aligned}
&2E_0-\sum_{j=1}^N\int_{|x|>1}2\big(1-\phi_x\big)|\varphi_{jx}|^2+\frac{2}{3}\sum_{j=1}^N\mu_j\int\big(1-\phi_x\big)|\varphi_j|^6\\
&-\frac{1}{2}\sum_{j=1}^N\int|\varphi_j|^2\phi^{(3)}+4\sum_{k<j}\theta_{kj}\int\big(1-\phi_x\big)|\varphi_k|^{p_k}|\varphi_j|^{p_j}\\
&\le 2E_0-\sum_{j=1}^N\int_{|x|>1}2\big(1-\phi_x\big)|\varphi_{jx}|^2+\frac{2}{3}\sum_{j=1}^N\mu_j\int\big(1-\phi_x\big)|\varphi_j|^6-\frac{1}{2}\sum_{j=1}^N\int|\varphi_j|^2\phi^{(3)}\\
&\quad+\frac{2}{3}\sum_{j<k}\beta_{jk}\int\big(1-\phi_x\big)|\varphi_k|^{p_k}+\frac{2}{3}\sum_{k<j}\beta_{kj}\int\big(1-\phi_x\big)|\varphi_j|^{p_j}.
\end{aligned}
\tag{3.7}
$$

Using (3.6) and the choice of M, we obtain that

$$
\begin{aligned}
&-\sum_{j=1}^N\operatorname{Im}\int\phi\varphi_j(t)\overline{\varphi}_{jx}(t)+\sum_{j=1}^N\operatorname{Im}\int\phi\psi_j\overline{\psi}_{jx}\\
&=\int_0^t\Big(2E_0+4M(1+M)^2\sum_{j=1}^N\|\varphi_j\|^6_{L^2(|x|>1)}+\frac{M}{2}\sum_{j=1}^N\|\varphi_j\|^2_{L^2(|x|>1)}\Big)d\tau\\
&\le\int_0^t\Big(2E_0+4M(1+M)^2\sum_{j=1}^N\|\varphi_j\|^6_{L^2}+\frac{M}{2}\sum_{j=1}^N\|\varphi_j\|^2_{L^2}\Big)d\tau\\
&=\Big(2E_0+4M(1+M)^2\sum_{j=1}^N\|\psi_j\|^6_{L^2}+\frac{M}{2}\sum_{j=1}^N\|\psi_j\|^2_{L^2}\Big)t.
\end{aligned}
\tag{3.8}
$$

The proof is complete. □

Proof of Theorem 1.1. We assume the solution $\varphi_j(t)$ of (1.1) exists for all $t\ge 0$ and then derive a contradiction. The proof is divided into two steps.

Step 1. In this step, we assume the initial data $\overrightarrow{\varphi}(0)=\overrightarrow{\psi}$ satisfies

$$\eta=-2E_0-4M(1+M)^2\sum_{j=1}^N\|\psi_j\|^6_{L^2}-\frac{M}{2}\sum_{j=1}^N\|\psi_j\|^2_{L^2}>0, \tag{3.9}$$

$$4\Big(\sum_{j=1}^N\int\Phi|\psi_j|^2\Big)^2\Big(\frac{4}{\eta}\sum_{j=1}^N\|\psi_{jx}\|^2_{L^2}+1\Big)^2\le a_0, \tag{3.10}$$

where M and a_0 are defined as in Proposition 3.3.

We first prove that if the initial data $\varphi_j(0)=\psi_j$ satisfies (3.9) and (3.10), then $\varphi_j(t)$ satisfies (3.3) for all $t\ge 0$. We prove this by contradiction. Since $\eta>0$ and $1\le 2\Phi(x)$ for $|x|>1$, we have from (3.10) that

$$\sum_{j=1}^N\|\psi_j\|^4_{L^2(|x|>1)}\le a_0. \tag{3.11}$$

Define T_0 as

$$T_0 = \sup\{t > 0;\ \sum_{j=1}^N \|\varphi_j(s)\|^4_{L^2(|x|>1)} \leq 2a_0, 0 \leq s < t\}.$$

By (3.11) we know that $T_0 > 0$. If $T_0 = +\infty$, then we are done. Assuming now that $T_0 < +\infty$, the continuity in L^2 of $\varphi_j(t)$ implies

$$\sum_{j=1}^N \|\varphi_j(T_0)\|^4_{L^2(|x|>1)} = 2a_0. \tag{3.12}$$

As $\varphi_j(t)$ satisfies all the assumptions in Proposition 3.3 on $[0, T_0)$, we get from (2.7), (3.9) and Proposition 3.3 that for $0 < t < T_0$,

$$\begin{aligned}\sum_{j=1}^N \int \Phi|\varphi_j(t)|^2 &\leq \sum_{j=1}^N \int \Phi|\psi_j|^2 - 2\int_0^t \operatorname{Im}\sum_{j=1}^N \int \phi\varphi_j\overline{\varphi}_{jx}\,dx\,d\tau \\ &\leq \sum_{j=1}^N \int \Phi|\psi_j|^2 - 2t\operatorname{Im}\sum_{j=1}^N \int \phi\psi_j\overline{\psi}_{jx} - \eta t^2.\end{aligned} \tag{3.13}$$

This inequality yields

$$\begin{aligned}\sum_{j=1}^N \int \Phi|\varphi_j(t)|^2 &\leq -\eta\Big(t + \frac{1}{\eta}\operatorname{Im}\sum_{j=1}^N \int \phi\psi_j\overline{\psi}_{jx}\Big)^2 \\ &\quad + \frac{1}{\eta}\Big(\operatorname{Im}\sum_{j=1}^N \int \phi\psi_j\overline{\psi}_{jx}\Big)^2 + \sum_{j=1}^N \int \Phi|\psi_j|^2.\end{aligned}$$

Noticing that

$$\Big(\operatorname{Im}\sum_{j=1}^N \int \phi\psi_j\overline{\psi}_{jx}\Big)^2 \leq 2\sum_{j=1}^N \int |\phi\psi_j|^2 \int |\psi_{jx}|^2 \tag{3.14}$$

and the fact of $\phi^2 \leq 2\Phi$, we deduce that

$$\sum_{j=1}^N \int \Phi|\varphi_j(t)|^2 \leq \Big(\frac{4}{\eta}\sum_{j=1}^N \|\psi_{jx}\|^2_{L^2} + 1\Big)\sum_{j=1}^N \int \Phi|\psi_j|^2, \quad 0 \leq t < T_0. \tag{3.15}$$

Since $1 \leq 2\Phi(x)$ for $|x| > 1$, (3.10) and (3.15) imply

$$\begin{aligned}\Big(\sum_{j=1}^N \|\varphi_j(t)\|^2_{L^2(|x|>1)}\Big)^2 &\leq \Big(2\sum_{j=1}^N \int \Phi|\varphi_j(t)|^2\Big)^2 \\ &\leq 4\Big(\sum_{j=1}^N \int \Phi|\psi_j|^2\Big)^2\Big(\frac{4}{\eta}\sum_{j=1}^N \|\psi_{jx}\|^2_{L^2} + 1\Big)^2 \\ &\leq a_0, \quad 0 \leq t < T_0.\end{aligned}$$

This and the continuity in L^2 of $\varphi_j(t)$ yield

$$\sum_{j=1}^N \|\varphi_j(T_0)\|^4_{L^2(|x|>1)} \leq a_0, \tag{3.16}$$

which contradicts to (3.12). So if the initial data $\overrightarrow{\varphi}(0) = \overrightarrow{\psi}$ satisfies (3.9) and (3.10), then $\varphi_j(t)$ satisfies (3.3) for all $t \geq 0$.

Therefore, since all the assumptions in Proposition 3.3 hold with $T = \infty$, $\varphi_j(t)$ satisfies (3.3) with $T_0 = \infty$, which implies that $\sum_{j=1}^N \int \Phi|\varphi_j(t)|^2$ goes to negative in finite time. This is a contradiction. Hence if the initial data $\overrightarrow{\varphi}(0) = \overrightarrow{\psi}$ satisfies (3.9) and (3.10), then $\overrightarrow{\varphi}(t)$ must blow up in finite time.

Step 2. In this step, we prove the theorem for all the initial data with negative energy. The main idea is to use the scaling invariance of the (1.1). In the first place, for $\varepsilon > 0$, let $\varphi_{\varepsilon j}(x,t) = \varepsilon^{-1/2}\varphi_j(x/\varepsilon, t/\varepsilon^2)$. Put $\varphi_{\varepsilon j}(x,0) = \psi_{\varepsilon j}(x) = \varepsilon^{-1/2}\psi_j(x/\varepsilon)$. Then $\varphi_{\varepsilon j}$ is also a solution of (1.1) with initial data $\psi_{\varepsilon j}$ in $C([0,+\infty), H^1(\mathbb{R}))$. Moreover, $\varphi_{\varepsilon j}(t)$ satisfies

$$\|\varphi_{\varepsilon j}(t)\|_{L^2} = \|\psi_{\varepsilon j}\|_{L^2} = \|\psi_j\|_{L^2}, \quad t \geq 0; \tag{3.17}$$

$$E(\overrightarrow{\varphi}_\varepsilon(t)) = \varepsilon^{-2}E(\overrightarrow{\psi}), \quad t \geq 0. \tag{3.18}$$

In the second place, we show that there exists an $\varepsilon > 0$ such that

$$\eta_\varepsilon = -2E(\overrightarrow{\psi}_\varepsilon) - 4M(1+M)^2\sum_{j=1}^N \|\psi_{\varepsilon j}\|_{L^2}^6 - \frac{M}{2}\sum_{j=1}^N \|\psi_{\varepsilon j}\|_{L^2}^2 > 0; \tag{3.19}$$

$$4\Big(\sum_{j=1}^N \int \Phi|\psi_{\varepsilon j}|^2\Big)^2\Big(\frac{4}{\eta_\varepsilon}\sum_{j=1}^N \|\psi_{\varepsilon j x}\|_{L^2}^2 + 1\Big)^2 \leq a_0. \tag{3.20}$$

Using (3.18), (3.19) follows by choosing $\varepsilon > 0$ such that

$$\varepsilon^2 < -2E_0\Big(4M(1+M)^2\sum_{j=1}^N \|\psi_j\|_{L^2}^6 + \frac{M}{2}\sum_{j=1}^N \|\psi_j\|_{L^2}^2\Big)^{-1}. \tag{3.21}$$

Now we have from (3.17) and (3.18) that for some $\varepsilon_1 > 0$ and $0 < \varepsilon < \varepsilon_1$,

$$\frac{4}{\eta}\sum_{j=1}^N \|\psi_{\varepsilon j x}\|_{L^2}^2 \leq C_0(\varepsilon_1),$$

$C_0(\varepsilon_1)$ denotes positive constant C_0 depending on ε_1. On the other hand, Lemma 3.2 implies that there exists an $\varepsilon_2 > 0$ with $\varepsilon_2 < \varepsilon_1$ and

$$\sum_{j=1}^N \int \Phi|\psi_{\varepsilon j}|^2 \leq 2\sum_{j=1}^N \|R\psi_{\varepsilon j}\|_{L^2}^2 \leq \frac{1}{4}(C_0(\varepsilon_1)+1)^{-1}a_0^{\frac{1}{2}} \tag{3.22}$$

for $0 < \varepsilon < \varepsilon_2$, where R is defined as in Lemma 3.2.

Thus if $0 < \varepsilon < \varepsilon_2$ and satisfying (3.21), then $\overrightarrow{\varphi}_\varepsilon(0) = \overrightarrow{\psi}$ satisfies (3.19) and (3.20). Therefore the proof of the theorem in the general case is reduced to Step 1 when we consider $\varphi_{\varepsilon j}(x,t)$ instead of $\varphi_j(x,t)$. The proof of Theorem 1.1 is complete. □

Acknowledgement. The authors want to thank the anonymous referee for the helpful comments.

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