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HOMOCLINIC ORBIT SOLUTIONS OF A ONE DIMENSIONAL WILSON-COWAN TYPE MODEL

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ABSTRACT. We analyze a time independent integral equation defined on a spatially extended domain which arises in the modelling of neuronal networks. In this paper, the coupling function is oscillatory and the firing rate is a smooth "heaviside-like" function. We will derive an associated fourth order ODE and establish that any bounded solution of the ODE is also a solution of the integral equation. We will then apply shooting arguments to prove that the ODE has N-bump homoclinic orbit solutions for any even-valued $N > 0$. homoclinic orbit.

1. INTRODUCTION

In 1972, Wilson and Cowan [23] derived the partial integro-differential equation

$$u_t = -u + \int_{-\infty}^{\infty} w(x-y)f(u(y,t)-th)dy \tag{1.1}$$

to describe the behavior of a single layer of neurons [23]. Here, $u(x,t)$ and $f(u(x,t)-th)$ represent the level of excitation (e.g. voltage) and the firing rate, respectively, of a neuron at position x and time t. The parameter $th \geq 0$ denotes the threshold of excitation. The term $w(x-y)$ determines the coupling between neurons at positions x and y.

In 1977, Amari [1] studied pattern formation in (1.1) for lateral inhibition type couplings. That is, w is assumed to be continuous, integrable and even, with $w(0) > 0$, and exactly one positive zero. Under the simplifying assumption that the firing rate f is a Heaviside step function, he analyzed the existence, multiplicity and stability of stationary one-bump solutions of the time independent equation

$$u = \int_{-\infty}^{\infty} w(x-y)f(u(y)-th)dy. \tag{1.2}$$

Equations (1.1) and (1.2) have been studied with respect to various combinations of firing rate functions and coupling functions.

Other investigations of (1.1) for lateral inhibition type couplings include those of Coombes et. al. [4] and Guo et. al. [9]. Coombes et. al. [4] obtain a closed form

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stable 1-bump homoclinic orbit solution of (1.2) with

$$f(u) = H(u),$$
$$w(x) = \kappa(1 - |x|)e^{-|x|}$$

and $H(u)$ is the Heaviside step function.

Guo and Chow [9] state criteria that result in two 1-bump solutions of (1.2) with

$$f(u) = (\alpha u + \beta)H(u) \quad \text{and}$$
$$w(x) = Ae^{-\alpha|x|} - e^{-|x|} \quad \text{where } A, a > 1.$$

A wide range of techniques have been used to study (1.1) and (1.2). Owen et. al. employ an Evans function approach to investigate instabilities of localized solutions of (1.1) and (1.2). Kishimoto and Amari [11] assume that f has a sigmoidal shape and use the Schauder Fixed Point Theorem [6] to prove the existence of a single bump stationary solution of (1.2). Ermentrout and McLeod [8] investigate the existence of traveling waves when w is strictly positive and Gaussian shaped, and f is a sigmoidal function. They use a homotopy argument based on the contraction mapping theorem to prove the existence of monotonic wave fronts. Subsequently, Pinto and Ermentrout [18] make use of the result in [8] and use singular perturbation methods to study wave front solutions in a related system of equations. Ermentrout [7] and Coombes [3] give an extensive review of theoretical methods and results.

In order to analyze more complicated solutions (e.g. multi-bump solutions), Laing et. al. [14], Coombes et. al. [4], and Guo et. al. [9] derive associated fourth order ODEs by applying Fourier Transform methods. In [14] and [4] conditions are given which show that when the integral equation (1.2) has a homoclinic orbit satisfying $u(\pm\infty) = 0$ then that solution also satisfies an associated ODE of the form

$$u'''' + q_1 u'' + h(u) = 0, \tag{1.3}$$

where q_1 is a real constant and h is a real-valued function.

In this paper we also derive a fourth order ODE associated with (1.2). This affords us the opportunity to employ the method of topological shooting to prove the existence of N-bump homoclinic orbit solutions. The method of topological shooting is a well known technique, and has been employed to prove the existence of a wide variety of solutions of two-point boundary value problems. For example, Shangbing [21] applies a shooting method to prove the existence of traveling waves of a bioremediation model. Peletier and Troy [17] derive a fourth order ODE by scaling the Extended Fisher-Kolmogorov equation. Subsequently, they proved that the fourth order ODE has two odd 1-bump periodic solutions.

In this paper we extend the results obtained in Krisner [13]. In [13], conditions were given which guarantee that (1.2) has two 1-bump periodic solutions. The primary goal in this paper is to develop techniques which allow us to prove the existence of N-bump homoclinic orbit solutions of (1.2) for any even-valued $N > 0$. In our survey the coupling function w is oscillatory shaped and the firing rate function f is a smooth step-like function.

The outline of the paper is as follows. In Section 2, we define our coupling and firing rate functions. These functions were originally introduced in Laing et al. [14]. In addition, we will state several previously established results including

(1) the link between (1.2) and a fourth order ODE,

(2) the initial conditions of the ODE which yield even solutions,
(3) a parameter regime that gives rise to a tractable setting for our construction of N-bump solutions,
(4) the existence of infinitely many critical values, and
(5) the existence of two 1-bump periodic solutions.

In Section 3, we will show that the critical numbers are continuous with respect to the initial conditions. This analysis will lay the framework for the construction of N-bump periodic solutions which is contained in Section 4.

2. Preliminary Results

In this section we will establish several results that are necessary for the construction of N-bump homoclinic orbit solutions of the time independent integral equation

$$u(x) = \int_{-\infty}^{\infty} w(x-y) f(u(y) - th) dy. \tag{2.1}$$

As in [13] the coupling and firing rate functions are defined by

$$w(x) = e^{-b|x|}(b \sin(|x|) + \cos(x)), \quad b > 0, \quad \text{and} \tag{2.2}$$

$$f(u - th) = 2e^{-r/(u-th)^2} H(u - th), \quad r > 0, \; th > 0 \tag{2.3}$$

respectively. Figure 1 depicts the essential characteristics of the functions w and f.

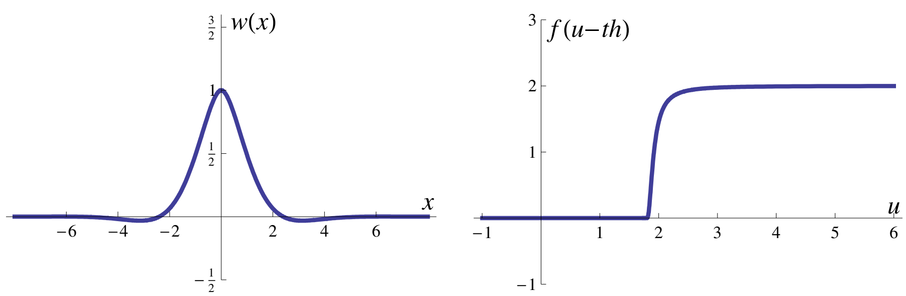

Figure 1. Left panel, example of (2.2) with $b = 1.1$. Right panel, example of (2.3) with $r = 0.02$, $th = 1.75$.

The Associated ODE. Here we state an important theorem which establishes a crucial connection between the ODE

$$u'''' - 2(b^2 - 1)u'' + (b^2 + 1)^2 u = 4b(b^2 + 1) f(u - th) \tag{2.4}$$

and the integral equation (2.1) with w defined by (2.2) and f defined by (2.3). Krisner [12] proves the following result.

Theorem 2.1. *Suppose that u is a solution of* (2.4)*, and that $u(t) = o(e^{b|t|})$ as $t \to \pm\infty$. Then u is a solution of* (2.1).

Hence, the goal of this paper will be fulfilled upon proving the existence of N-bump homoclinic orbit solutions of the IVP

$$\begin{gathered} u'''' - 2(b^2 - 1)u'' + (b^2 + 1)^2 u = 4b(b^2 + 1) f(u - th) \\ u(0) = \alpha, u'(0) = 0, u''(0) = \beta, u'''(0) = 0. \end{gathered} \tag{2.5}$$

Even Solutions of the ODE. An immediate consequence of [13, Lemma 4.1] is the following.

Lemma 2.2. *Any solution u of (2.5) satisfies $u(x) = u(-x)$ for all x in the domain of existence.*

Thus, we need only to consider the behavior of solutions on $[0, \omega)$ for some $\omega > 0$ (possibly $\omega = \infty$).

The First Integral Equation. We now derive an associated 3rd order equation. By multiplying both sides of (2.4) by u' and integrating over $[0, x]$ we obtain

$$u'''u' - \frac{(u'')^2}{2} - (b^2 - 1)(u')^2 + (b^2 + 1)^2 Q(u) = E \tag{2.6}$$

for some constant E and

$$Q(u) = \int_0^u \Big(s - \frac{4b}{b^2 + 1} f(s - th)\Big) ds. \tag{2.7}$$

Since we are interested in solutions that satisfy $(u, u', u'', u''') \to (0, 0, 0, 0)$ as $x \to \infty$, then $E = 0$ in (2.6). Thus, we obtain

$$u'''u' - \frac{(u'')^2}{2} - (b^2 - 1)(u')^2 + (b^2 + 1)^2 Q(u) = 0. \tag{2.8}$$

The Initial Conditions. The conditions $u(0) = \alpha$, $u'(0) = 0$, $u''(0) = \beta$, and $u'''(0) = 0$ substituted into (2.8) yields

$$\beta^2 = 2(b^2 + 1)^2 Q(\alpha) \iff \beta = \pm(b^2 + 1)\sqrt{2Q(\alpha)}. \tag{2.9}$$

In our construction of even bump solutions we consider $\alpha < 0$ and $\beta > 0$. This together with (2.9) and the fact that $Q(u) = \frac{u^2}{2}$ yields

$$\beta = -(b^2 + 1)\alpha. \tag{2.10}$$

This crucial result reduces our problem to that of one dimensional shooting. That is, all solutions of (2.5) that we consider are uniquely determined by the value of α. Furthermore, we occasionally use the notation $u(\cdot, \alpha)$ whenever its necessary to emphasize the solution's dependence on α. Lastly, we will assume that (2.10) holds throughout the remainder of this paper.

Range of Parameters. In Krisner [13] we defined a parameter regime that ensured the existence of 1-bump periodic solutions. These periodic solutions will be used to construct the multi-bump homoclinic orbit solutions. Hence, the conditions required to construct the periodic solutions of [13] will also be required for the construction of the homoclinic orbit solutions.

We begin by defining the same parameter regime that guaranteed the existence of periodic solutions. We will assume throughout the remainder of this paper that $(r, b, th) \in \Lambda$ where

$$\Lambda = \{(r, b, th) \in X : Q(u) = 0 \text{ has a unique positive solution}\}, \tag{2.11}$$

and $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0, \text{ and } x_3 > 0\}$, (see Figure 2).

The following lemma is an immediate consequence of [13, Theorem 3.5]

Lemma 2.3. *For each fixed $th \in (0, 2)$ Λ contains a continuum provided that $r > 0$ is sufficiently small.*

For a proof of the above lemma, see [13, Theorem 3.5].

The Function Q. As noted above, the function Q defined by (2.7) plays a pivotal role in determining the parameter regime that we consider. Figure 2 depicts a rather typical qualitative picture of the function Q with $(r, b, th) \in \Lambda$. We see that $Q'(u)$, where $' = \frac{d}{du}$, has two positive roots, which we denote by u_s and u_{ss}. Specifically, we define $u_{ss} > 0$ so that $Q'(u_{ss}) = 0 < Q(u_{ss})$ and $u_s > u_{ss}$ so that $Q'(u_s) = Q(u_s) = 0$.

The function Q with our choice of parameters given in (2.11) also determines the constant solutions of (2.5). Note that $u \equiv 0$ and $u \equiv u_s$ are the only constant solutions of (2.5). This is easily seen by expressing the equation in (2.5) as

$$u'''' - 2(b^2 - 1)u'' = -(b^2 + 1)^2 Q'(u).$$

We can also deduce from (2.8) that $u \equiv u_{ss}$ is **not** a constant solution.

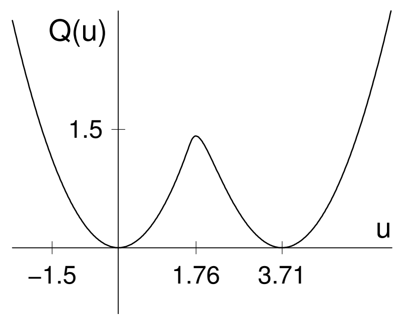

FIGURE 2. The function Q when $(r, b_r, th) \in \Lambda$. Here the parameters are $r = .05, b_r = 1.44$, and $th = 1.5$. Also, $u_{ss} \approx 1.76$ and $u_s \approx 3.71$.

Oscillatory Behavior of Solutions. Much of the work in this paper deals with the behavior of solutions near critical points. Thus, it is necessary to guarantee their existence. The following theorem, proved in [13], guarantees the existence of infinitely many critical points.

Theorem 2.4. *Suppose that $(r, b, th) \in \Lambda$ with $r \le \frac{th^4}{16}$. Also, let u be a nontrivial solution of (2.5) with interval of existence $[0, \omega)$. Then u' changes sign on (X, ω), for any $X \in (0, \omega)$.*

For a proof of the above theorem, see [13, Theorem 5.1].

Limiting Values as $r \to 0$. As stated in Lemma 2.3, "sufficiently small" r gave rise to the existence of a continuum in Λ. In Theorem 2.4 it was seen that sufficiently small r ensured the existence of infinitely many critical points. Our construction of N-bump homoclinic orbit solutions will also rely on sufficiently small r as well as precise limiting values of b, u_s and u_{ss} as $r \to 0$. To emphasize their dependence on r, we write $b_r, u_s(r)$ and $u_{ss}(r)$. In later sections we will omit the r and simply write b, u_s and u_{ss}.

In the following lemma we begin by obtaining the limiting value of $u_s(r)$ and b_r as $r \to 0^+$.

Lemma 2.5. *Suppose that $th \in (0,2)$ is fixed and that $(r, b_r, th) \in \Lambda$. Then*

$$u_s(r) \to 2th^+, \quad \text{and} \tag{2.12}$$

$$b_r \to \frac{2}{th} \pm \frac{\sqrt{4-th^2}}{th} \tag{2.13}$$

as $r \to 0^+$.

Proof. Equation (2.12) is proved in [13, Lemma 3.4]. We now prove (2.13). Since $u_s(r)$ satisfies $Q'(u_s(r)) = 0$, i.e.,

$$Q'(u_s(r)) = u_s(r) - \frac{8b_r}{b_r^2+1} e^{-r/(u_s(r)-th)^2} = 0,$$

then it follows from (2.12) that

$$\frac{8b_r}{b_r^2+1} = u_s(r) e^{-r/(u_s(r)-th)^2} \to 2th \quad \text{as } r \to 0^+. \tag{2.14}$$

Hence, (2.13) follows after a little algebra. □

Lemma 2.6. *Let $0 < th < 2$ be fixed, and for small $r > 0$ let $(r, b_r, th) \in \Lambda$. Then there exists $r^* > 0$ such that*

$$0 < u_{ss}(r) - th \le r^{1/4} \quad \text{for all } r \in (0, r^*), \quad \text{and} \tag{2.15}$$

$$Q(u_{ss}(r)) \to Q(th) \quad \text{as } r \to 0^+. \tag{2.16}$$

Remarks: (i) The power $r^{1/4}$ in (2.15) can be improved but for our purposes this power will suffice. (ii) In our proofs of multi-bump solutions we will make use of properties (2.15) and (2.16).

Proof of Lemma 2.6. We prove (2.15) by contradiction. That is, suppose that there exists a sequence $\{r_n\}_{n=1}^{\infty}$ such that

$$r_n \to 0^+ \quad \text{and} \quad u_{ss}(r_n) - th > r_n^{1/4} \quad \text{for all } n \ge 1. \tag{2.17}$$

Throughout this proof we shall write b_n instead of b_{r_n}. To obtain a contradiction we define

$$u_n = th + r_n^{1/4} \quad \text{for each } n \ge 1. \tag{2.18}$$

By (2.17) and (2.18) we obtain

$$th < u_n < u_{ss}(r_n) \quad \text{for all } n \ge 1.$$

This means that $0 < Q'(u_n)$ for all $n \ge 1$, or equivalently

$$0 < u_n - \frac{8b_n}{b_n^2+1} e^{-r_n/(u_n-th)^2} = u_n - \frac{8b_n}{b_n^2+1} e^{-r_n^{1/2}} \quad \forall n \ge 1. \tag{2.19}$$

From (2.14) and (2.18) it follows that

$$u_n - \frac{8b_n}{b_n^2+1} e^{-r_n^{1/2}} \to -th < 0 \quad \text{as } n \to \infty.$$

This contradicts (2.19) for large $n > 1$. Therefore, (2.15) must hold as claimed.

It remains to prove (2.16). By definition of Q it follows that

$$\begin{aligned} Q(u_{ss}(r)) - Q(th) &= \frac{u_{ss}^2(r)}{2} - \frac{th^2}{2} - \frac{8b_r}{b_r^2+1} \int_{th}^{u_{ss}(r)} e^{-r/(s-th)^2} ds \\ &\le (u_{ss}(r) - th)\Big(\frac{u_{ss}(r)+th}{2}\Big). \end{aligned} \tag{2.20}$$

Since $Q(u_{ss}(r)) > Q(th)$, then (2.16) is a consequence of (2.15) and (2.20). This concludes the proof. $\square$

Periodic Solutions. An important prerequisite of our construction of N-bump homoclinic solutions is the existence of periodic solutions. Next, we state Theorem 6.1 of [13].

Theorem 2.7. *Suppose that $(r, b, th) \in \Lambda$ with $r \le \frac{th^4}{16}$, and that $\alpha < 0$ and $\beta = -(b^2+1)\alpha$. Then, there exists $\alpha^* < \alpha_* < 0$ such that $u(\cdot, \alpha^*)$ and $u(\cdot, \alpha_*)$ are 1-bump periodic solutions of* (2.5)*. Moreover, we can choose α^* and α_* so that*

$$th < ||u(\cdot, \alpha_*)||_\infty < u_s < ||u(\cdot, \alpha^*)||_\infty \tag{2.21}$$

where $Q(u_s) = 0$.

The existence of these periodic solutions provides a means of "controlling" the bumps of the solution. For example, if $u(\cdot, \alpha^*)$ is a periodic solution described in Theorem 2.7, then $N \to \infty$ as $\alpha \to \alpha^*$.

3. Continuity of Critical Values

In this subsection we will lay the foundation of the shooting method that we use to prove the existence of N-bump homoclinic orbit solutions for any even N. To accomplish this we must first assume that the conditions of Theorem 2.4 hold. Recall that this theorem ensures the existence of infinitely many critical numbers on the domain of existence. Also, recall that we will assume that $u(0) = \alpha < 0$ and $u''(0) = \beta = -(b^2+1)\alpha$. Hence, solutions of (2.5) are uniquely determined by α.

In particular, critical numbers of solutions of (2.5) are uniquely determined by α. The primary objective of this section is to prove that these critical numbers *continuously* depend on α. Hence, it is necessary to develop a rigorous naming scheme of the critical numbers in a way that emphasizes their dependence on α. As in Theorem 2.4, we will assume that u is a nontrivial solution of (2.5), i.e., $0 \ne \alpha \ne u_s$. For notational convenience, we will define the critical numbers in terms of the sets

$$\Gamma_-(x_0) = \{x > x_0 \mid u'(\cdot, \alpha) < 0 \text{ on } (x_0, x)\}$$
$$\Gamma_+(x_0) = \{x > x_0 \mid u'(\cdot, \alpha) > 0 \text{ on } (x_0, x)\}.$$

Since $u'(0) = 0$ and $u''(0) = \beta > 0$, then $u'(x, \alpha) > 0$ in a right neighborhood $(0, \delta)$. Thus, for any $u(0) = \alpha < 0$ we define

$$\xi_1(\alpha) = \sup \Gamma_+(0), \tag{3.1}$$

$$\eta_k(\alpha) = \begin{cases} \sup \Gamma_-(\xi_k(\alpha)) & \text{if } u''(\xi_k(\alpha), \alpha) < 0 \text{ or } u'''(\xi_k(\alpha), \alpha) < 0 \\ \xi_k(\alpha) & \text{if } u''(\xi_k(\alpha), \alpha) = 0 \text{ and } u'''(\xi_k(\alpha), \alpha) > 0 \end{cases} \tag{3.2}$$

for $k \ge 1$, and

$$\xi_k(\alpha) = \begin{cases} \sup \Gamma_+(\eta_{k-1}(\alpha)) & \text{if } u''(\eta_{k-1}(\alpha), \alpha) > 0 \text{ or } u'''(\eta_{k-1}(\alpha), \alpha) > 0 \\ \eta_{k-1}(\alpha) & \text{if } u''(\eta_{k-1}(\alpha), \alpha) = 0 \text{ and } u'''(\eta_{k-1}(\alpha), \alpha) < 0 \end{cases} \tag{3.3}$$

for $k \ge 2$.

Theorem 2.4 guarantees that ξ_k and η_k are well defined for all $k \ge 1$ provided that $u'(x, \alpha) > 0$ on an interval of the form $(0, \delta)$. We will now show that the conditions in the piecewise defined functions given in (3.2) and (3.3) encompass all possibilities. That is, for (3.2), we will show that the negation of the statement $u''(\xi_k(\alpha),\alpha) < 0$ or $u'''(\xi_k(\alpha),\alpha) < 0$ is $u''(\xi_k(\alpha),\alpha) = 0$ and $u'''(\xi_k(\alpha),\alpha) > 0$. It follows from (3.1) and (3.3) that $u''(\xi_k(\alpha),\alpha) \leq 0$ for all $k \geq 1$. This reduces the negation of the statement $u''(\xi_k(\alpha),\alpha) < 0$ or $u'''(\xi_k(\alpha),\alpha) < 0$ to $u''(\xi_k(\alpha),\alpha) = 0$ and $u'''(\xi_k(\alpha),\alpha) \geq 0$. In the following lemma we will show that $u'''(\xi_k(\alpha),\alpha) > 0$ if $u''(\xi_k(\alpha),\alpha) = 0$.

Lemma 3.1. *Suppose that u is a nontrivial solution of* (2.5) *with $u'(x_0) = u''(x_0) = 0$ for some x_0 in the domain of existence. Then,*

- (i) *$u'(x) < 0$ in a left neighborhood of x_0 implies that $u'''(x_0) < 0$, and*
- (ii) *$u'(x) > 0$ in a left neighborhood of x_0 implies that $u'''(x_0) > 0$.*

Proof. of (i): It follows from (2.8) that $Q(u(x_0)) = 0$. That is, $u(x_0)$ is one of the two roots of Q, namely 0 or u_s. Since u is assumed to be nontrivial, then uniqueness of solutions, (Theorem 7.1 of [5]), guarantees that $u'''(x_0) \neq 0$. This together with the assumption that $u'(x) < 0$ on $(x_0 - \delta, x_0)$ say, leads to $u'''(x_0) < 0$.

A similar argument can be used to prove (ii). □

Hence, the negation of $u''(\xi_k(\alpha),\alpha) < 0$ or $u'''(\xi_k(\alpha),\alpha) < 0$ is $u''(\xi_k(\alpha),\alpha) = 0$ and $u'''(\xi_k(\alpha),\alpha) > 0$.

Similarly, the negation of $u''(\eta_{k-1}(\alpha),\alpha) > 0$ or $u'''(\eta_{k-1}(\alpha),\alpha) > 0$ is $u''(\eta_{k-1}(\alpha),\alpha) = 0$ and $u'''(\eta_{k-1}(\alpha),\alpha) < 0$. Thus, $\xi_k(\alpha)$ and $\eta_k(\alpha)$ are defined in every possible case.

Another question that may arise in the above definitions concerns the need for the sign of $u'''(\xi_k(\alpha),\alpha)$ in (3.2) and $u'''(\eta_{k-1}(\alpha),\alpha)$ in (3.3). Without addressing the sign of the third derivative in these definitions it is possible that $\xi_k(\alpha) = \xi_{k+1}(\alpha) = \xi_{k+2}(\alpha)\dots$ The following lemma ensures that this does not happen.

Lemma 3.2. *For any $\alpha < 0$ and any $k \geq 1$ it follows that*

$$\xi_k(\alpha) \leq \eta_k(\alpha) \leq \xi_{k+1}(\alpha). \tag{3.4}$$

Furthermore, if $\xi_k(\alpha) = \eta_k(\alpha)$, then $\eta_k(\alpha) < \xi_{k+1}(\alpha)$. Also, if $\eta_k(\alpha) = \xi_{k+1}(\alpha)$, then $\xi_k(\alpha) < \eta_k(\alpha)$.

Proof. First, (3.4) is a direct consequence of (3.2) and (3.3). Assume that $\xi_k(\alpha) = \eta_k(\alpha)$. Then (3.2) implies that $u''(\xi_k(\alpha),\alpha) = 0$ and $u'''(\xi_k(\alpha),\alpha) > 0$. Thus, $u'''(\eta_k(\alpha),\alpha) > 0$ and (3.3) result in $\xi_{k+1}(\alpha) = \sup\Gamma_+(\eta_k(\alpha)) > \eta_k(\alpha)$.

Using a similar argument we can show that $\xi_{k+1}(\alpha) = \eta_k(\alpha)$ implies $\xi_k(\alpha) < \eta_k(\alpha)$. □

For a better understanding of (3.2) and (3.3) we apply these two definitions to the illustration given in Figure 3.

We now prove that ξ_k and η_k are continuous functions of α for all $k \geq 1$. We will do this with the help of the following lemma.

Lemma 3.3. *Suppose that $u(x,\alpha_*)$ is a nonconstant solution of* (2.5) *such that $u'(x_*,\alpha_*) = u''(x_*,\alpha_*) = 0 \neq u'''(x_*,\alpha_*)$ for some $x_* > 0$ and some $\alpha_* \in \mathbb{R}$. Then for any $\epsilon > 0$ such that*

$$u'''(x,\alpha_*) \neq 0 \quad \text{on } [x_* - \epsilon, x_* + \epsilon] \tag{3.5}$$

it follows that

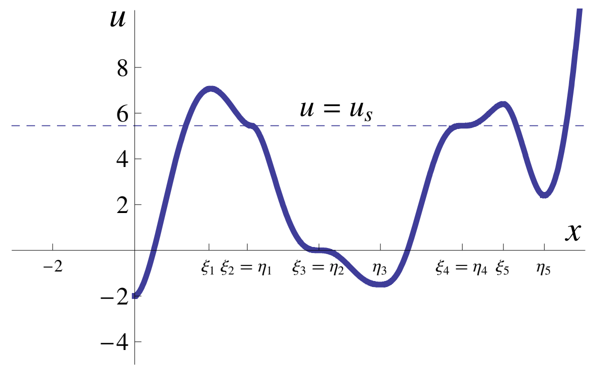

FIGURE 3. An illustration of a function u such that $u''(\xi_k(\alpha),\alpha) = 0$ for $k = 2,3,4$, $u'''(\eta_k(\alpha),\alpha) < 0$ for $k = 1,2$, and $u'''(\xi_4(\alpha),\alpha) > 0$. The above graph is not an actual solution of (2.5), but merely an interpolating polynomial derived using Mathematica.

(i) $u''(x_* - x, \alpha_*)u''(x_* + x, \alpha_*) < 0$ *on* $(0, \epsilon]$.

In addition, assume that $\{\alpha_n\}$ *is a sequence such that*

$$\alpha_n \to \alpha_* \quad \text{as } n \to \infty, \tag{3.6}$$

and that $u(x, \alpha_n)$ *is a nonconstant solution of* (2.5) *for each* $n \geq 1$. *Then there exists* $N > 0$ *such that*

(ii) $u'''(x, \alpha_n)u'''(x, \alpha_*) > 0$ *on* $[x_* - \epsilon, x_* + \epsilon]$,
(iii) $u''(x_* - \epsilon, \alpha_n)u''(x_* + \epsilon, \alpha_n) < 0$, *and*
(iv) *there exists a unique* $\tau_n \in (x_* - \epsilon, x_* + \epsilon)$ *such that* $u''(\tau_n, \alpha_n) = 0$

for all $n \geq N$. *Furthermore, it also follows that*

(v) $\tau_n \to x_*$ *as* $n \to \infty$.

The above lemma is [13, Lemma 5.10].

Theorem 3.4. *The function* ξ_1 *as defined in* (3.1) *is a continuous function of* α.

For a proof of the above theorem, see [13, Theorem 5.11].

Theorem 3.4 begins the inductive process that we use to prove continuity of ξ_k and η_k. Proving continuity of ξ_1 is sufficient to begin the inductive chain. That is, we will assume that ξ_k is continuous to prove that η_k is continuous for arbitrary $k \geq 1$. To complete the induction process we will show that ξ_{k+1} is continuous under the assumption that η_k is continuous.

We now prove that η_k is continuous, given that ξ_k is continuous, by considering three cases. Lemma 3.2 asserts that the three cases are

(1) $\xi_k(\alpha) < \eta_k(\alpha) < \xi_{k+1}(\alpha)$,
(2) $\eta_k(\alpha) = \xi_{k+1}(\alpha)$, and
(3) $\xi_k(\alpha) = \eta_k(\alpha)$.

We begin with the simplest case, given by **(1)**, after the following lemma.

Lemma 3.5. *If* $u''(\eta_k(\alpha_*), \alpha_*) > 0$, *then* η_k *is continuous at* $\alpha = \alpha_*$. *Likewise, if* $u''(\xi_k(\alpha_*), \alpha_*) < 0$, *then* ξ_k *is continuous at* $\alpha = \alpha_*$.

Proof. Since $\frac{d}{dx}u'(x,\alpha_*)|_{x=\eta_k(\alpha_*)} \neq 0$, then η_k is continuous by way of the Implicit Function Theorem.

A similar argument holds for continuity of ξ_k. $\square$

We now state conditions that yield $u''(\xi_k(\alpha_*),\alpha_*) < 0 < u''(\eta_k(\alpha_*),\alpha_*)$.

Lemma 3.6. *If $\xi_k(\alpha_*) < \eta_k(\alpha_*) < \xi_{k+1}(\alpha_*)$, then $u''(\eta_k(\alpha_*),\alpha_*) > 0$, and η_k is continuous at $\alpha = \alpha_*$. Moreover, if $\eta_{k-1}(\alpha_*) < \xi_k(\alpha_*) < \eta_k(\alpha_*)$, then $u''(\xi_k(\alpha_*),\alpha_*) < 0$, and ξ_k is continuous at $\alpha = \alpha_*$.*

Proof. Suppose that $\xi_k(\alpha_*) < \eta_k(\alpha_*) < \xi_{k+1}(\alpha_*)$. By equation (3.2), it follows that $u''(\eta_k(\alpha_*),\alpha_*) \geq 0$. For a contradiction, assume that $u''(\eta_k(\alpha_*),\alpha_*) = 0$. Since $\xi_{k+1}(\alpha_*) > \eta_k(\alpha_*)$ and $u''(\eta_k(\alpha_*),\alpha_*) = 0$, then $u'''(\eta_k(\alpha_*),\alpha_*) > 0$ as a consequence of (3.3). From this we infer that $u'(x,\alpha_*) > 0$ on an interval of the form $(\eta_k(\alpha_*) - \delta, \eta_k(\alpha_*))$ for some $\delta > 0$. But $\xi_k(\alpha_*) < \eta_k(\alpha_*)$ implies that $u'(x,\alpha_*) < 0$ on $(\xi_k(\alpha_*), \eta_k(\alpha_*))$. Thus, we have obtained a contradiction. Therefore, $u''(\eta_k(\alpha_*),\alpha_*) > 0$, and continuity of η_k at $\alpha = \alpha_*$ follows from Lemma 3.5.

We can prove that $u''(\xi_k(\alpha_*),\alpha_*) < 0$ and continuity of ξ_k at $\alpha = \alpha_*$ in a similar manner. $\square$

To finish proving continuity of η_k at $\alpha = \alpha_*$ we consider the cases where $\xi_{k+1}(\alpha_*) = \eta_k(\alpha_*)$, and $\xi_k(\alpha_*) = \eta_k(\alpha_*)$. In the following lemma, we consider the case where $\xi_{k+1}(\alpha_*) = \eta_k(\alpha_*)$.

Lemma 3.7. *Suppose that ξ_k is continuous. Then η_k is continuous at any $\alpha = \alpha_*$ such that that $\xi_{k+1}(\alpha_*) = \eta_k(\alpha_*)$, (see Figure 4).*

Proof. Let $\epsilon > 0$ and suppose that $\{\alpha_n\}$ is a sequence such that $\alpha_n \to \alpha_*$ as $n \to \infty$. We must show that there exists $N > 0$ such that

(i) $\eta_k(\alpha_n) < \eta_k(\alpha_*) + \epsilon$
(ii) $\eta_k(\alpha_n) > \eta_k(\alpha_*) - \epsilon$

for all $n \geq N$.

(i): For a contradiction, assume that there exists a sequence $\{\alpha_n\}$ and $\epsilon > 0$ such that $\alpha_n \to \alpha_*$ as $n \to \infty$ and that $\eta_k(\alpha_n) \geq \eta_k(\alpha_*) + \epsilon$ for all $n \geq 1$. Lemma 3.2 and $\xi_{k+1}(\alpha_*) = \eta_k(\alpha_*)$ guarantee that $\xi_k(\alpha_*) < \eta_k(\alpha_*)$, and therefore $u'(x,\alpha_*) < 0$ on $(\xi_k(\alpha_*), \eta_k(\alpha_*))$. Since $\xi_k(\alpha_*) < \eta_k(\alpha_*)$, then (3.2) ensures that

$$u''(\xi_k(\alpha_*),\alpha_*) < 0 \quad \text{or} \quad u'''(\xi_k(\alpha_*),\alpha_*) < 0. \tag{3.7}$$

We now show that there exists $N > 0$ such that $\xi_k(\alpha_n) < \eta_k(\alpha_n)$ for all $n \geq N$. Otherwise, if there is a subsequence $\{\alpha_{n_j}\}$ such that $\xi_k(\alpha_{n_j}) = \eta_k(\alpha_{n_j})$ for each n_j, then (3.2) asserts that $u''(\xi_k(\alpha_{n_j}),\alpha_{n_j}) = 0$ and $u'''(\xi_k(\alpha_{n_j}),\alpha_{n_j}) > 0$ for each n_j. By continuity of ξ_k in α and u'' and u''' in (x,α), (Theorem 7.1 of [5]), it follows that

$$\begin{gathered} 0 = \lim_{n_j\to\infty} u''(\xi_k(\alpha_{n_j}),\alpha_{n_j}) = u''(\xi_k(\alpha_*),\alpha_*) \quad \text{and} \\ 0 \leq \lim_{n_j\to\infty} u'''(\xi_k(\alpha_{n_j}),\alpha_{n_j}) = u'''(\xi_k(\alpha_*),\alpha_*). \end{gathered}$$

But this contradicts (3.7) and proves our claim that $N > 0$ can be chosen such that $\xi_k(\alpha_n) < \eta_k(\alpha_n)$ for all $n \geq N$. Without loss of generality, we can assume that $\xi_k(\alpha_n) < \eta_k(\alpha_n)$ for all $n \geq 1$. Thus, $u'(x,\alpha_n) < 0$ on $(\xi_k(\alpha_n), \eta_k(\alpha_n))$ for all $n \geq 1$.

Our assumption that $\xi_{k+1}(\alpha_*) = \eta_k(\alpha_*)$ together with (3.3) imply that

$$u'''(\eta_k(\alpha_*), \alpha_*) < 0 = u'(\eta_k(\alpha_*), \alpha_*) = u''(\eta_k(\alpha_*), \alpha_*).$$

We can further restrict ϵ if necessary so that

$$u'''(x, \alpha_*) < 0 \quad \text{on } [\eta_k(\alpha_*) - \epsilon, \eta_k(\alpha_*) + \epsilon].$$

By Lemma 3.3, there exists $N_1 > 0$ and a sequence $\{\tau_n\}$ such that

(a_1) $u''(\tau_n, \alpha_n) = 0$
(b_1) $\eta_k(\alpha_*) - \epsilon < \tau_n < \eta_k(\alpha_*) + \epsilon$
(c_1) $\tau_n \to \eta_k(\alpha_*)$ as $n \to \infty$

whenever $n \geq N_1$. Throughout the remainder of this proof we denote

$$u_n = u(\tau_n, \alpha_n), u'_n = u'(\tau_n, \alpha_n),\ u''_n = u''(\tau_n, \alpha_n), \quad \text{and} \quad u'''_n = u'''(\tau_n, \alpha_n).$$

It follows from continuity of u' and u''' in (x, α) together with (c_1) that $u'''_n \to u'''(\eta_k(\alpha_*), \alpha_*) < 0$ and $u'_n \to u'(\eta_k(\alpha_*), \alpha_*) = 0$ as $n \to \infty$. Hence, we can choose $N_2 \geq N_1$ so that

(a_2) $\xi_k(\alpha_n) < \xi_k(\alpha_*) + \epsilon$, (by continuity of ξ_k),
(b_2) $\xi_k(\alpha_*) + \epsilon < \tau_n < \eta_k(\alpha_*) + \epsilon$, and
(c_2) $u'''_n - (b^2 - 1)u'_n < 0$

for all $n \geq N_2$. Now, (a_2), (b_2) and the contradiction hypothesis $\eta_k(\alpha_n) \geq \eta_k(\alpha_*)+\epsilon$ ensure that $\xi_k(\alpha_n) < \tau_n < \eta_k(\alpha_n)$ for each $n \geq N_2$, and hence $u'_n < 0$ for each $n \geq N_2$. To obtain the desired contradiction, we consider Eq. (2.8) at $x = \tau_n$ which can be written as

$$u'_n(u'''_n - (b^2 - 1)u'_n) + (b^2 + 1)^2 Q(u_n) = 0.$$

This is a contradiction since $Q(u_n) \geq 0$ for any u_n, $u'_n < 0$, and (c_2) for all $n \geq N_2$. This concludes the proof part (i).

(ii): Since $\xi_k(\alpha_*) < \eta_k(\alpha_*)$ we can restrict ϵ so that $\xi_k(\alpha_*) + \epsilon < \eta_k(\alpha_*) - \epsilon$. By continuity of ξ_k, we can choose $N_1 > 0$ so that

$$\xi_k(\alpha_*) - \epsilon < \xi_k(\alpha_n) < \xi_k(\alpha_*) + \epsilon \quad \text{for all } n \geq N_1. \tag{3.8}$$

To complete the proof of (ii) we must prove that there exists $N_2 \geq N_1$ so that $u'(x, \alpha_n) < 0$ on $(\xi_k(\alpha_n), \eta_k(\alpha_*) - \epsilon]$ whenever $n \geq N_2$. This will guarantee that $\eta_k(\alpha_n) > \eta_k(\alpha_*) - \epsilon$ for all $n \geq N_2$. We will do this in two steps. We will show that there exists $N_2 \geq N_1$ such that

(a) $u'(x, \alpha_n) < 0$ on $(\xi_k(\alpha_n), \xi_k(\alpha_*) + \epsilon]$ and
(b) $u'(x, \alpha_n) < 0$ on $[\xi_k(\alpha_*) + \epsilon, \eta_k(\alpha_*) - \epsilon]$

provided that $n \geq N_2$.

We begin by proving (a). Recall from (3.3) that $u''(\xi_k(\alpha_*), \alpha_*) \leq 0$. First, consider the simpler of the two cases where $u''(\xi_k(\alpha_*), \alpha_*) < 0$. We can further restrict ϵ if necessary to ensure that $u''(x, \alpha_*) < 0$ on the compact interval $[\xi_k(\alpha_*) - \epsilon, \xi_k(\alpha_*) + \epsilon]$. By continuity of u'' in (x, α) we can choose $N_2 \geq N_1$ so that $u''(x, \alpha_n) < 0$ on $[\xi_k(\alpha_*) - \epsilon, \xi_k(\alpha_*) + \epsilon]$ for $n \geq N_2$. It follows from (3.8) and $N_2 \geq N_1$ that

$$u'(x, \alpha_n) = \int_{\xi_k(\alpha_n)}^{x} u''(t, \alpha_n)dt < 0 \quad \text{whenever} \quad n \geq N_2,$$

and $\xi_k(\alpha_n) < x \leq \xi_k(\alpha_*)+\epsilon$. Hence, ($a$) holds for the case where $u''(\xi_k(\alpha_*), \alpha_*) < 0$.

Now assume that $u''(\xi_k(\alpha_*), \alpha_*) = 0$. Thus, (3.2) and $\xi_k(\alpha_*) < \eta_k(\alpha_*)$ imply that $u'''(\xi_k(\alpha_*), \alpha_*) < 0$. Again, we restrict ϵ so that $u'''(x, \alpha_*) < 0$ on the compact interval $[\xi_k(\alpha_*)-\epsilon, \xi_k(\alpha_*)+\epsilon]$. Continuity of u''' in (x, α) ensures that we can choose $N_2 \geq N_1$ so that $u'''(x, \alpha_n) < 0$ on $[\xi_k(\alpha_*) - \epsilon, \xi_k(\alpha_*) + \epsilon]$ whenever $n \geq N_2$. Now, if $\xi_k(\alpha_n) < x \leq \xi_k(\alpha_n) + \epsilon$ and $n \geq N_2$, then

$$\begin{aligned} 0 &> \int_{\xi_k(\alpha_n)}^{x} \int_{\xi_k(\alpha_n)}^{t} u'''(s, \alpha_n)\, ds\, dt \\ &= u'(x, \alpha_n) - u''(\xi_k(\alpha_n), \alpha_n)(x - \xi_k(\alpha_n)). \end{aligned} \tag{3.9}$$

Combining (3.9) with the fact that $u''(\xi_k(\alpha_n), \alpha_n) \leq 0$ we obtain $u'(x, \alpha_n) < u''(\xi_k(\alpha_n), \alpha_n)(x - \xi_k(\alpha_n)) \leq 0$ whenever $\xi_k(\alpha_n) < x \leq \xi_k(\alpha_n) + \epsilon$ and $n \geq N_2$. This completes the proof of part (a).

To prove (b) note that $u'(x, \alpha_*) < 0$ on $[\xi_k(\alpha_*) + \epsilon, \eta_k(\alpha_*) - \epsilon]$. Since u' is continuous in (x, α), we can choose $N > 0$ so that $u'(x, \alpha_n) < 0$ on the interval $[\xi_k(\alpha_*) + \epsilon, \eta_k(\alpha_*) - \epsilon]$ for each $n \geq N$. This completes the proof of part (b) as well as the proof of the lemma. □

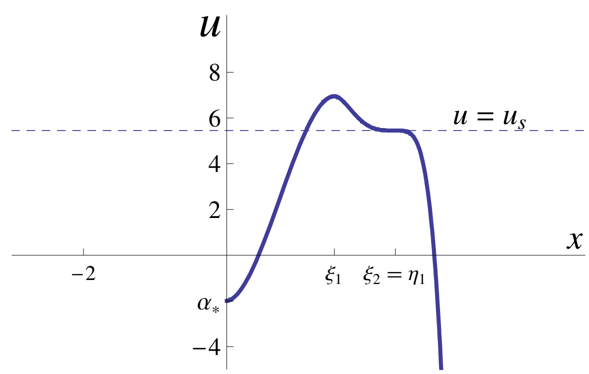

FIGURE 4. The case where $\xi_k(\alpha_*) = \eta_{k-1}(\alpha_*)$ for $k = 2$.

In the following lemma, we prove continuity of η_k at $\alpha = \alpha_*$ for $\xi_k(\alpha_*) = \eta_k(\alpha_*)$.

Lemma 3.8. *Assume that ξ_k is continuous. Then η_k is continuous at any α_* such that $\xi_k(\alpha_*) = \eta_k(\alpha_*)$.*

Proof. Let $\epsilon > 0$ and suppose that $\{\alpha_n\}$ is a sequence such that $\alpha_n \to \alpha_*$ as $n \to \infty$. We must show that there exists $N > 0$ so that

(i) $\eta_k(\alpha_n) > \eta_k(\alpha_*) - \epsilon$ and
(ii) $\eta_k(\alpha_n) < \eta_k(\alpha_*) + \epsilon$

for all $n \geq N$. To prove (i) we use continuity of ξ_k and the fact that $\eta_k(\alpha_n) \geq \xi_k(\alpha_n)$ for any α_n. Hence,

$$\eta_k(\alpha_n) \geq \xi_k(\alpha_n) \to \xi_k(\alpha_*) = \eta_k(\alpha_*).$$

This proves (i).

(ii): We proceed by contradiction and assume that

$$\eta_k(\alpha_n) \geq \eta_k(\alpha_*) + \epsilon. \tag{3.10}$$

By continuity of ξ_k, there exists $N > 0$ so that $\xi_k(\alpha_n) < \xi_k(\alpha_*) + \epsilon/2$ whenever $n \geq N$. Combining this estimate with (3.10) yields

$$\xi_k(\alpha_n) < \xi_k(\alpha_*) + \epsilon/2 = \eta_k(\alpha_*) + \epsilon/2 < \eta_k(\alpha_n) \quad \text{whenever} \quad n \geq N. \tag{3.11}$$

Therefore, $u'(x, \alpha_n) < 0$ on $(\xi_k(\alpha_n), \eta_k(\alpha_n))$. In particular, $u'(\xi_k(\alpha_*)+\epsilon/2, \alpha_n) < 0$ for all $n \geq N$ as a consequence of (3.11). Now by Lemma 3.2 and the assumption that $\xi_k(\alpha_*) = \eta_k(\alpha_*)$ it follows that $\xi_{k+1}(\alpha_*) > \eta_k(\alpha_*)$. If necessary, we restrict ϵ so that $\xi_k(\alpha_*)+\epsilon/2 < \xi_{k+1}(\alpha_*)$. Doing so guarantees that $u'(\xi_k(\alpha_*)+\epsilon/2, \alpha_*) > 0$. This and continuity of u' in (x, α) contradicts that $u'(\xi_k(\alpha_*)+\epsilon/2, \alpha_n) < 0$ for each $n \geq N$. That is,

$$0 < u'(\xi_k(\alpha_*) + \epsilon/2, \alpha_*) = \lim_{n\to\infty} u'(\xi_k(\alpha_*) + \epsilon/2, \alpha_n) \leq 0.$$

□

We now summarize Lemmas 3.6, 3.7, and 3.8 with the following theorem.

Theorem 3.9. *If ξ_k is continuous, then η_k is continuous.*

We now prove that ξ_{k+1} is continuous given that η_k is continuous. As we did in proving continuity of η_k, we will consider three separate cases. The case in which $\eta_k(\alpha) < \xi_{k+1}(\alpha) < \eta_{k+1}(\alpha)$ follows from Lemma 3.6. Thus, all that remains are the cases where $\eta_k(\alpha) = \xi_{k+1}(\alpha)$ and $\xi_{k+1}(\alpha) = \eta_{k+1}(\alpha)$. The proofs that follow are very similar to those in Lemmas 3.7 and 3.8. For this reason we will merely sketch the proofs of continuity of ξ_{k+1}.

Lemma 3.10. *Suppose that η_k is continuous. Then ξ_{k+1} is continuous at any $\alpha = \alpha_*$ such that $\eta_k(\alpha_*) = \xi_{k+1}(\alpha_*)$.*

Proof. Let $\epsilon > 0$ and $\{\alpha_n\}$ be a sequence such that $\alpha_n \to \alpha_*$ as $n \to \infty$. We must show that there exists $N > 0$ such that

(i) $\xi_{k+1}(\alpha_n) > \xi_{k+1}(\alpha_*) - \epsilon$
(ii) $\xi_{k+1}(\alpha_n) < \xi_{k+1}(\alpha_*) + \epsilon$

whenever $n \geq N$.

(i): Note that $\xi_{k+1}(\alpha_n) \geq \eta_k(\alpha_n)$ for any α_n. This together with our assumption that η_k is continuous yield

$$\xi_{k+1}(\alpha_n) \geq \eta_k(\alpha_n) \to \eta_k(\alpha_*) = \xi_{k+1}(\alpha_*) \quad \text{as } n \to \infty.$$

This completes the proof of (i).

(ii): To obtain a contradiction, assume that $\xi_{k+1}(\alpha_n) \geq \xi_{k+1}(\alpha_*) + \epsilon$. It follows from Lemma 3.2 and $\eta_k(\alpha_*) = \xi_{k+1}(\alpha_*)$ that $\xi_{k+1}(\alpha_*) < \eta_{k+1}(\alpha_*)$, and hence $u'(x, \alpha_*) < 0$ on $(\xi_{k+1}(\alpha_*), \eta_{k+1}(\alpha_*))$. We will restrict ϵ if necessary so that $\xi_{k+1}(\alpha_*) + \epsilon/2 < \eta_{k+1}(\alpha_*)$. This guarantees that $u'(\xi_{k+1}(\alpha_*) + \epsilon/2, \alpha_*) < 0$. Now choose $N > 0$ so that $\eta_k(\alpha_n) < \eta_k(\alpha_*) + \epsilon/2 = \xi_{k+1}(\alpha_*) + \epsilon/2$ for $n \geq N$. Hence, $\eta_k(\alpha_n) < \xi_{k+1}(\alpha_*) + \epsilon/2 < \xi_{k+1}(\alpha_n)$ for each $n \geq N$. But this implies that $u'(\xi_{k+1}(\alpha_*) + \epsilon/2, \alpha_n) > 0$ for each $n \geq N$. This and $u'(\xi_{k+1}(\alpha_*) + \epsilon/2, \alpha_*) < 0$ lead to a contradiction. This completes the proof of (ii) as well as the lemma. □

All that remains is the case where $\eta_{k+1}(\alpha_*) = \xi_{k+1}(\alpha_*)$.

Lemma 3.11. *Suppose that η_k is continuous. Then ξ_{k+1} is continuous at any $\alpha = \alpha_*$ such that $\eta_{k+1}(\alpha_*) = \xi_{k+1}(\alpha_*)$.*

Proof. Let $\epsilon > 0$ and $\{\alpha_n\}$ be a sequence such that $\alpha_n \to \alpha_*$ as $n \to \infty$. We must show that there exists $N > 0$ such that

(i) $\xi_{k+1}(\alpha_n) > \xi_{k+1}(\alpha_*) - \epsilon$
(ii) $\xi_{k+1}(\alpha_n) < \xi_{k+1}(\alpha_*) + \epsilon$

whenever $n \geq N$.

(i): We first show that there exists $N > 0$ so that $\eta_k(\alpha_n) < \xi_{k+1}(\alpha_n)$ for all $n \geq N$. Since $\xi_{k+1}(\alpha_*) = \eta_{k+1}(\alpha_*)$, then Lemma 3.2 implies that $\eta_k(\alpha_*) < \xi_{k+1}(\alpha_*)$. Hence, by (3.3), it follows that either $u''(\eta_k(\alpha_*), \alpha_*) > 0$ or $u'''(\eta_k(\alpha_*), \alpha_*) > 0$. It now follows from continuity of η_k in α and u'' and u''' in (x, α) that there exists $N > 0$ so that either $u''(\eta_k(\alpha_n), \alpha_n) > 0$ or $u'''(\eta_k(\alpha_n), \alpha_n) > 0$ for all $n \geq N$. Thus, $\eta_k(\alpha_n) < \xi_{k+1}(\alpha_n)$ for all $n \geq N$, and $u'(x, \alpha_n) > 0$ on $(\eta_k(\alpha_n), \xi_{k+1}(\alpha_n))$ for all $n \geq N$. Without loss of generality, assume that $\eta_k(\alpha_n) < \xi_{k+1}(\alpha_n)$ for all $n \geq 1$.

Our goal now is to show that there exists $N > 0$ so that $u'(x, \alpha_n) > 0$ on $(\eta_k(\alpha_n), \xi_{k+1}(\alpha_*) - \epsilon]$ for all $n \geq N$. This will be done in two steps. That is, we will show that there exists $N > 0$ so that

(*a*) $u'(x, \alpha_n) > 0$ on $(\eta_k(\alpha_n), \eta_k(\alpha_*) + \epsilon]$, and
(*b*) $u'(x, \alpha_n) > 0$ on $[\eta_k(\alpha_*) + \epsilon, \xi_{k+1}(\alpha_*) - \epsilon]$

for each $n \geq N$. For technical purposes we restrict ϵ so that $\eta_k(\alpha_*) + \epsilon < \xi_{k+1}(\alpha_*) - \epsilon$. By continuity of η_k there exists $N_1 > 0$ so that $\eta_k(\alpha_*) - \epsilon < \eta_k(\alpha_n) < \eta_k(\alpha_*) + \epsilon$ for all $n \geq N_1$.

By (3.2) it follows that $u''(\eta_k(\alpha_*), \alpha_*) \geq 0$. We begin proving (*a*) by assuming that $u''(\eta_k(\alpha_*), \alpha_*) > 0$. We further restrict ϵ so that $u''(x, \alpha_*) > 0$ on the interval $[\eta_k(\alpha_*) - \epsilon, \eta_k(\alpha_*) + \epsilon]$. By continuity of u'' there exists $N_2 \geq N_1$ so that $u''(x, \alpha_n) > 0$ on $[\eta_k(\alpha_*) - \epsilon, \eta_k(\alpha_*) + \epsilon]$ whenever $n \geq N_2$. Hence,

$$u'(x, \alpha_n) = \int_{\eta_k(\alpha_n)}^{x} u''(t, \alpha_n) dt > 0$$

whenever $\eta_k(\alpha_n) < x \leq \eta_k(\alpha_*) + \epsilon$ and $n \geq N_2$. This proves (*a*) for the case $u''(\eta_k(\alpha_*), \alpha_*) > 0$.

Now assume that $u''(\eta_k(\alpha_*), \alpha_*) = 0$. Then $u'''(\eta_k(\alpha_*), \alpha_*) > 0$ as a consequence of $\xi_{k+1}(\alpha_*) > \eta_k(\alpha_*)$ and (3.3). Again we restrict ϵ to ensure that $u'''(x, \alpha_*) > 0$ on $[\eta_k(\alpha_*) - \epsilon, \eta_k(\alpha_*) + \epsilon]$. If follows by continuity of u''' in (x, α) that there exists $N_2 \geq N_1$ so that $u'''(x, \alpha_n) > 0$ on $[\eta_k(\alpha_*) - \epsilon, \eta_k(\alpha_*) + \epsilon]$ whenever $n \geq N_2$. Thus,

$$\begin{aligned} 0 &< \int_{\eta_k(\alpha_n)}^{x} \int_{\eta_k(\alpha_n)}^{t} u'''(s, \alpha_n)\, ds\, dt \\ &= u'(x, \alpha_n) - u''(\eta_k(\alpha_n), \alpha_n)(x - \eta_k(\alpha_n)) \end{aligned} \tag{3.12}$$

whenever $\eta_k(\alpha_n) < x \leq \eta_k(\alpha_*) + \epsilon$ and $n \geq N_2$. Since $u''(\eta_k(\alpha_n), \alpha_n) \geq 0$, then $u'(x, \alpha_n) > 0$ on $(\eta_k(\alpha_n), \eta_k(\alpha_*) + \epsilon]$ follows from (3.12). This completes (*a*).

To prove (*b*), note that $u'(x, \alpha_*) > 0$ on $[\eta_k(\alpha_*) + \epsilon, \xi_{k+1}(\alpha_*) - \epsilon]$. Then continuity of u' in (x, α) ensures that there exists $N > 0$ so that $u'(x, \alpha_n) > 0$ on $[\eta_k(\alpha_*) + \epsilon, \xi_{k+1}(\alpha_*) - \epsilon]$ whenever $n \geq N$. This completes the proof of part (i).

(ii): For a contradiction, assume that $\xi_{k+1}(\alpha_n) \geq \xi_{k+1}(\alpha_*) + \epsilon$ for all $n \geq 1$. Our premise that $\eta_{k+1}(\alpha_*) = \xi_{k+1}(\alpha_*)$ and Lemma 3.2 imply that $\eta_k(\alpha_*) < \xi_{k+1}(\alpha_*)$.

For technical purposes we restrict ϵ so that

$$\eta_k(\alpha_*) + \epsilon < \xi_{k+1}(\alpha_*) - \epsilon. \tag{3.13}$$

Since η_k is continuous, there exists $N_1 > 0$ so that

$$\eta_k(\alpha_n) < \eta_k(\alpha_*) + \epsilon \quad \text{for all } n \geq N_1. \tag{3.14}$$

This together with our contradiction premise, $\xi_{k+1}(\alpha_n) \geq \xi_{k+1}(\alpha_*) + \epsilon$, yield that $\eta_k(\alpha_n) < \xi_{k+1}(\alpha_n)$ for all $n \geq N_1$, and therefore $u'(x, \alpha) > 0$ on $(\eta_k(\alpha_n), \xi_{k+1}(\alpha_n))$ for all $n \geq N_1$.

By (3.2), $\eta_{k+1}(\alpha_*) = \xi_{k+1}(\alpha_*)$ guarantees that $u'''(\xi_{k+1}(\alpha_*), \alpha_*) > 0$. We can further restrict ϵ so that $u'''(x, \alpha_*) > 0$ on $[\xi_{k+1}(\alpha_*) - \epsilon, \xi_{k+1}(\alpha_*) + \epsilon]$. It now follows from Lemma 3.3 that there exists $N_2 \geq N_1$, and a sequence $\{\tau_n\}$ so that

(*a*) $u''(\tau_n, \alpha_n) = 0$,
(*b*) $\xi_{k+1}(\alpha_*) - \epsilon < \tau_n < \xi_{k+1}(\alpha_*) + \epsilon$, and
(*c*) $\tau_n \to \xi_{k+1}(\alpha_*)$ as $n \to \infty$

for all $n \geq N_2$.

Combining the contradiction premise with (3.13), (3.14), and (*b*) yield that $\eta_k(\alpha_n) < \tau_n < \xi_{k+1}(\alpha_n)$ for each $n \geq N_2$. Using the notation

$$u_n = u(\tau_n, \alpha_n), u_n' = u'(\tau_n, \alpha_n), \ u_n'' = u''(\tau_n, \alpha_n), \quad \text{and} \quad u_n''' = u'''(\tau_n, \alpha_n),$$

we obtain $u_n' > 0$ for each $n \geq N_2$.

Combining (*c*) with continuity of u', u''' results in $u_n' \to u'(\xi_{k+1}(\alpha_*), \alpha_*) = 0$ and $u_n''' \to u'''(\xi_{k+1}(\alpha_*), \alpha_*) > 0$. Thus, we can choose $N_3 \geq N_2$ so that $u_n''' - (b^2 - 1)u_n' > 0$ for all $n \geq N_3$. Now Eq. (2.8) can be written as

$$u_n'(u_n''' - (b^2 - 1)u_n') + (b^2 + 1)^2 Q(u_n) = 0.$$

But the left side of the above equation is strictly positive for all $n \geq N_3$. This is the desired contradiction. This concludes the proof of the lemma. □

We now summarize Lemmas 3.6, 3.10, and 3.11 in the following theorem.

Theorem 3.12. *If η_k is continuous, then ξ_{k+1} is continuous.*

Theorems 3.4, 3.9 and 3.12 show that ξ_k and η_k are continuous for all $k \geq 1$.

4. N-Bump Solutions

In this section we prove the existence of N-bump solutions of (2.5) for any even valued $N > 0$. Recall that a solution is an N-bump solution if there are exactly N disjoint intervals in which $u > th$, i.e., u exceeds threshold, and u must also satisfy the limiting property $(u, u', u'', u''') \to (0, 0, 0, 0)$. We will continue to assume that the initial values of (2.5) are $\alpha < 0$ and $\beta = -(b^2 + 1)\alpha > 0$. In addition, we will assume that the conditions of Theorem 2.4 hold. This guarantees that the functions η_k and ξ_k in (3.1), (3.2), and (3.3) are well defined.

We now proceed to develop a shooting method to prove the existence of multi-bump homoclinic solutions. This shooting method requires that we precisely determine the behavior of $u(x, \alpha)$ at (or near) the parameter value $\alpha = \alpha'$ where

(*a*) $u(\xi_k(\alpha'), \alpha') = u_s$, (or $\ u(\xi_k(\alpha'), \alpha') = 0$), for some $k \geq 1$,
(*b*) $u(a_0, \alpha') = th$, where $a_0 = \pm\xi_k(\alpha')$.

Solutions with the critical values $u = 0$ or $u = u_s$. Recall that when $(r, b, th) \in \Lambda$, then (2.5) has exactly two constant solutions, namely $u \equiv 0$ and $u \equiv u_s > 0$. Recall also that these constant solutions correspond to the only two roots of the function Q defined by (2.7). That is, $Q(0) = Q(u_s) = 0$, (see Figure 2).

In Lemma 4.1 we let u_* denote one of the two roots of the function Q, i.e., either $u_* = 0$ or $u_* = u_s$. It will be shown that if $u(\xi_k(\alpha), \alpha)$ crosses the line $u = u_*$ from below at some $\alpha = \alpha' < 0$ and $k \geq 1$, then $u(\eta_k(\alpha), \alpha)$ also crosses the line $u = u_*$ from below. In other words, if $u(\xi_k(\alpha'), \alpha') = u_*$ with $u'''(\xi_k(\alpha'), \alpha') > 0$, (see Figure 5), then $\xi_k(\alpha') = \eta_k(\alpha')$ and $u(\xi_k(\alpha), \alpha) - u_*$ has the same sign as $u(\eta_k(\alpha), \alpha) - u_*$ on an interval of the form $(\alpha' - \delta, \alpha')$ for some $\delta > 0$. Using similar arguments we can show that $u(\xi_k(\alpha), \alpha) - u_*$ has the same sign as $u(\eta_k(\alpha), \alpha) - u_*$ on an interval of the form $(\alpha', \alpha' + \delta)$. There are other cases which occur. There is the case where $u(\xi_k(\alpha), \alpha)$ crosses the line $u = u_*$ at $\alpha = \alpha'$ from above. That is, $u(\xi_k(\alpha'), \alpha') = u_*$ with $u'''(\xi_k(\alpha'), \alpha') < 0$, and hence $\xi_k(\alpha') = \eta_{k-1}(\alpha')$. In this case, we can apply similar arguments to show that $u(\eta_{k-1}(\alpha), \alpha)$ must also cross $u = u_*$ from above. The method employed in the following lemma can be easily modified to prove the other cases.

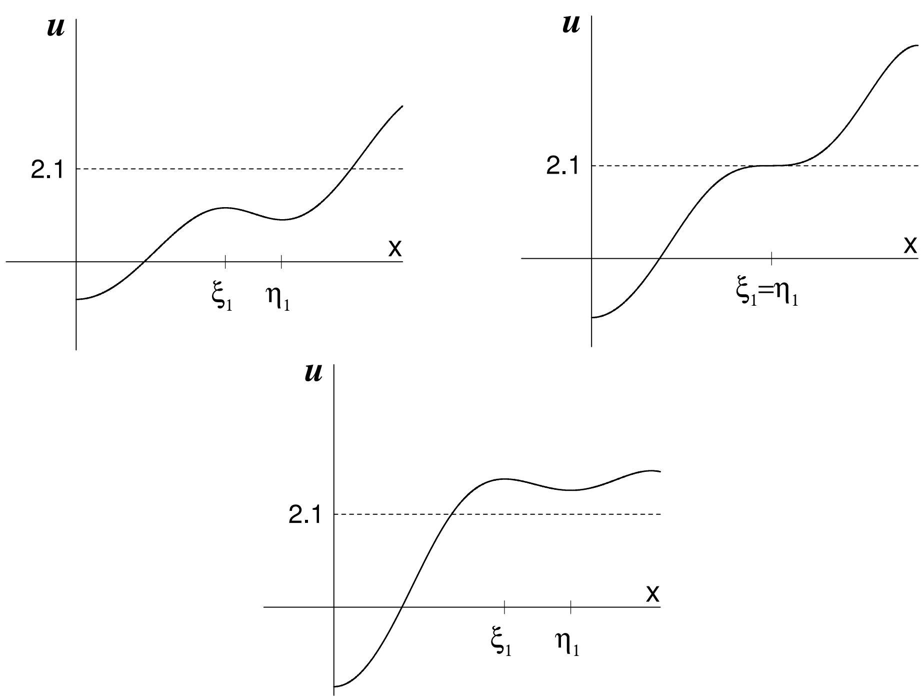

FIGURE 5. See text

Lemma 4.1. *Suppose that $(r, b, th) \in \Lambda$, (recall Eq. (2.11)), and that $Q(u_*) = 0$, (that is, $u_* = 0$ or $u_* = u_s$). Also, assume that $\alpha' < 0$ and $\beta' = -(b^2 + 1)\alpha' > 0$ determine a solution $u(\cdot, \alpha')$ of (2.5) such that*

$$u(\xi_k(\alpha'), \alpha') = u_* \quad \text{for some} \quad k \geq 1.$$

If there exists $\delta > 0$ *such that*

$$u(\xi_k(\alpha), \alpha) > u_* \quad \text{for} \quad \alpha' - \delta < \alpha < \alpha', \tag{4.1}$$

then there exists $\epsilon > 0$ *so that*

(i) $u(\xi_k(\alpha), \alpha) > u(\eta_k(\alpha), \alpha) > u_*$ for $\alpha' - \epsilon < \alpha < \alpha'$
given that $u'''(\xi_k(\alpha'), \alpha') > 0$, *and*

(ii) $u(\xi_k(\alpha), \alpha) > u(\eta_{k-1}(\alpha), \alpha) > u_*$, (provided that $k \geq 2$), for $\alpha' - \epsilon < \alpha < \alpha'$
given that $u'''(\xi_k(\alpha'), \alpha') < 0$.

Proof. (i): Assume that $u'''(\xi_k(\alpha'), \alpha') > 0$. Since $u(\xi_k(\alpha'), \alpha') = u_*$, $Q(u_*) = 0$, and $u'(\xi_k(\alpha'), \alpha') = 0$, then Eq. (2.8) shows that $u''(\xi_k(\alpha'), \alpha') = 0$. Hence, $\xi_k(\alpha') = \eta_k(\alpha')$ as a consequence of (3.2).

The proof of (i) is by contradiction, i.e., assume that there is an increasing sequence α_n such that

(a) $\alpha_n \to \alpha'$ as $n \to \infty$,
(b) $\alpha' - \delta < \alpha_n < \alpha'$, and
(c) $u(\eta_k(\alpha_n), \alpha_n) \leq u_*$ for all $n \geq 1$.

Our first task is to show that $u(\eta_k(\alpha_n), \alpha_n) = u_*$ is impossible whenever n is sufficiently large. For a contradiction assume that there is a subsequence α_{n_i} such that $u(\eta_k(\alpha_{n_i}), \alpha_{n_i}) = u_*$ for each $i \geq 1$. At $x = \eta_k(\alpha_{n_i})$ Eq. (2.8) gives

$$u''(\eta_k(\alpha_{n_i}), \alpha_{n_i}) = 0 \quad \text{for each } i \geq 1. \tag{4.2}$$

Since (b) holds, then (4.1) ensures that $u(\xi_k(\alpha_{n_i}), \alpha_{n_i}) > u_*$ for each $i \geq 1$. Hence, $u'(x, \alpha_{n_i}) < 0$ on $(\xi_k(\alpha_{n_i}), \eta_k(\alpha_{n_i}))$. By combining this result with Lemma 3.1 and (4.2) we see that $u'''(\eta_k(\alpha_{n_i}), \alpha_{n_i}) < 0$ for each $i \geq 1$. Continuity of u''' in (x, α) and η_k in α yields

$$\lim_{i \to \infty} u'''(\eta_k(\alpha_{n_i}), \alpha_{n_i}) = u'''(\eta_k(\alpha'), \alpha') \leq 0. \tag{4.3}$$

But this is impossible since $u'''(\eta_k(\alpha'), \alpha') = u'''(\xi_k(\alpha'), \alpha') > 0$. Therefore, we may assume that

$$u(\eta_k(\alpha_n), \alpha_n) < u_* \quad \text{for all } n \geq 1. \tag{4.4}$$

Next we show that

$$Q(u(\eta_k(\alpha_n), \alpha_n)) > 0 \quad \text{and} \quad Q(u(\xi_k(\alpha_n), \alpha_n)) > 0 \quad \text{for all } n \geq 1. \tag{4.5}$$

Recall that u_* denotes one of the two roots of Q, namely $u_* = 0$ or $u_* = u_s$. First assume that $u_* = u_s$. Then $Q(u(\xi_k(\alpha_n), \alpha_n)) > 0$ follows from (4.1) and (b). Continuity of η_k in α and u in (x, α), gives

$$\lim_{n \to \infty} u(\eta_k(\alpha_n), \alpha_n) = u(\eta_k(\alpha'), \alpha') = u(\xi_k(\alpha'), \alpha') = u_s > 0.$$

This and (4.4) show that there exists $N > 0$ so that $u_s > u(\eta_k(\alpha_n), \alpha_n) > 0$ for all $n \geq N$. Thus, $Q(u(\eta_k(\alpha_n), \alpha_n)) > 0$ for all $n \geq N$. If $u_* = 0$, then a similar argument shows that $N > 0$ exists so that $u_s > u(\xi_k(\alpha_n), \alpha_n) > 0$ whenever $n \geq N$. This guarantees that $Q(u(\xi_k(\alpha_n), \alpha_n)) > 0$ whenever $n \geq N$. Without loss of generality, we assume that (4.5) holds for all $n \geq 1$.

We now show that there exists $x_n \in (\xi_k(\alpha_n), \eta_k(\alpha_n))$ at which

$$-\frac{(u''(x_n, \alpha_n))^2}{2} + (b^2 + 1)^2 Q(u(x_n, \alpha_n)) = 0 \tag{4.6}$$

for all $n \geq 1$. An immediate consequence of (4.5) and (2.8) at $x = \xi_k(\alpha_n)$ is that $u''(\xi_k(\alpha_n), \alpha_n) < 0$ for each $n \geq 1$. Likewise, it follows that $u''(\eta_k(\alpha_n), \alpha_n) > 0$ for each $n \geq 1$. Thus, there exists a value $y_n \in (\xi_k(\alpha_n), \eta_k(\alpha_n))$ such that $u''(y_n, \alpha_n) = 0$. Note that $u(\xi_k(\alpha_n), \alpha_n) > u_* > u(\eta_k(\alpha_n), \alpha_n)$ implies the existence of $z_n \in (\xi_k(\alpha_n), \eta_k(\alpha_n))$ such that $u(z_n, \alpha) = u_*$, hence $Q(u(z_n, \alpha_n)) = 0$ for each $n \geq 1$. Thus, we have

$$-\frac{(u''(x,\alpha_n))^2}{2} + (b^2+1)^2 Q(u(x,\alpha_n)) \quad \begin{cases} \leq 0 & \text{at} \quad x = z_n \\ \geq 0 & \text{at } x = y_n. \end{cases} \tag{4.7}$$

This ensures the existence of a value x_n between y_n, z_n for which (4.6) holds. Also, $\xi_k(\alpha_n) < x_n < \eta_k(\alpha_n)$ implies that $u'(x_n, \alpha_n) < 0$ for each $n \geq 1$. This fact together with (2.8) yields

$$u'''(x_n, \alpha_n) - (b^2 - 1)u'(x_n, \alpha_n) = 0 \quad \text{for each} \quad n \geq 1. \tag{4.8}$$

Since $\xi_k(\alpha_n) \to \xi_k(\alpha')$ and $\eta_k(\alpha_n) \to \eta_k(\alpha') = \xi_k(\alpha')$ as $n \to \infty$, then $\xi_k(\alpha_n) < x_n < \eta_k(\alpha_n)$ implies that $x_n \to \xi_k(\alpha')$ as $n \to \infty$. Therefore,

$$u'(x_n, \alpha_n) \to u'(\xi_k(\alpha'), \alpha') = 0 \quad \text{as } n \to \infty$$

by continuity of u' in (x, α). This and (4.8) imply that

$$u'''(x_n, \alpha_n) \to 0 \quad \text{as } n \to \infty. \tag{4.9}$$

But this contradicts $u'''(x_n, \alpha_n) \to u'''(\xi_k(\alpha'), \alpha') > 0$ as $n \to \infty$. This concludes the proof of (i).

We can prove (ii) in a similar fashion. □

Solutions with a local maximum at $u = th$. A solution that plays a crucial role in our shooting argument is one that satisfies

$$u(a_0, \alpha') = th, \quad \text{where } a_0 = \pm\xi_k(\alpha') \tag{4.10}$$

for some $k \geq 1$ (see Figure 6). The object of Lemmas 4.2-4.3 is to show that $|u(\zeta)| > th$ for every $\zeta > a_0$ such that $u'(\zeta) = 0$. More precisely, Lemma 4.2 is used in the proof of Lemma 4.3 to establish the sharper estimate $u(b_0) \leq -u_{ss}(r) < -th$ where

$$b_0 = \sup\{x > a_0 \mid u'(x) < 0 \quad \text{on } (a_0, x)\}.$$

Throughout Lemmas 4.2-4.4 it is not necessary to continually refer to the initial condition parameter α. Hence, we omit the initial condition parameter and write $u(x)$ instead of $u(x, \alpha)$.

The following lemma relies on several preliminary results established in Section 2. First, Lemma 2.5 states that for any fixed $0 < th < 2$ there exists $b_r > 1$ such that $(r, b_r, th) \in \Lambda$ whenever $r > 0$ is sufficiently small. and

$$b_r \to \frac{2}{th} + \frac{\sqrt{4 - th^2}}{th} \quad \text{as } r \to 0^+.$$

The fact that

$$\frac{2}{th} + \frac{\sqrt{4 - th^2}}{th} > 1$$

for any fixed $0 < th < 2$ implies that b_r is bounded away from 1 as $r \to 0^+$. In particular, there exists a value $R > 0$ such that

$$b_r \geq 1 + m \quad \text{for all } r \in (0, R) \tag{4.11}$$

where

$$m = \frac{1}{2}\Big(\frac{2}{th} + \frac{\sqrt{4-th^2}}{th} - 1\Big). \tag{4.12}$$

In addition, Lemma 4.2 relies on the fact that

$$u_{ss}(r) - th \to 0^+ \quad \text{as } r \to 0^+, \tag{4.13}$$

which follows immediately from Lemma 2.6.

Lemma 4.2. *Suppose that u is a nontrivial solution of* (2.5) *where $0 < th < 2$ is fixed. Choose $R > 0$ so that* (4.11) *holds, and that*

$$u_{ss}(r) - th \le \frac{2(b_r^2 - 1)th}{b_r^2 + 1} \quad \text{for all } r \in (0, R). \tag{4.14}$$

Assume that $(-\omega, \omega)$ is the maximal interval in $\mathbb{R}$ on which u exists and that $a_0 \in (-\omega, \omega)$ satisfies

$$u(a_0) = th, u'(a_0) = 0, u''(a_0) = -(b_r^2 + 1)th \quad \text{and} \quad u'''(a_0) \le 0. \tag{4.15}$$

Then there exists $\bar{x} > a_0$ such that

(i) $u'(x) < 0 \quad$ on $(a_0, \bar{x}]$,
(ii) $u(x) > 0$ *on* $(a_0, \bar{x})$*, and* $u(\bar{x}) = 0$*, and*
(iii) $u''(\bar{x}) \le -(b_r^2 + 1)u_{ss}(r)$.

Remark: Throughout the proof, we shall write b and u_{ss} instead of b_r and $u_{ss}(r)$.

Proof of Lemma 4.2. By (2.5), $u''''(a_0) < 0$, thus u''', u'', u', u are decreasing in a right neighborhood of $x = a_0$. Note that (2.5) also implies that $u'''' < 0$ as long as $0 < u \le th$, and $u'' < 0$. Hence, there exists $\bar{x} > a_0$ such that (i) and (ii) hold.

We prove *(iii)* by contradiction. That is, assume that

$$u''(x) > -(b^2 + 1)u_{ss} \quad \text{on } (a_0, \bar{x}]. \tag{4.16}$$

By (2.5) it follows that

$$u'''' = 2(b^2 - 1)u'' - (b^2 + 1)^2 u < 2(b^2 - 1)u'' \quad \text{on } (a_0, \bar{x}). \tag{4.17}$$

Recalling that $u(a_0) = th, u''(a_0) = -(b^2 + 1)th$ and $u'''(a_0) \le 0$ we obtain

$$u''(x) \le (1 - 3b^2)th + 2(b^2 - 1)u \tag{4.18}$$

upon two integrations of (4.17) over $(a_0, x) \subset (a_0, \bar{x})$. Combining (4.16) with (4.18) yields

$$-(b^2 + 1)u_{ss} < (1 - 3b^2)th + 2(b^2 - 1)u \quad \text{on } (a_0, \bar{x}),$$

and

$$-(b^2 + 1)u_{ss} < (1 - 3b^2)th \quad \text{at} \quad x = \bar{x}. \tag{4.19}$$

Denoting $m = u_{ss} - th$ (4.19) is equivalent to

$$-(b^2 + 1)(th + m) + (3b^2 - 1)th < 0$$

which results in

$$\frac{2(b^2 - 1)th}{b^2 + 1} < m = u_{ss} - th.$$

This contradicts (4.14), hence the proof is complete. □

In the following lemma, we consider a solution u of (2.5) that has a critical point (a_0, th), i.e., $u'(a_0) = 0$ and $u(a_0) = th$. Estimates of u at subsequent critical numbers are given.

Lemma 4.3. *Suppose that u is a nontrivial solution of (2.5) with $0 < th < 2$. Choose $R > 0$ so that for all $r < R$, $b \geq 1 + m$ where m is defined by (4.12) and*

$$u_{ss} - th \leq \frac{2(b^2 - 1)th}{b^2 + 1}.$$

Assume that $(-\omega, \omega)$ is the maximal interval in $\mathbb{R}$ on which u exists and that $a_0 \in (-\omega, \omega)$ satisfies

$$u(a_0) = th, u'(a_0) = 0, u''(a_0) \leq 0 \quad \text{and} \quad u'''(a_0) \leq 0. \tag{4.20}$$

Then, $u(b_0) < -u_{ss}$ and $u(a_1) > u_s$ where

$$b_0 = \sup\{x > a_0 \mid u'(t) \neq 0 \quad \text{on } (a_0, x)\}, \quad \text{and} \tag{4.21}$$

$$a_1 = \sup\{x > b_0 \mid u'(t) \neq 0 \quad \text{on } (b_0, x)\}. \tag{4.22}$$

Furthermore, for any $\zeta > a_0$ where $u'(\zeta) = 0$ we have

$$|u| > th, sgn\ u'' = -sgn\ u, \quad \text{and} \quad sgn\ u''' = -sgn\ u \quad \text{at} \quad x = \zeta. \tag{4.23}$$

In particular, if a local maximum occurs at the critical number $x = \zeta$, then $u(\zeta) > u_s$.

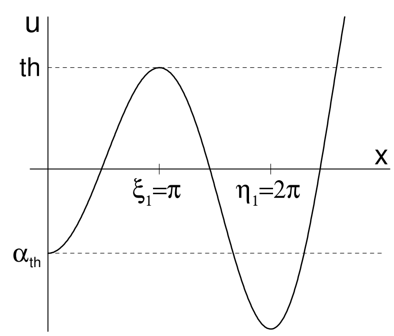

FIGURE 6. A solution with critical value $u = th$.

Proof. To prove this lemma we invoke an iterative procedure to which we apply Induction.

To start, we determine the exact value of $u''(a_0)$. By substituting $u(a_0) = th$ and $u'(a_0) = 0$ into (2.8) we find that $u''(a_0) = \pm(b^2 + 1)th$. Specifically, the assumption $u''(a_0) \leq 0$ implies that $u''(a_0) = -(b^2 + 1)th$. Thus, $u'(x) < 0$ on (a_0, b_0) and $u'(b_0) = 0$. The fact that $u''''(a_0) < 0$ implies that $u'''(x) < 0$ in a right-neighborhood of $x = a_0$, and hence

$$y_0 = \sup\{x > a_0 \mid u'''(t) < 0 \quad \text{on } (a_0, x)\} \tag{4.24}$$

is well defined and $y_0 \in (a_0, b_0)$.

We now show that $u(b_0) < -u_{ss}$ and that (4.23) holds at $x = b_0$.

The fact that $u''''(a_0) < 0$ implies that u'''' vanishes on the interval (a_0, y_0). By (2.5) and the fact that $u < th$ on (a_0, b_0) it follows that $u(\bar{x}) = 0$ for some $a_0 < \bar{x} < y_0$. Since $u'''(y_0) = 0$ the first integral equation (2.8) leads to

$$-\frac{(u'')^2}{2} - (b^2-1)(u')^2 + (b^2+1)^2 Q(u) \geq 0 \quad \text{at} \quad x = y_0,$$

or since $b > 1$,

$$2(b^2+1)^2 Q(u(y_0)) \geq (u''(y_0))^2. \tag{4.25}$$

Furthermore, $u'''(x) < 0$ on (a_0, y_0) yields $0 > u''(a_0) > u''(\bar{x}) > u''(y_0)$, hence

$$(u''(y_0))^2 > (u''(\bar{x}))^2.$$

This fact together with Lemma 4.2 and (4.25) give

$$2(b^2+1)^2 Q(u(y_0)) > (b^2+1)^2 u_{ss}^2.$$

The fact that

$$Q(u) = \frac{u^2}{2} \quad \text{on } (-\infty, th]$$

leads to $u(y_0) < -u_{ss}$. Since u decreases further until it reaches the next critical number, $x = b_0$, we must have

$$u(b_0) < u(y_0) < -u_{ss}. \tag{4.26}$$

It remains to show that $u''(b_0) > 0$ and $u'''(b_0) > 0$. The fact that $u'(b_0) = 0$ and $u'(x) < 0$ in a left-neighborhood of $x = b_0$ implies that $u''(b_0) \geq 0$. Also, since $u(b_0) < -u_{ss}$ it follows from (2.8) that

$$\frac{(u''(b_0))^2}{2} = (b^2+1)^2 Q(u(b_0)) \neq 0,$$

thus $u''(b_0) > 0$.

To show that $u'''(b_0) > 0$ first note that $u''(y_0) < 0 < u''(b_0)$. Thus a value $x_0 \in (y_0, b_0)$ exists such that $u''(x_0) = 0$ and $u'''(x_0) \geq 0$. Also, $u'(x) < 0$ on (a_0, b_0) and $\bar{x} < y_0 < x_0$, imply that $u(x) < 0$ on (x_0, b_0). An integration of (2.5) over (x_0, b_0) results in

$$u'''(b_0) = -(b^2+1)\int_{x_0}^{b_0} u\,dx - 2(b^2-1)u'(x_0) + u'''(x_0) > 0$$

as desired.

In a similar fashion, we shall show that $u(a_1) > u_s$ and (4.23) at $x = a_1$ where a_1 is defined by (4.22). First, recall that $u''(b_0) > 0$ and $u'''(b_0) > 0$ so that $u'(x) > 0$ on (b_0, a_1). Also, note that $u'(x) > 0$ holds so long as $u'''(x) > 0$. Thus,

$$y_1 = \sup\{x > b_0 \mid u'''(t) > 0 \quad \text{on } (b_0, x)\} \tag{4.27}$$

is well defined and $y_1 \in (b_0, a_1)$. Since $b > 1$ and $u'''(y_1) = 0$, then

$$-\frac{(u'')^2}{2} + (b^2+1)^2 Q(u) > 0 \quad \text{at} \quad x = y_1 \tag{4.28}$$

follows from (2.8). Because $u''(b_0) > 0$ and $u'''(x) > 0$ on (b_0, y_1), it follows that

$$(u''(y_1))^2 > (u''(b_0))^2. \tag{4.29}$$

Combining (4.28) with (4.29) yields

$$Q(u(y_1)) > \frac{(u''(y_1))^2}{2(b^2+1)^2} > \frac{(u''(b_0))^2}{2(b^2+1)^2} = Q(u(b_0)). \tag{4.30}$$

Now recall from (4.26) that $u(b_0) < -u_{ss}$, thus

$$Q(u(b_0)) > Q(-u_{ss}). \tag{4.31}$$

Since

$$Q(u) = \frac{u^2}{2} - \frac{8b}{b^2+1}\int_0^u e^{-r/(s-th)^2} H(s-th)ds,$$

then $Q(-u) > Q(u)$ for all $u > th$. Specifically, $Q(-u_{ss}) > Q(u_{ss})$ and by (4.31) we obtain

$$Q(u(b_0)) > Q(u_{ss}).$$

This and (4.30) yield

$$Q(u(y_1)) > Q(u_{ss}). \tag{4.32}$$

The fact that $u'(x) > 0$ on (b_0, a_1) together with (4.30) and (4.32) yield

$$u(a_1) > u(y_1) > u_s. \tag{4.33}$$

It remains to show that $u''(a_1) < 0$ and $u'''(a_1) < 0$. We start by showing that $u''(a_1) < 0$. The fact that $u'(x) > 0$ on (b_0, a_1) with $u'(a_1) = 0$ guarantees that $u''(a_1) \le 0$. Eq. (2.8) along with $u(a_1) > u_s$ leads to

$$\frac{(u'')^2}{2} = (b^2+1)^2 Q(u) > 0 \quad \text{at} \quad x = a_1.$$

Therefore, $u''(a_1) < 0$ as desired.

We proceed to show that $u'''(a_1) < 0$. First, note that $u''(b_0) > 0 > u''(a_1)$, hence there exists a value $x_1 \in (y_1, a_1)$ such that $u''(x_1) = 0$ and $u'''(x_1) \le 0$. Here we use the fact that

$$Q'(u) = u - \frac{4b}{b^2+1} f(u-th) > 0 \quad \text{for all } u > u_s.$$

By (4.30) and (4.32) it follows that $u(y_1) > u_s$. Since $u'(x) > 0$ on (b_0, a_1) and $x_1 > y_1$, then $u(x) > u_s$ on (x_1, a_1). Thus, $Q'(u(x)) > 0$ on (x_1, a_1). An integration of (2.5) over (x_1, a_1) gives

$$u'''(a_1) = u'''(x_1) - 2(b^2-1)u'(x_1) - (b^2+1)^2 \int_{x_1}^{a_1} Q'(u(s))ds < 0$$

as desired.

In a similar fashion, one can show that (4.23) holds at $x = b_1$ where

$$b_1 = \sup\{x > a_1 \mid u'(t) < 0 \quad \text{on } (a_1, x)\}.$$

In general, this procedure can be continued inductively to show that (4.23) holds for all $\zeta > a_0$ for which $u'(\zeta) = 0$. $\square$

Lemma 4.3 addresses the behavior of a solution u that satisfies

$$u(a_0) = th, u'(a_0) = 0, u''(a_0) \le 0, \quad \text{and} \quad u'''(a_0) \le 0.$$

In the following lemma we show that there is no loss in generality in assuming that $u'''(a_0) \le 0$ if $a_0 > 0$.

Lemma 4.4. *Suppose that u is a non-constant solution of* (2.5) *with* $0 < th < 2$. *Choose $R > 0$ so that for all $r < R$, $b \ge 1 + \delta$ for some $\delta > 0$, and*

$$u_{ss} - th \le \frac{2(b^2-1)th}{b^2+1}.$$

Assume that $(-\omega, \omega)$ *is the maximal interval in* $\mathbb{R}$ *on which* u *exists and that* $a_0 \in (0, \omega)$ *satisfies*

$$u(a_0) = th, u'(a_0) = 0, \quad \text{and} \quad u''(a_0) \leq 0. \tag{4.34}$$

Then $u'''(a_0) \leq 0$.

Proof. Assume on the other hand that $u'''(a_0) > 0$. Define $v(x) = u(-x)$. By symmetry of u, (see Lemma 2.2), $v(x) = u(x)$, and hence v is a non-constant solution of (2.5). Furthermore,

$$v(-a_0) = th, v'(-a_0) = 0, v''(-a_0) \leq 0, \quad \text{and} \quad v'''(-a_0) \leq 0.$$

That is, v satisfies the condition of Lemma 4.3. Thus, we conclude from Lemma 4.3 that for any $\zeta > -a_0$ where $v'(\zeta) = 0$ we have that $|v(\zeta)| > th$. Particularly, for $\zeta = a_0$, we have $th = u(a_0) = v(a_0) > th$ a contradiction. Therefore, $u'''(a_0) \leq 0$ and the lemma is proved. □

Construction of N-Bump Homoclinic Solutions. We now proceed in proving the existence of N-bump homoclinic orbit solutions for any even N. As usual, we consider $\alpha < 0$ and $\beta = -(b^2 + 1)\alpha > 0$. This choice of initial conditions ensures that $\alpha = u(0, \alpha) < 0$ and that $u(t, \alpha)$ increases on $(0, \xi_1(\alpha))$.

The shooting argument we use to construct N-bump solutions for even N is based on the periodic solution proved in Theorem 2.7 of Section 2. This theorem guarantees the existence of a one bump periodic solution $u(\cdot, \alpha^*)$ such that

$$\begin{gathered} \alpha^* = u(0, \alpha^*) = u(\eta_k(\alpha^*), \alpha^*) < 0, \quad \text{and} \\ u(\xi_1(\alpha^*), \alpha^*) = u(\xi_k(\alpha^*), \alpha^*) > u_s \quad \text{for all } k \geq 1 \end{gathered}$$

Our construction of even bump solutions by shooting methods will make use of the following technical lemma.

Lemma 4.5. *If* $u(\cdot, \alpha)$ *is a solution of* (2.5)*, then*

$$u(\xi_1(\alpha), \alpha) \to 0^+ \quad \text{as } \alpha \to 0^-.$$

Proof. Since $\alpha < 0$, then the equation in (2.5) is linear so long as $u \leq th$. Hence, we find that

$$u(x, \alpha) = \alpha(\cosh(bx)\cos(x) - b\sinh(bx)\sin(x)), \quad \text{and} \tag{4.35}$$

$$u'(x, \alpha) = -\alpha(b^2 + 1)\cosh(bx)\sin(x) \tag{4.36}$$

so long as $u(x, \alpha) \leq th$. By (4.35) we obtain that $u(x, \alpha) \leq th$ on $[0, \pi]$ whenever $|\alpha|$ is sufficiently small. Therefore, $\xi_1(\alpha) = \pi$ is an immediate consequence of (4.36). □

Lemmas 4.7 and 4.8 below will be used to generate the induction process. These lemmas make repeated use of Lemma 4.1. Before stating and proving Lemmas 4.7 and 4.8, we will prove the following lemma which yields an equivalent condition of Lemma 4.1.

Lemma 4.6. *Suppose that* $Q(u(\eta_k(\alpha), \alpha)) = 0$ *for some* $\alpha < 0$ *and* $k \geq 1$*. That is, either* $u(\eta_k(\alpha), \alpha) = 0$*, or* u_s*. Then, either* $\eta_k(\alpha) = \xi_k(\alpha)$ *or* $\eta_k(\alpha) = \xi_{k+1}(\alpha)$*. In particular,*

(i) $\eta_k(\alpha) = \xi_{k+1}(\alpha)$ *if and only if* $u'''(\eta_k(\alpha), \alpha) < 0$*, and*
(ii) $\eta_k(\alpha) = \xi_k(\alpha)$ *if and only if* $u'''(\eta_k(\alpha), \alpha) > 0$*.*

Proof. The condition $Q(u(\eta_k(\alpha),\alpha)) = 0$ and (2.8) imply that $u''(\eta_k(\alpha),\alpha) = 0$. By Lemma 3.1 it follows that $u'''(\eta_k(\alpha),\alpha) \neq 0$. Now, (3.2) leads to $\eta_k(\alpha) = \xi_{k+1}(\alpha)$ if and only if $u'''(\eta_k(\alpha),\alpha) < 0$, and (3.3) yields that $\eta_k(\alpha) = \xi_k(\alpha)$ if and only if $u'''(\eta_k(\alpha),\alpha) > 0$. □

Lemma 4.6 implies that if $u(\eta_{k-1}(\alpha_0),\alpha_0) < 0$, for example, and $u(\xi_k(\alpha_0),\alpha_0) = 0$, then $u'''(\xi_k(\alpha_0),\alpha_0) > 0$ and $\xi_k(\alpha_0) = \eta_k(\alpha_0)$. We now prove the following two induction lemmas.

Lemma 4.7. *Let $k \geq 2$. Suppose that $[c,d] \subset (\alpha^*,0)$ is an interval such that*

$$th = u(\xi_k(c),c) > u(\xi_k(\alpha),\alpha) > u(\xi_k(d),d) = 0 \quad \text{for all } \alpha \in (c,d),$$

and

$$u(\eta_{k-1}(\alpha),\alpha) < 0 \quad \text{for all } \alpha \in [c,d].$$

Then there exists an interval $[\hat{c},\hat{d}] \subset (c,d)$ such that

$$th = u(\xi_{k+1}(\hat{c}),\hat{c}) > u(\xi_{k+1}(\alpha),\alpha) > u(\xi_{k+1}(\hat{d}),\hat{d}) = 0 \quad \text{for all } \alpha \in (\hat{c},\hat{d}),$$

and

$$u(\eta_k(\alpha),\alpha) < 0 \quad \text{for all } \alpha \in [\hat{c},\hat{d}].$$

Remark: See the left panel of Figure 7 for an illustration of the main ideas of the following proof.

Proof of 4.7. Since $u(\eta_{k-1}(d),d) < 0 = u(\xi_k(d),d)$, then we deduce from Lemma 4.6 that $\eta_k(d) = \xi_k(d)$ and $u'''(\xi_k(d),d) > 0$. Hence, Lemma 4.1 and our assumption that $u(\xi_k(\alpha),\alpha) > 0$ for each $\alpha \in (c,d)$ imply that $u(\eta_k(\alpha),\alpha) > 0$ whenever $d - \alpha > 0$ is sufficiently small. Furthermore, the assumption that $u(\xi_k(c),c) = th$ and Lemma 4.3 lead to $u(\eta_k(c),c) < 0$. It now follows by continuity of $u(\eta_k(\alpha),\alpha)$ in α that

$$b = \sup\{\hat{\alpha} > c \mid u(\eta_k(\alpha),\alpha) < 0 \quad \text{on } (c,\hat{\alpha})\}$$

is well defined, $u(\eta_k(b),b) = 0$, and $c < b < d$. This result together with Lemma 4.6 and the assumption that $u(\xi_k(\alpha),\alpha) > 0$ for each $\alpha \in (c,d)$ all lead to $\xi_{k+1}(b) = \eta_k(b)$ and $u'''(\xi_{k+1}(b),b) < 0$. Another application of Lemma 4.1 yields that $u(\xi_{k+1}(\alpha),\alpha) < 0$ whenever $b - \alpha > 0$ is sufficiently small. Once again we invoke Lemma 4.3 to obtain $u(\xi_{k+1}(c),c) > th$. Thus, it follows from continuity of $u(\xi_{k+1}(\alpha),\alpha)$ in α that

$$\hat{d} = \sup\{\hat{\alpha} > c \mid u(\xi_{k+1}(\alpha),\alpha) > 0 \quad \text{on } (c,\hat{\alpha})\}$$

and

$$\hat{c} = \inf\{\hat{\alpha} < \hat{d} \mid u(\xi_{k+1}(\alpha),\alpha) < th \quad \text{on } (\hat{\alpha},\hat{d})\}$$

are well defined, $th = u(\xi_{k+1}(\hat{c}),\hat{c}) > u(\xi_{k+1}(\alpha),\alpha) > u(\xi_{k+1}(\hat{d}),\hat{d}) = 0$, for all $\alpha \in (\hat{c},\hat{d})$. Also note that $\hat{d} < b$, and therefore, $u(\eta_k(\alpha),\alpha) < 0$ for all $\alpha \in [\hat{c},\hat{d}]$. This concludes the proof of the lemma. □

Lemma 4.8. *Assume that $\alpha^* < 0$ denotes the initial value that gave rise to the one bump periodic solution of Theorem 2.7. That is,*

$$u(0,\alpha^*) = u(\eta_k(\alpha^*),\alpha^*) < 0 \quad \text{and}$$
$$u_s < u(\xi_1(\alpha^*),\alpha^*) = u(\xi_k(\alpha^*),\alpha^*)$$

for all $k \geq 1$. Suppose that there exists $A_k \in (\alpha^*,0)$ and $b_k \in (\alpha^*,A_k)$ such that

(i) *$u(\xi_k(\alpha),\alpha) > u_s$ on $[\alpha^*,A_k)$ and $u(\xi_k(A_k),A_k) = u_s$, and*

(ii) $u(\eta_k(\alpha), \alpha) < 0$ *on* $[\alpha^*, b_k)$ *and* $u(\eta_k(b_k), b_k) = 0$.

Then there exists $A_{k+1} \in (\alpha^*, b_k)$ *and* $b_{k+1} \in (\alpha^*, A_{k+1})$ *such that*

(i) $u(\xi_{k+1}(\alpha), \alpha) > u_s$ *on* $[\alpha^*, A_{k+1})$ *and* $u(\xi_{k+1}(A_{k+1}), A_{k+1}) = u_s$, *and*
(ii) $u(\eta_{k+1}(\alpha), \alpha) < 0$ *on* $[\alpha^*, b_{k+1})$ *and* $u(\eta_{k+1}(b_{k+1}), b_{k+1}) = 0$.

Remark: See the right panel of Figure 7 for an illustration of the main ideas of the following proof.

Proof of Lemma 4.8. Since $u(\eta_k(b_k), b_k) = 0 < u_s < u(\xi_k(b_k), b_k)$, then $\eta_k(b_k) = \xi_{k+1}(b_k)$ and $u'''(\xi_{k+1}(b_k), b_k) < 0$ as a result of Lemma 4.6. Thus, it follows from Lemma 4.1 that $u(\xi_{k+1}(\alpha), \alpha) < 0$ whenever $b_k - \alpha > 0$ is sufficiently small. Further, since $u(\xi_{k+1}(\alpha^*), \alpha^*) > u_s$, then continuity of $u(\xi_{k+1}(\alpha), \alpha)$ in α implies that

$$A_{k+1} = \sup\{\hat{\alpha} > \alpha^* \mid u(\xi_{k+1}(\alpha), \alpha) > u_s \quad \text{on } (\alpha^*, \hat{\alpha})\}$$

is well defined and $u(\xi_{k+1}(A_{k+1}), A_{k+1}) = u_s$. Our definition of A_{k+1} together with the fact that $u(\xi_{k+1}(\alpha), \alpha) < 0$ whenever $b_k - \alpha > 0$ is sufficiently small ensures that $\alpha^* < A_{k+1} < b_k < A_k$.

Since $u(\eta_k(A_{k+1}), A_{k+1}) < 0$, then Lemma 4.6 implies that $\eta_{k+1}(A_{k+1}) = \xi_{k+1}(A_{k+1})$ and $u'''(\xi_{k+1}(A_{k+1}), A_{k+1}) > 0$. We once again apply Lemma 4.1 which guarantees that $u(\eta_{k+1}(\alpha), \alpha) > u_s$ for sufficiently small $A_{k+1} - \alpha > 0$. Since $u(\eta_{k+1}(\alpha^*), \alpha^*) < 0$, then continuity of $u(\eta_{k+1}(\alpha), \alpha)$ in α guarantees that

$$b_{k+1} = \sup\{\hat{\alpha} > \alpha^* \mid u(\eta_{k+1}(\alpha), \alpha) < 0 \quad \text{on } (\alpha^*, \hat{\alpha})\}$$

is well defined and $u(\eta_{k+1}(b_{k+1}), b_{k+1}) = 0$. The fact that $u(\eta_{k+1}(\alpha), \alpha) > u_s > 0$ for sufficiently small $A_{k+1} - \alpha > 0$ implies that $b_{k+1} < A_{k+1}$ concluding the proof of the lemma. $\square$

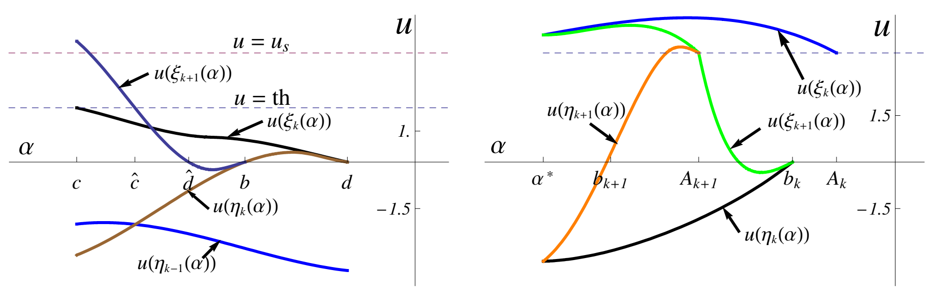

FIGURE 7. An illustration of the behavior of the critical values $u(\eta_{k-1}(\alpha)), u(\eta_k(\alpha)), u(\xi_k(\alpha))$, and $u(\xi_{k+1}(\alpha))$. The left panel corresponds to Lemma 4.7, and the right panel corresponds to Lemma 4.8.

We proceed by proving the existence of $A_1 \in (\alpha^*, 0)$ and $b_1 \in (\alpha^*, A_1)$ that satisfy the hypotheses $(i), (ii)$ of Lemma 4.8. First, it follows from Lemma 4.5 that $u(\xi_1(\alpha), \alpha) < th < u_s$ for sufficiently small $-\alpha > 0$. Since $u(\xi_1(\alpha^*), \alpha^*) > u_s$, then continuity of $u(\xi_1(\alpha), \alpha)$ in α ensures that

$$A_1 = \sup\{\hat{\alpha} > \alpha^* \mid u(\xi_1(\alpha), \alpha) > u_s \quad \text{on } (\alpha^*, \hat{\alpha})\} \tag{4.37}$$

is well defined and $u(\xi_1(A_1), A_1) = u_s$. Furthermore, $A_1 < 0$ follows from the fact that $u(\xi_1(\alpha), \alpha) < th < u_s$ for sufficiently small $-\alpha > 0$.

To construct b_1 we note that $\eta_1(A_1) = \xi_1(A_1)$, and $u'''(\xi_1(A_1), A_1) > 0$. Thus, $u(\eta_1(\alpha), \alpha) > u_s$ whenever $A_1 - \alpha > 0$ is sufficiently small as a consequence of Lemma 4.1. Continuity of $u(\eta_1(\alpha), \alpha)$ in α and the fact that $u(\eta_1(\alpha^*), \alpha^*) < 0$ implies that

$$b_1 = \sup\{\hat{\alpha} > \alpha^* \mid u(\eta_1(\alpha), \alpha) < 0 \quad \text{on } (\alpha^*, \hat{\alpha})\} \tag{4.38}$$

is well defined and $u(\eta_1(b_1), b_1) = 0$. The fact that $u(\eta_1(\alpha), \alpha) > u_s$ for sufficiently small $A_1 - \alpha > 0$ gives $\alpha^* < b_1 < A_1$ as desired.

We have now shown that the hypothesis of Lemma 4.8 for $j = 1$, i.e., A_1 and b_1 exist as required. Proceeding inductively, it follows that the entire sequence $\{A_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ exist and satisfy Lemma 4.8.

Throughout the remainder of this paper we define $k \in \mathbb{Z}^+$ by $2k = N$. To obtain an N-bump homoclinic orbit solution we must ensure that $\alpha_N = u(0, \alpha_N) < 0$ exists such that $u(\xi_j(\alpha_N), \alpha_N) > th$ for $j = 1, \dots, k$ and that $u(\xi_j(\alpha_N), \alpha_N) < th$ for all $j > k$. In particular, the N-bump solution that we construct will have the properties

$$u(\xi_j(\alpha_N), \alpha_N) > u_s > th \quad \text{for} \quad j = 1, \dots, k, \tag{4.39}$$

$$0 < u(\xi_j(\alpha_N), \alpha_N) < th \quad \text{for} \quad j > k, \quad \text{and} \tag{4.40}$$

$$u(\eta_j(\alpha_N), \alpha_N) < 0 \quad \text{for all } j \geq 1. \tag{4.41}$$

We will implement a topological shooting method that generates a nested sequence $\{I_j\}_{j=1}^{\infty}$ of non-empty compact intervals which possess the following properties:

- (P1) $I_{j+1} \subset I_j^0$ for all $j \geq 1$, (I_j^0 denotes the interior of I_j),
- (P2) for each $\alpha \in I_1$ and $j = 1, \dots, k$, $u(\xi_j(\alpha), \alpha) > u_s$ and $u(\eta_j(\alpha), \alpha) \leq 0$. Moreover, $u(\eta_j(\alpha), \alpha) < 0$ for all $\alpha \in I_1^0$ and $j = 1, \dots, k$,
- (P3) for each $j > k$, $0 < u(\xi_j(\alpha), \alpha) < th$ for all $\alpha \in I_j^0$ and $u(\eta_{j-1}(\alpha), \alpha) < 0$ for all $\alpha \in I_j$.

We proceed by constructing the first interval, I_1. First, recall from (4.37) that $u(\xi_1(\alpha), \alpha) > u_s$ for all $\alpha \in [\alpha^*, A_1)$ and $u(\xi_1(A_1), A_1) = u_s$. Furthermore, $u(\eta_1(\alpha), \alpha) < 0$ for $\alpha \in [\alpha^*, b_1)$ and $u(\eta_1(b_1), b_1) = 0$ where $b_1 \in (\alpha^*, A_1)$ is defined by (4.38). By Lemma 4.8, there exists $A_2 \in (\alpha^*, b_1)$ and $b_2 \in (\alpha^*, A_2)$ such that

- (i) $u(\xi_j(\alpha), \alpha) > u_s$ on $[\alpha^*, A_2)$ and $u(\xi_2(A_2), A_2) = u_s$,
- (ii) $u(\eta_j(\alpha), \alpha) \leq 0$ on $[\alpha^*, b_2)$ and $u(\eta_2(b_2), b_2) = 0$

for $j = 1, 2$. An inductive application of Lemma 4.8 yields the following lemma.

Lemma 4.9. *For $2k \equiv N$ and $j = 1, 2, \dots, k$ it follows that*

$$u(\xi_j(\alpha), \alpha) > u_s \quad \text{for} \quad \alpha \in [\alpha^*, A_k), \quad \text{and} \quad u(\xi_k(A_k), A_k) = u_s,$$
$$u(\eta_j(\alpha), \alpha) \leq 0 \quad \text{for} \quad \alpha \in [\alpha^*, b_k), \quad \text{and} \quad u(\eta_k(b_k), b_k) = 0.$$

An immediate consequence of Lemma 4.9 and the fact that $[\alpha^*, b_k] \subset [\alpha^*, A_k)$, is that property $(P2)$ holds for

$$I_1 \equiv [\alpha^*, b_k]. \tag{4.42}$$

The aim of the following lemma is to construct the second interval, I_2. Subsequently, we will inductively employ Lemma 4.7 to define $I_3, I_4, \dots$

Lemma 4.10. *Let $d_2 = A_{k+1}$ where A_{k+1} is defined by Lemma 4.8 and define*

$$c_2 = \inf\{\hat{\alpha} < d_2 \mid u(\xi_{k+1}(\alpha), \alpha) < th \quad \text{on } (\hat{\alpha}, d_2)\}.$$

Also, define $I_2 = [c_2, d_2]$. *Then* $I_2 \subset I_1^0$,

$$0 = u(\xi_{k+1}(d_2), d_2) < u(\xi_{k+1}(\alpha), \alpha) < u(\xi_{k+1}(c_2), c_2) = th \quad \text{for all } \alpha \in I_2^0,$$
$$u(\eta_k(\alpha), \alpha) < 0 \quad \text{for all } \alpha \in I_2^0.$$

Proof. It follows from Lemma 4.8 that $d_2 = A_{k+1} < b_k$. Also, $\alpha^* < c_2$ is a consequence of $u(\xi_{k+1}(\alpha^*), \alpha^*) > u_s > th$, $u(\xi_{k+1}(A_{k+1}), A_{k+1}) = 0$, and the fact that $u(\xi_{k+1}(\alpha), \alpha)$ is continuous with respect to α. Thus, $I_2 \subset I_1^0$.

Now,

$$0 = u(\xi_{k+1}(d_2), d_2) < u(\xi_{k+1}(\alpha), \alpha) < u(\xi_{k+1}(c_2), c_2) = th \quad \text{for all } \alpha \in I_2^0$$

follows immediately from the definitions of c_2 and d_2, and

$$u(\eta_k(\alpha), \alpha) < 0 \quad \text{for all } \alpha \in I_2$$

follows by Lemma 4.9 and the fact that $I_1 \subset I_2^0$. This concludes the proof of the lemma. □

In Lemma 4.10 we defined $I_2 = [c_2, d_2]$ and described the properties of solutions $u(\cdot, \alpha)$ with $\alpha \in I_2$. Lemma 4.7 guarantees the existence of $I_3 = [c_3, d_3] \subset I_2^0$ such that

$$0 = u(\xi_{k+2}(d_3), d_3) < u(\xi_{k+2}(\alpha), \alpha) < u(\xi_{k+2}(c_3), c_3) = th$$
$$\text{for all } \alpha \in I_3^0, \quad \text{and} \quad u(\eta_{k+1}(\alpha), \alpha) < 0 \quad \text{for all } \alpha \in [c_3, d_3].$$

An inductive application of Lemma 4.7 yields $I_j = [c_j, d_j]$ for $j \geq 2$ such that

$$0 = u(\xi_{k+j-1}(d_j), d_j) < u(\xi_{k+j-1}(\alpha), \alpha) < u(\xi_{k+j-1}(c_j), c_j) = th$$
$$\text{for all } \alpha \in (c_j, d_j), \quad \text{and}$$
$$u(\eta_{k+j-2}(\alpha), \alpha) < 0 \quad \text{for all } \alpha \in [c_j, d_j].$$

Hence, to construct our N-bump solution, we let $\alpha_N \in \bigcap_{j=1}^{\infty} I_j$ and consider $u(\cdot, \alpha_N)$. We have just shown that the nested sequence of intervals I_j exists which satisfies properties $(P1)-(P3)$ described above. In particular, $u(\xi_j(\alpha_N), \alpha_N) > u_s$ for $j = 1, 2, \dots, k$, $u(x, \alpha_N) < th$ on $[\eta_k(\alpha_N), \infty)$, and $u(x, \alpha_N)$ satisfies the linear equation with constant coefficients

$$u'''' - 2(b^2 - 1)u'' + (b^2 + 1)^2 u = 0 \quad \text{on } [\eta_k(\alpha_N), \infty).$$

The general solution of this linear equation is

$$u(x, \alpha_N) = c_1 e^{-bx} \cos(x) + c_2 e^{-bx} \sin(x) + c_3 e^{bx} \cos(x) + c_4 e^{bx} \sin(x) \quad \text{on } [\eta_k(\alpha), \infty)$$

for some constants $c_1 - c_4$. The only way u can remain below th on $[\eta_k(\alpha_N), \infty)$ is that $c_3 = c_4 = 0$. From this we conclude that

$$u(x, \alpha_N) \to 0 \quad \text{as } x \to \infty.$$

This completes the proof of existence of a N-bump solution.

Numerical Solutions

The solutions in Figure 8 were obtained by using Mathematica. Our choice of parameters are $r = 0.02$, $th = 1.75$ and $b = 1.095669029014$. Furthermore, Mathematica was used to verify that $(r, b, th) \in \Lambda$ (see (2.11)) for this choice of parameters. The solution in the left panel was obtained by manually applying the shooting method as described in this paper. Special thanks to the referee for computing the solution in the right panel. These numerics suggest that both solutions are highly sensitive to the initial conditions as indicated by the large number of decimal places that are *required* for the value α. For this reason, we predict that the N-bump solutions are unstable.

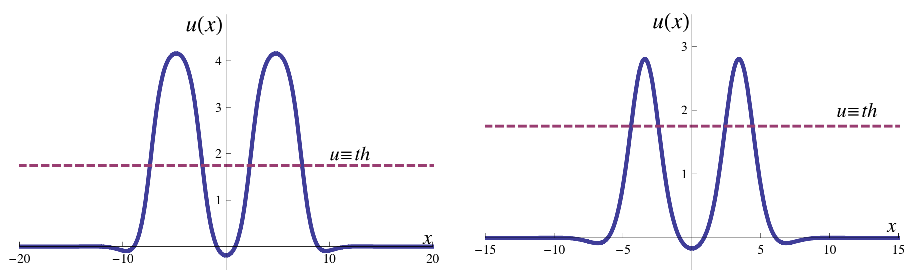

Figure 8. Two 2-bump solutions with $r = 0.02, th = 1.75$, and $b = 1.095669029014$. The initial conditions in the left panel are $\alpha = -0.192218655$, and $\beta = 0.422975$. In the right panel the initial conditions are $\alpha = -0.166290142$, and $\beta = 0.36592$.

Lastly, as mentioned in the previous section, the solution is of the form

$$u(x, \alpha_2) = c_1 e^{-bx}\cos(x) + c_2 e^{-bx}\sin(x) + c_3 e^{bx}\cos(x) + c_4 e^{bx}\sin(x) \quad \text{on } [\eta_1(\alpha), \infty)$$

since $u < th$ on $[\eta_1(\alpha), \infty)$. This is only possible if $c_3 = c_4 = 0$. Specifically, our computations estimate that $c_3 = c_4 = 0$, $c_1 \approx 2729.389291$, and $c_2 \approx 4900.231464$ for the solution shown in the left panel and $c_3 = c_4 = 0$, $c_1 \approx -41.07182903$, and $c_2 = -217.0976993$ for the solution on the right.

Conclusion

In this paper we have analyzed a subclass of stationary solutions of (1.1). In previous studies, (see [4, 14]), the Fourier transform was applied to both sides of (1.2) to obtain a fourth order ODE. Then ODE methods were implemented to obtain a thorough numerical investigation of homoclinic orbit solutions. For technical reasons, the Fourier transform does not give rise to other types of interesting solutions such as periodic, heteroclinic, or chaotic solutions. The fundamental aim of this paper was to use the results of Krisner [12, 13] to prove that (1.1) does have N-bump homoclinic orbit solutions. In fact, under the parameter regime derived in Section 2, it was shown in Section 4 that (1.1) has N-bump homoclinic orbit solutions for any positive even-valued integer N.

A natural extension of this result would be to prove that N-bump solutions exist for a larger parameter regime. In particular, Lemma 4.3 was proved under the assumption that $b > 1$. It still remains to prove that N-bump homoclinic orbit solutions exist for $0 < b < 1$ either by modifying the proof of Lemma 4.3 or by adopting an entirely different approach.

Another extension is to incorporate "noise" into the Wilson-Cowan model (1.1) to account for random fluctuations. It would be interesting to investigate the existence of special solutions, such as periodic and/or homoclinic orbit solutions, with the presence of noise.

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