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UNIQUE SOLVABILITY FOR A SECOND ORDER NONLINEAR SYSTEM VIA TWO GLOBAL INVERSION THEOREMS

ROBERT DALMASSO

Abstract. In this paper we use two global inversion theorems to establish the existence and uniqueness for a nonlinear second order homogeneous Dirichlet system.

1. Introduction

Let $n \geq 1$ and let $f = (f_1, \dots, f_n) : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. We consider the system

$$\begin{gathered} u''(x) + f(x, u(x)) = 0\,, \quad 0 \leq x \leq 1, \\ u(0) = u(1) = 0\,. \end{gathered} \tag{1.1}$$

We first introduce some notations:

$$\|u\| = \max_{1 \leq j \leq n} (|u_j|), \quad u = (u_1, \dots, u_n) \in \mathbb{R}^n\,,$$

$M(n)$ is the space of $n \times n$ matrices with real entries and $\rho(M)$ is the spectral radius of $M \in M(n)$,

$$\|M\| = \max_{1 \leq j \leq n} \sum_{k=1}^{n} |m_{jk}|, \quad M = (m_{jk})_{1 \leq j,k \leq n} \in M(n)\,,$$

$$\|y\|_p = (\int_0^1 |y(t)|^p \, dt)^{1/p}, \quad y \in L^p(0,1)\,,\ 1 \leq p < +\infty\,,$$

$$\|y\|_\infty = \operatorname*{ess\,sup}_{(0,1)} |y|, \quad y \in L^\infty(0,1)\,,$$

$$\|y\|_p = \max_{1 \leq j \leq n} (\|y_j\|_p), \quad y = (y_1, \dots, y_n) \in L^p((0,1), \mathbb{R}^n)\,,\ 1 \leq p \leq +\infty,$$

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n;\ x_j \geq 0\,,\ j = 1, \dots, n\}\,.$$

Recently the author proved the following theorem.

Theorem 1.1 ([4]). *Assume that the partial derivatives $\partial f_j/\partial u_k$ exist and are continuous on $[0,1] \times \mathbb{R}^n$ for $j,k = 1, \dots, n$. Let $\Lambda = (\lambda_{jk})_{1 \leq j,k \leq n} : \mathbb{R}_+ \to M(n)$ be*

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a continuous map with λ_{jk} nondecreasing and bounded for $j,k = 1,\dots,n$. Assume that

$$\Big|\frac{\partial f_j}{\partial u_k}(x,u)\Big| \le \lambda_{jk}(\|u\|) \quad \forall\,(x,u) \in [0,1]\times\mathbb{R}^n\,,\ 1 \le j,k \le n\,, \tag{1.2}$$

$$\rho(\Lambda(t)) < \pi^2 \quad \forall\, t \ge 0\,, \tag{1.3}$$

$$\int_0^{+\infty} \det(\pi^2 I - \Lambda(t))\,dt = +\infty\,. \tag{1.4}$$

Then the boundary value problem (1.1) *has a unique solution.*

The purpose of this paper is to improve and complement Theorem 1.1. We have the following results.

Theorem 1.2. *Assume that the partial derivatives $\partial f_j/\partial u_k$ exist and are continuous on $[0,1]\times\mathbb{R}^n$ for $j,k = 1,\dots,n$. Let $\Lambda = (\lambda_{jk})_{1\le j,k\le n} : \mathbb{R}_+ \to M(n)$ be a continuous map with λ_{jk} nondecreasing for $j,k = 1,\dots,n$. Assume that* (1.2) *and* (1.3) *hold and that*

$$\int_0^{+\infty} \frac{dt}{\|(\pi^2 I - \Lambda(t))^{-1}\|} = +\infty\,. \tag{1.5}$$

Then the boundary value problem (1.1) *has a unique solution.*

Theorem 1.3. *Assume that the partial derivatives $\partial f_j/\partial u_k$ exist and are continuous on $[0,1]\times\mathbb{R}^n$ for $j,k = 1,\dots,n$. Let $b \in \mathbb{R}^n_+$ and let $A = (a_{jk})_{1\le j,k\le n} : \mathbb{R}_+ \to M(n)$ be a continuous map with a_{jk} nondecreasing for $j,k = 1,\dots,n$. Assume that*

$$u_j f_j(x,u) \le \sum_{k=1}^n a_{jk}(\|u\|)|u_j u_k| + b_j|u_j|\,, \tag{1.6}$$

for all $(x,u) \in [0,1]\times\mathbb{R}^n$, $1 \le j \le n$,

$$\rho(A(t)) < \pi^2 \quad \forall\, t \ge 0\,, \tag{1.7}$$

$$\lim_{t\to+\infty} \frac{t}{\|(\pi^2 I - A(t))^{-1}\|} = +\infty\,. \tag{1.8}$$

Let $\Lambda = (\lambda_{jk})_{1\le j,k\le n} : \mathbb{R}_+ \to M(n)$ be a continuous map with λ_{jk} nondecreasing for $j,k = 1,\dots,n$. Assume that (1.2) *and* (1.3) *hold. Then the boundary value problem* (1.1) *has a unique solution.*

In Section 2 we recall some results from the theory of nonnegative matrices. We also give two global inversion theorems. We prove Theorem 1.2 in Section 3 and Theorem 1.3 in section 4. Finally in Section 5 we conclude with some examples.

2. Preliminaries

We begin with some results from the theory of nonnegative matrices. We refer the reader to [1] for proofs.

Definition 2.1. $A \in M(n)$ is called $\mathbb{R}^n_+$-monotone if $Ax \in \mathbb{R}^n_+$ implies $x \in \mathbb{R}^n_+$.
$N = (n_{jk})_{1\le j,k\le n}$ is nonnegative if $n_{jk} \ge 0$ for $j,k = 1,\dots,n$.

Theorem 2.2 ([1, p. 113])**.** *$A \in M(n)$ is $\mathbb{R}^n_+$-monotone if and only if A is nonsingular and A^{-1} is nonnegative.*

Theorem 2.3 ([1, p. 113]). *Let $A = \alpha I - N$ where $\alpha \in \mathbb{R}$ and $N \in M(n)$ is nonnegative. Then the following are equivalent:*

(i) *A is $\mathbb{R}^n_+$-monotone;*
(ii) *$\rho(N) < \alpha$.*

Remark 2.4. With the notations of Theorem 2.3, assume that (i) (or (ii)) holds. Then $\det A > 0$.

The proof of Theorem 1.2 makes use of the following global inversion theorem of Hadamard-Lévy type established by M. Rădulescu and S. Rădulescu [5, Theorem 2].

Theorem 2.5. *Let (Y, N_0) be a Banach space and let $L : D(L) \to Y$ be a linear operator with closed graph, where $D(L)$ is a linear subspace of Y. Then $D(L)$ is a Banach space with respect to the norm defined by*

$$N_1(u) = N_0(u) + N_0(Lu)\,, \quad u \in D(L)\,.$$

Further, let $K : (Y, N_0) \to (Y, N_0)$ be a C^1 map and let X be a linear subspace of $D(L)$ which is closed in the norm N_1. Consider the nonlinear map $S : (X, N_1) \to (Y, N_0)$ defined by $S = L - K$, and assume that S is a local diffeomorphism. If there exists a continuous map $c : \mathbb{R}_+ \to \mathbb{R}^\star_+$ such that

$$\int_0^{+\infty} c(t)\,dt = +\infty\,,$$

$$N_0(S'(u)(h)) \geq c(N_0(u))N_0(h) \quad \forall\, u\,,\, h \in X\,,$$

then S is a global diffeomorphism.

The proof of Theorem 1.3 makes use of the Banach-Mazur-Caccioppoli global inversion theorem ([2], [3] and [6]).

Theorem 2.6. *Let E and F be two Banach spaces. Then $S : E \to F$ is a global homeomorphism if and only if S is a local homeomorphism and a proper map.*

3. Proof of Theorem 1.2

We begin with two lemmas.

Lemma 3.1. *Let $w \in C^1([0,1], \mathbb{R})$ be such that $w(0) = w(1) = 0$. Then*

$$\|w'\|_2 \geq \pi\|w\|_2 \quad \textit{and} \quad \|w'\|_2 \geq 2\|w\|_\infty\,.$$

The first inequality is known as the Wirtinger inequality and the second inequality is known as the Lees inequality.

Lemma 3.2. *Let*

$$X = \{h \in C^2([0,1], \mathbb{R}^n);\, h(0) = h(1) = 0\}\,,$$

and let $V = (v_{jk})_{1\leq j,k\leq n} : [0,1] \to M(n)$ be a continuous map. Assume that there exists $N = (n_{jk})_{1\leq j,k\leq n} \in M(n)$ such that $\rho(N) < \pi^2$ and

$$|v_{jk}(x)| \leq n_{jk} \quad \forall\, x\, \in [0,1]\,,\ 1 \leq j,k \leq n\,.$$

If $T : X \to C([0,1], \mathbb{R}^n)$ is the operator defined by

$$T(h)(x) = h''(x) + V(x)h(x)\,, \quad h \in X\,,\ x \in [0,1]\,,$$

then

$$\|T(h)\|_\infty \geq \frac{2}{\pi\|(\pi^2 I - N)^{-1}\|}\|h\|_\infty \quad \forall\, h \in X\,.$$

Proof. Let $h = (h_1, \dots, h_n) \in X$ and let $j \in \{1, \dots, n\}$. Integrating by parts we get

$$\begin{aligned}
\int_0^1 h_j(x)T(h)_j(x)\,dx &= \int_0^1 h_j(x)(h_j''(x) + \sum_{k=1}^n v_{jk}(x)h_k(x))\,dx \\
&= -\int_0^1 h_j'(x)^2\,dx + \sum_{k=1}^n \int_0^1 v_{jk}(x)h_j(x)h_k(x)\,dx\,.
\end{aligned}$$

Then using the Cauchy-Schwarz inequality and Lemma 3.1 we can write

$$\begin{aligned}
\|h_j\|_2\|T(h)_j\|_2 &\geq -\int_0^1 h_j(x)T(h)_j(x)\,dx \\
&= \|h_j'\|_2^2 - \sum_{k=1}^n \int_0^1 v_{jk}(x)h_j(x)h_k(x)\,dx \\
&\geq \pi\|h_j\|_2\|h_j'\|_2 - \sum_{k=1}^n n_{jk}\|h_j\|_2\|h_k\|_2 \\
&\geq \pi\|h_j\|_2\|h_j'\|_2 - \frac{1}{\pi}\sum_{k=1}^n n_{jk}\|h_j\|_2\|h_k'\|_2\,,
\end{aligned}$$

from which we deduce that

$$\|T(h)_j\|_2 \geq \pi\|h_j'\|_2 - \frac{1}{\pi}\sum_{k=1}^n n_{jk}\|h_k'\|_2\,, \tag{3.1}$$

for $j = 1, \dots, n$. Let a, b denote the vectors

$$a = (\|h_j'\|_2)_{1\leq j\leq n} \quad \text{and} \quad b = (\pi\|T(h)_j\|_2)_{1\leq j\leq n}\,.$$

Inequality (3.1) can be written

$$b - (\pi^2 I - N)a \in \mathbb{R}_+^n\,.$$

Theorem 2.3 implies that $\pi^2 I - N$ is $\mathbb{R}_+^n$-monotone. Then using Theorem 2.2 we obtain

$$(\pi^2 I - N)^{-1}b - a \in \mathbb{R}_+^n\,, \tag{3.2}$$

which implies that

$$\pi\|(\pi^2 I - N)^{-1}\|\,\|T(h)\|_2 \geq \|h_j'\|_2\,,$$

for $j = 1, \dots, n$. Using Lemma 3.1 and the fact that $\|T(h)\|_2 \leq \|T(h)\|_\infty$ we deduce that

$$\|T(h)\|_\infty \geq \frac{2}{\pi\|(\pi^2 I - N)^{-1}\|}\|h\|_\infty\,,$$

and the lemma is proved. □

Now we can complete the proof of Theorem 1.2. Let $Y = C([0,1], \mathbb{R}^n)$ be equipped with the sup norm $\|.\|_\infty$ and let $L : D(L) \to Y$ be the linear operator defined by

$$Lu = u''\,, \quad u \in D(L)\,,$$

where $D(L) = C^2([0,1],\mathbb{R}^n)$. Since L has closed graph, it follows from Theorem 2.5 that $D(L)$ is a Banach space with respect to the norm N_1 defined by

$$N_1(u) = \|u\|_\infty + \|Lu\|_\infty\,, \quad u \in D(L)\,.$$

Let $K : (Y, \|.\|_\infty) \to (Y, \|.\|_\infty)$ be given by

$$K(u)(x) = -f(x, u(x))\,, \quad u \in Y\,,\, x \in [0,1]\,.$$

The regularity assumptions on f imply that K is of class C^1. The set $X = \{u \in D(L); u(0) = u(1) = 0\}$ is a closed subspace of $D(L)$ in the norm N_1. Let $S = L-K$. Clearly $S : (X, N_1) \to (Y, \|.\|_\infty)$ is of class C^1. Let $u \in X$ be fixed and let $V = (v_{jk})_{1\le j,k\le n} : [0,1] \to M(n)$ be such that

$$v_{jk}(x) = \frac{\partial f_j}{\partial u_k}(x, u(x))\,, \quad x \in [0,1]\,,\ 1 \le j,k \le n\,.$$

We have

$$S'(u)(h)(x) = h''(x) + V(x)h(x)\,, \quad h \in X\,,\, x \in [0,1]\,.$$

Also (1.2) implies

$$|v_{jk}(x)| \le \lambda_{jk}(\|u\|_\infty) \quad \forall\, x \in [0,1]\,,\ 1 \le j,k \le n\,.$$

Then using Lemma 3.2, we get

$$\|S'(u)(h)\|_\infty \ge \frac{2}{\pi\|(\pi^2 I - \Lambda(\|u\|_\infty))^{-1}\|}\|h\|_\infty \quad \forall\, h \in X\,. \tag{3.3}$$

Let $Q : X \to Y$ be defined by

$$Q(h)(x) = -V(x)h(x)\,, \quad h \in X\,,\, x \in [0,1]\,.$$

The operator $L : X \to Y$ is one to one and onto. We have $S'(u) = L - Q = L(I - L^{-1}Q)$. By (3.3) $\ker(S'(u)) = \{0\}$. Then $\ker(I - L^{-1}Q) = \{0\}$. Since $L^{-1} : (Y, \|.\|_\infty) \to (X, \|.\|_\infty)$ is compact, $L^{-1}Q$ is compact too. By the Fredholm alternative we obtain that $I - L^{-1}Q$ is onto. Therefore $S'(u) : (X, N_1) \to (Y, \|.\|_\infty)$ is an invertible operator. By the local inversion theorem we have that S is a local diffeomorphism. Now let

$$c(t) = \frac{2}{\pi\|(\pi^2 I - \Lambda(t))^{-1}\|}\,, \quad t \ge 0\,.$$

This function satisfies the hypotheses of Theorem 2.5. Therefore S is a global diffeomorphism and consequently the equation $Su = 0$ has a unique solution $u \in X$. This is also the unique solution of the boundary value problem (1.1).

4. Proof of Theorem 1.3

We keep the notations introduced in Section 3. In the same way we show that $S : (X, N_1) \to (Y, \|\cdot\|_\infty)$ is a local diffeomorphism. Now let $u = (u_1, \dots, u_n) \in X$ and let $j \in \{1, \dots, n\}$. Integrating by parts we get

$$\begin{aligned}\int_0^1 u_j(x) S(u)_j(x)\,dx &= \int_0^1 u_j(x)(u_j''(x) + f_j(x, u(x))\,dx \\ &= -\int_0^1 u_j'(x)^2\,dx + \int_0^1 u_j(x) f_j(x, u(x))\,dx\,.\end{aligned}$$

Then using the Cauchy-Schwarz inequality, (1.6) and Lemma 3.1 we can write

$$
\begin{aligned}
\|u_j\|_2\|S(u)_j\|_2 &\geq -\int_0^1 u_j(x)S(u)_j(x)\,dx \\
&\geq \|u_j'\|_2^2 - \sum_{k=1}^n \int_0^1 a_{jk}(\|u(x)\|)|u_j(x)u_k(x)|\,dx - b_j\int_0^1 |u_j(x)|\,dx \\
&\geq \|u_j'\|_2^2 - \sum_{k=1}^n a_{jk}(\|u\|_\infty)\|u_j\|_2\|u_k\|_2 - b_j\|u_j\|_2 \\
&\geq \pi\|u_j\|_2\|u_j'\|_2 - \frac{1}{\pi}\sum_{k=1}^n a_{jk}(\|u\|_\infty)\|u_j\|_2\|u_k'\|_2 - b_j\|u_j\|_2\,,
\end{aligned}
$$

from which we deduce that

$$
\|S(u)_j\|_2 \geq \pi\|u_j'\|_2 - \frac{1}{\pi}\sum_{k=1}^n a_{jk}(\|u\|_\infty)\|u_k'\|_2 - b_j\,, \tag{4.1}
$$

for $j = 1, \dots, n$. Let r, s and b denote the vectors

$$
r = (\|u_j'\|_2)_{1\leq j\leq n}\,, \quad s = (\pi\|S(u)_j\|_2)_{1\leq j\leq n}, \quad b = (\pi b_j)_{1\leq j\leq n}.
$$

Inequality (4.1) can be written as

$$
s - (\pi^2 I - A(\|u\|_\infty))r + b \in \mathbb{R}^n_+\,.
$$

Theorem 2.3 implies that $\pi^2 I - A(\|u\|_\infty)$ is $\mathbb{R}^n_+$-monotone. Then using Theorem 2.2 we obtain

$$
(\pi^2 I - A(\|u\|_\infty))^{-1}(s + b) - r \in \mathbb{R}^n_+\,, \tag{4.2}
$$

which implies that

$$
\pi\|(\pi^2 I - A(\|u\|_\infty))^{-1}\|(\|S(u)\|_2 + \|b\|) \geq \|u_j'\|_2\,,
$$

for $j = 1, \dots, n$. Using Lemma 3.1 and the fact that $\|S(u)\|_2 \leq \|S(u)\|_\infty$ we deduce that

$$
\|S(u)\|_\infty \geq \frac{2\|u\|_\infty}{\pi\|(\pi^2 I - A(\|u\|_\infty))^{-1}\|} - \|b\|\,. \tag{4.3}
$$

We shall prove that (4.3) implies that $S : (X, N_1) \to (Y, \|.\|_\infty)$ is a proper map. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X and $v \in Y$ such that $S(u_n) \to v$ as $n \to +\infty$. (1.8) and (4.3) imply that there exists a constant $M > 0$ such that $\|u_n\|_\infty \leq M$ for every $n \in \mathbb{N}$. Since $K : (X, N_1) \to (Y, \|.\|_\infty)$ is a compact operator, it follows that the sequence $(K(u_n))_{n\in\mathbb{N}}$ contains a convergent subsequence. Without loss of generality we may assume that $(K(u_n))_{n\in\mathbb{N}}$ is convergent to $w \in Y$. Letting $n \to +\infty$ in the equality

$$
u_n = L^{-1}S(u_n) + L^{-1}K(u_n)\,,
$$

we obtain

$$
\lim_{n\to+\infty} \|u_n - L^{-1}(v) - L^{-1}(w)\|_\infty = 0\,. \tag{4.4}
$$

Then we have

$$
\begin{aligned}
&\lim_{n\to+\infty} \|L(u_n) - L(L^{-1}(v) + L^{-1}(w))\|_\infty \\
&= \lim_{n\to+\infty} \|(S(u_n) - v) + (K(u_n) - w)\|_\infty = 0\,.
\end{aligned}
$$

From this equality and (4.4), we deduce that

$$\lim_{n\to+\infty} N_1(u_n - L^{-1}(v+w)) = 0\,.$$

Therefore $S : (X, N_1) \to (Y, \|\cdot\|_\infty)$ is a proper map. Using Theorem 2.6 we conclude that S is a global homeomorphism and consequently the equation $Su = 0$ has a unique solution $u \in X$. This is also the unique solution of the boundary value problem (1.1).

5. Examples

In this section we give two examples to illustrate Theorems 1.2 and 1.3. Define $a,\, h : \mathbb{R} \to \mathbb{R}_+$ by

$$a(t) = \begin{cases} 0 & \text{if } t \le 1\,, \\ 1 - \frac{1}{t^\alpha} & \text{if } t \ge 1 \end{cases} \quad \text{and} \quad h(t) = \int_1^t a(s)\,ds\,, \quad t \in \mathbb{R}\,,$$

where $\alpha > 0$.

Example 5.1. Let $n = 2$. We set

$$f_1(x,u) = \pi^2 h(u_1) + g_1(x), \quad f_2(x,u) = |u_1|^\beta + \pi^2 h(u_2) + g_2(x)\,,$$

for $(x,u) \in [0,1] \times \mathbb{R}^2$. $\beta > 1$ is a constant and $g_1,\, g_2 \in C([0,1],\mathbb{R})$. Then we can take

$$a_{11} = a_{22} = \pi^2 a\,, \quad a_{12} = 0\,, \quad a_{21}(t) = t^{\beta-1}\,, \quad t \ge 0\,,$$
$$b_j = \|g_j\|_\infty\,, \quad j = 1,\,2\,,$$
$$\lambda_{11} = \lambda_{22} = \pi^2 a\,, \quad \lambda_{12} = 0\,, \quad \lambda_{21}(t) = \beta t^{\beta-1}, \quad t \ge 0\,.$$

We easily verify that $\rho(A(t)) = \rho(\Lambda(t)) = \pi^2 a(t) < \pi^2$ for $t \ge 0$,

$$\frac{t}{\|(\pi^2 I - A(t))^{-1}\|} = \frac{\pi^4 t^{1-\alpha}}{\pi^2 + t^{\alpha+\beta-1}} \quad \text{for} \quad t \ge 1\,,$$
$$\|(\pi^2 I - \Lambda(t))^{-1}\| = \frac{t^\alpha}{\pi^2} + \frac{\beta}{\pi^4} t^{2\alpha+\beta-1} \quad \text{for } t \ge 1\,.$$

Note that a_{21} and λ_{21} are unbounded. If $2\alpha + \beta < 2$ we can use either Theorem 1.2 or Theorem 1.3. Now let $2\alpha < 1$ and $\beta = 2(1-\alpha)$. Then Theorem 1.2 applies but Theorem 1.3 does not apply.

Example 5.2. Let $n = 2$. We set

$$f_1(x,u) = \pi^2 h(u_1) + g_1(x), \quad f_2(x,u) = \cos|u_1|^\beta + \pi^2 h(u_2) + g_2(x)\,,$$

for $(x,u) \in [0,1] \times \mathbb{R}^2$. $\beta > 1$ is a constant and $g_1,\, g_2 \in C([0,1],\mathbb{R})$. Then we can take

$$a_{11} = a_{22} = \pi^2 a\,, \quad a_{12} = a_{21} = 0\,, \quad b_1 = \|g_1\|_\infty\,, \quad b_2 = 1 + \|g_2\|_\infty\,,$$
$$\lambda_{11} = \lambda_{22} = \pi^2 a\,, \quad \lambda_{12} = 0\,, \quad \lambda_{21}(t) = \beta t^{\beta-1}, \quad t \ge 0\,.$$

We easily verify that $\rho(A(t)) = \rho(\Lambda(t)) = \pi^2 a(t) < \pi^2$ for $t \ge 0$,

$$\frac{t}{\|(\pi^2 I - A(t))^{-1}\|} = \frac{t^{1-\alpha}}{\pi^2} \quad \text{for } t \ge 1\,,$$
$$\|(\pi^2 I - \Lambda(t))^{-1}\| = \frac{t^\alpha}{\pi^2} + \frac{\beta}{\pi^4} t^{2\alpha+\beta-1} \quad \text{for } t \ge 1\,.$$

Notice that λ_{21} is unbounded. If $2\alpha+\beta\le 2$, then Theorem 1.2 and Theorem 1.3 apply. If $2\alpha+\beta>2$ and $\alpha<1$, Theorem 1.3 still applies but not Theorem 1.2.

We conclude this paper with the following remark.

Remark 5.3. With the notations of Theorems 1.2 and 1.3, assume that λ_{jk} are bounded for j, $k=1,\dots,n$. Then (1.4) implies (1.5). Indeed we have

$$(\pi^2 I-\Lambda(t))^{-1}=\frac{1}{\det(\pi^2 I-\Lambda(t))}B(t),\quad t\ge 0\,,$$

where $B(t)\in M(n)$ is nonnegative and $\det(\pi^2 I-\Lambda(t))>0$ (see Remark 2.4). Since λ_{jk} are bounded for j, $k=1,\dots,n$, there exists a constant $d>0$ such that $\|B(t)\|\le d$ for all $t\ge 0$. Then we can write

$$\frac{1}{\|(\pi^2 I-\Lambda(t))^{-1}\|}=\frac{\det(\pi^2 I-\Lambda(t))}{\|B(t)\|}\ge\frac{1}{d}\det(\pi^2 I-\Lambda(t))\,,\quad t\ge 0\,,$$

and our claim follows.

It is easily seen that (1.5) does not imply (1.4) in general. Indeed let

$$\lambda_{11}(t)=\lambda_{22}(t)=\pi^2(1-\frac{1}{t})\,,\quad t\ge 1$$

and $\lambda_{12}=\lambda_{21}=0$. Then we have

$$\frac{1}{\|(\pi^2 I-\Lambda(t))^{-1}\|}=\frac{\pi^2}{t}\quad\text{and}\quad\det(\pi^2 I-\Lambda(t))=\frac{\pi^4}{t^2}\,,\quad t\ge 1\,.$$

References

[1] A. Berman and J. Plemmons; *Nonnegative matrices in the mathematical sciences*, Academic Press, New York, 1979.

[2] S. Banach and S. Mazur; *Über mehrdeutige stetige Abbildungen*, Studia Math. 5 51934, 174-178.

[3] R. Caccioppoli; *sugli elementi uniti delle transformazioni funzionali: un teorema di esistenza e di unicita ed alcune sue applicazioni*, Rend. Seminar Mat. Padova, 3 (1932), 1-15.

[4] R. Dalmasso; *An existence and uniqueness theorem for a second order nonlinear system*, Journal of Mathematical Analysis and Applications 327 (2007), 715-722.

[5] M. Rădulescu and S. Rădulescu; *An application of a global inversion theorem to a Dirichlet problem for a second order differential equation*, Rev. Roumaine Math. Pures Appl. 37 (1992), 929–933.

[6] M. Rădulescu and S. Rădulescu; *Applications of a global inversion theorem to unique solvability of second order Dirichlet problems*, Annals of University of Craiova, Math. Comp. Sci. Ser. 30 (1) (2003), 198-203.

Robert Dalmasso

Laboratoire Jean Kuntzmann, Equipe EDP, Tour IRMA, BP 53, 38041 Grenoble Cedex 9, France

E-mail address: robert.dalmasso@imag.fr