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AN APPLICATION OF A GLOBAL BIFURCATION THEOREM TO THE EXISTENCE OF SOLUTIONS FOR INTEGRAL INCLUSIONS

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Abstract. We prove the existence of solutions to Hammerstein integral inclusions of weakly completely continuous type. As a consequence we obtain an existence theorem for differential inclusions, with Sturm-Liouville boundary conditions,

$$u''(t) \in -F(t, u(t), u'(t)) \quad \text{for a.e. } t \in (a, b)$$
$$l(u) = 0.$$

1. Introduction

The purpose of this paper is to prove existence theorems for the integral inclusion of weakly completely continuous type

$$u(t) \in \int_a^b K(t,s)F(s,u(s))ds \quad \text{for all } t \in [a,b].$$

Integral equations (inclusions) have been studied by many authors; see, for example [17], where the nonlinear alternative for multi-valued mappings is used for obtaining existence results for Volterra and Hammerstein type equations. Our approach is rather different and is based on a global bifurcation theorem for convex-valued completely continuous mappings; see [7, Theorem 1].

This paper will be divided into four sections. In the second section we will introduce a class of integral inclusions of weakly completely continuous type, and next we will state the main theorem. In the third section we will prove an existence theorem using a global bifurcation theorem for convex-valued completely continuous mappings [7, Theorem 1]. In this part we will assume that a multi-valued mapping $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ satisfies appropriate conditions close to zero and infinity. The fourth part contains some applications of the results given in the second section, and selectors theorems. As consequences we will obtain existence theorems for integral inclusions and for boundary value problems for differential inclusions.

In this paper we will use the following notation. Let E be a real Banach space. By $\mathrm{cl}(E)$ we will denote the family of all non-empty, closed and bounded subsets

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of E. By $\mathrm{cf}(E)$ we will denote the family of all non-empty, closed, bounded and convex subsets of E. For two sets $A, B \in \mathrm{cl}(E)$ we will denote by $\mathrm{D}(A,B)$ the Hausdorff distance between A and B. In particular we put $|A| = \mathrm{D}(A, \{0\})$.

Let E_1, E_2 be two Banach spaces and $X \subseteq E_1$. A multi-valued mapping $\varphi : X \to \mathrm{cl}(E_2)$ has a closed graph provided for all sequences $\{x_n\} \subset X$ and $\{y_n\} \subset E_2$ the conditions $x_n \to x$, $y_n \to y$ and $y_n \in \varphi(x_n)$ for every $n \in \mathbb{N}$ imply $y \in \varphi(x)$.

We call a multi-valued mapping $\varphi : X \to \mathrm{cl}(E_2)$ completely continuous if φ has a closed graph and for any bounded subset $A \subseteq X$ the set $\varphi(A) = \bigcup_{x\in A} \varphi(x)$ is a relatively compact subset of E_2.

A multi-valued mapping $\varphi : X \to \mathrm{cl}(E_2)$ has a strongly-weakly (s-w) closed graph provided for all sequences $\{x_n\} \subset X$ and $\{y_n\} \subset E_2$ the conditions $x_n \to x$, $y_n \rightharpoonup y$ and $y_n \in \varphi(x_n)$ for every $n \in \mathbb{N}$ imply $y \in \varphi(x)$ ($y_n \rightharpoonup y$ denotes weak convergence).

We call a multi-valued mapping $\varphi : X \to \mathrm{cl}(E_2)$ weakly completely continuous if φ has a strongly-weakly closed graph and for any bounded subset $A \subseteq X$ the set $\varphi(A) = \cup_{x\in A}\varphi(x)$ is a relatively weakly compact subset of E_2.

We will also need the following notations. For $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ we call x non-negative (and write $x \geq 0$) when $x_i \geq 0$ for $i = 1, \dots, k$. Let $\|\cdot\|_0$ be the supremum norm in $C[a,b]$ and $\|\cdot\|_k$ be the norm in $C([a,b], \mathbb{R}^k)$ given by $\|u\|_k = \sum_{i=1}^k \|u_i\|_0$ for $u = (u_1, \dots, u_k) \in C([a,b], \mathbb{R}^k)$. For $v \in C([a,b], \mathbb{R}^k)$ we say $v \geq 0$ if and only if $v(t) \geq 0$ for every $t \in [a,b]$. For $u, v \in C([a,b], \mathbb{R}^k)$ we write $\langle u, v\rangle = \int_a^b \sum_{i=1}^k u_i(t)v_i(t)dt$. Let the mapping $p : \mathbb{R}^k \to \mathbb{R}^k$ be given by $p(x_1, \dots, x_k) = (|x_1|, \dots, |x_k|)$.

2. Integral inclusions of weakly completely continuous type

In what follows we consider the integral inclusions of weakly completely continuous type,

$$u(t) \in \int_a^b K(t,s)F(s,u(s))ds, \quad t \in [a,b], \tag{2.1}$$

where the kernel $K : [a,b]^2 \to \mathbb{R}$ satisfies the following conditions:

$$K(t,s) = K(s,t), \quad \forall t, s \in [a,b] \tag{2.2}$$

$$K(t,\cdot) \in L^\infty((a,b), \mathbb{R}); \quad \forall t \in [a,b] \tag{2.3}$$

$$\mathcal{K}(t)(s) = K(t,s) \text{ is continuous, } \mathcal{K} : [a,b] \to L^\infty((a,b), \mathbb{R}) \tag{2.4}$$

$$v \geq 0 \text{ implies } \int_a^b K(t,s)v(s)ds \geq 0, \quad \forall v \in C([a,b], \mathbb{R}^k); \tag{2.5}$$

the set of eigenvalues of $v(t) = \lambda \int_a^b K(t,s)v(s)ds$ corresponding to non-negative eigenvectors is nonempty and is finite. Let us denote this set by

$$\Lambda = \{\mu_1 \dots, \mu_N\}, \quad \text{with } \mu_1 < \mu_2 < \dots < \mu_N; \tag{2.6}$$

the multi-valued mapping $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ satisfies the condition: There exists a multi-valued mapping $\varphi : C([a,b], \mathbb{R}^k) \to \mathrm{cf}(L^1((a,b), \mathbb{R}^k))$ with a s-w closed graph such that

$$\varphi(v) \subseteq \{w \in L^1((a,b), \mathbb{R}^k) : w(t) \in F(t, v(t)) \quad \text{a.e. on } [a,b]\} \tag{2.7}$$

for each $v \in C([a,b], \mathbb{R}^k)$.

Recall that a multi-valued mapping $F : [a,b]\times\mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ is integrably bounded if: For each $R > 0$ there exists a function $m_R \in L^1((a,b),\mathbb{R})$ such that

$$|F(t,x)| \le m_R(t) \quad \text{for a.e. } t \in [a,b] \text{ and every } x \in \mathbb{R}^k \text{ with } |x| \le R. \tag{2.8}$$

A solution of the integral inclusion (2.1) is a continuous function $u : [a,b] \to \mathbb{R}^k$ which satisfies (2.1).

Theorem 2.1. *Let $K : [a,b]^2 \to \mathbb{R}$ satisfies* (2.2)–(2.6) *and let a multi-valued mapping $F : [a,b]\times\mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ satisfies* (2.7), (2.8) *and for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$D(F(t,x),\{m_1 p(x)\}) \le \varepsilon|x| \quad \text{for } t \in [a,b]\ |x| \le \delta; \tag{2.9}$$

for every $\varepsilon > 0$ there exists $R_0 > 0$ such that

$$D(F(t,x),\{m_2 p(x)\}) \le \varepsilon|x| \quad \text{for } t \in [a,b]\ |x| \ge R_0; \tag{2.10}$$

with constants m_1, m_2 such that $m_1 > \max\Lambda$ and $m_2 < \min\Lambda$. Then there exists at least one non-trivial solution of integral inclusion (2.1).

3. Proof of Theorem 2.1

We need some notation. Let $\Psi : (0,\infty)\times C([a,b],\mathbb{R}^k) \to \mathrm{cf}(C([a,b],\mathbb{R}^k))$ be a completely continuous mapping such that $0 \in \Psi(\lambda,0)$ for all $\lambda \in (0,\infty)$. Let $f : (0,\infty)\times C([a,b],\mathbb{R}^k) \to \mathrm{cf}(C([a,b],\mathbb{R}^k))$ be given by

$$f(\lambda,u) = u - \Psi(\lambda,u). \tag{3.1}$$

We call $(\mu,0) \in (0,\infty)\times C([a,b],\mathbb{R}^k)$ a bifurcation point of f if for each neighbourhood U of $(\mu,0)$ in $(0,\infty)\times C([a,b],\mathbb{R}^k)$ there exists a point $(\lambda,u) \in U$ such that $u \ne 0$ and $0 \in f(\lambda,u)$. Let us denote the set of all bifurcation points of f by $\mathcal{B}_f$. Let $\mathcal{R}_f \subset (0,\infty)\times C([a,b],\mathbb{R}^k)$ be the closure (in $(0,\infty)\times C([a,b],\mathbb{R}^k)$) of the set of non-trivial solutions of the inclusion $0 \in f(\lambda,u)$.

Let V be a bounded open subset of a Banach space E and let the multi-valued mapping $g : \overline{V} \to \mathrm{cf}(E)$ be given by $g(x) = x - G(x)$, where $G : \overline{V} \to \mathrm{cf}(E)$ is a completely continuous multi-valued mapping such that, for $x \in \partial V$, the relation $x \notin \partial V$ holds. It is well known that in such situation we may define the Laray-Schauder degree $\deg(g,V,0)$ [2, 4, 12, 14, 16].

For each λ satisfying $(\lambda,0) \notin \mathcal{B}_f$ there exists $r_0 > 0$, such that $0 \notin f(\lambda,u)$ for $\|u\|_k = r \in (0,r_0]$, so the value $\deg(f(\lambda,\cdot),B(0,r),0)$ is defined. Assume that for an interval $[c,d] \subset (0,\infty)$ there exists $\delta > 0$ such, that

$$\Big(([c-\delta,c)\cup(d,d+\delta])\times\{0\}\Big)\cap\mathcal{B}_f = \emptyset.$$

Then we may define the bifurcation index $s[f,c,d]$ of the mapping f, with respect to the interval $[c,d]$ as

$$s[f,c,d] = \lim_{\lambda\to d^+}\deg(f(\lambda,\cdot),B(0,r),0) - \lim_{\lambda\to c^-}\deg(f(\lambda,\cdot),B(0,r),0),$$

where $r = r(\lambda) > 0$ is small enough.

The main tool used in this section is Theorem 3.1 below. It is a global bifurcation result for convex-valued completely continuous mappings being a consequence of the generalized of the Rabinovitz global bifurcation alternative (see [5, 22]).

Theorem 3.1 ([7]). *Let $f : (0,\infty) \times C([a,b],\mathbb{R}^k) \to \mathrm{cf}(C([a,b],\mathbb{R}^k))$ be given by (3.1), and assume that there exists an interval $[c,d] \subset (0,\infty)$ such that $\mathcal{B}_f \subset [c,d] \times \{0\}$ and $s[f,c,d] \neq 0$. Then there exists a non-compact component $\mathcal{C} \subset \mathcal{R}_f$ satisfying $\mathcal{C} \cap \mathcal{B}_f \neq \emptyset$.*

In what follows we will use the integral operator $S : L^1((a,b),\mathbb{R}^k) \to C([a,b],\mathbb{R}^k)$ given by

$$S(u)(t) = \int_a^b K(t,s)u(s)ds \tag{3.2}$$

where K is as above.

Remark 3.2. Let us observe that the operator S is well-defined and S is completely continuous.

Proposition 3.3. *Let $\varphi : E_1 \to \mathrm{cl}(E_2)$ be a weakly completely continuous multi-valued mapping and let $T : E \to E_1$ be a continuous linear mapping, and let $S : E_2 \to E_3$ be a continuous linear mapping such that for every bounded subset B of E_1, $\overline{S\varphi(B)}$ is a compact subset of a Banach space E_3. Then the composition $S \circ \varphi \circ T : E \to \mathrm{cl}(E_3)$ is completely continuous.*

Now we prove the main result.

Proof of Theorem 2.1. By (2.7) and (2.8) there exists a weakly completely continuous multi-valued mapping $\varphi : C([a,b],\mathbb{R}^k) \to \mathrm{cf}(L^1((a,b),\mathbb{R}^k))$ such that

$$\varphi(u) \subseteq \{w \in L^1((a,b),\mathbb{R}^k) : w(t) \in F(t,u(t)) \text{ a.e. on}[a,b]\} \tag{3.3}$$

for each $u \in C([a,b],\mathbb{R}^k)$. It follows from Remark 1 and Proposition 3.3 that the multi-valued mapping $S \circ \varphi : C([a,b],\mathbb{R}^k) \to \mathrm{cf}(C([a,b],\mathbb{R}^k))$ is completely continuous. Let $f : (0,\infty) \times C([a,b],\mathbb{R}^k) \to \mathrm{cf}(C([a,b],\mathbb{R}^k))$ be given by the formula

$$f(\lambda,u) = u - \lambda S\varphi(u).$$

Let us observe that if $0 \in f(1,u)$ then u is the solution of integral inclusion (2.1). So it is enough to show that there exists $u \in C([a,b],\mathbb{R}^k)$, $u \neq 0$ such that $0 \in f(1,u)$. To prove this we apply Theorem 3.1.

The proof will be given in three steps.

Step 1. We show that $\mathcal{B}_f \subseteq \{(\frac{\mu_i}{m_1},0); i = 1,\dots,N\}$. Let $(\lambda_0,0) \in \mathcal{B}_f$, and let $\{(\lambda_n,u_n)\} \subset (0,+\infty) \times C([a,b],\mathbb{R}^k)$ be the sequence of non-trivial solutions of the inclusion

$$u_n \in \lambda_n S\varphi(u_n)$$

such that $\lambda_n \to \lambda_0 \in (0,+\infty)$ and $u_n \to 0$. Let the mapping $P : C([a,b],\mathbb{R}^k) \to L^1((a,b),\mathbb{R}^k)$ be given by $P(u)(t) = p(u(t))$. So we have

$$u_n \in \lambda_n S\varphi(u_n) - m_1\lambda_n SP(u_n) + m_1\lambda_n SP(u_n).$$

Let us denote $v_n = \frac{u_n}{\|u_n\|_k}$. Then we have

$$v_n \in \lambda_n S(\frac{\varphi(u_n) - m_1P(u_n)}{\|u_n\|_k}) + \lambda_n m_1 SP(v_n).$$

By (2.9), we have $|\frac{\varphi(u_n)-m_1P(u_n)}{\|u_n\|_k}| \to 0$. Since the sequence $\{\lambda_n m_1 P(v_n)\}$ is bounded, there exists a subsequence of $\{v_n\}$ convergent to v_0 in $C([a,b],\mathbb{R}^k)$, where $\|v_0\|_k = 1$. So letting $n \to +\infty$ we get

$$v_0 = \lambda_0 m_1 SP(v_0).$$

Because $P(v_0) \geq 0$ then by (2.5) $SP(v_0) \geq 0$ and $v_0 \geq 0$. Hence $P(v_0) = v_0$ and $v_0 = \lambda_0 m_1 S(v_0)$. Then by (2.6) $\lambda_0 = \frac{\mu_i}{m_1}$ for some $i \in \{1, \dots, N\}$ that implies $\mathcal{B}_f \subseteq \{(\frac{\mu_i}{m_1}, 0);\ i = 1, \dots, N\}$.

Step 2. We show that $s[f, \frac{\mu_1}{m_1}, \frac{\mu_N}{m_1}] = -1$. For this purpose let us observe first that for $\lambda \notin \{\frac{\lambda}{m_1} :\ \lambda \in \Lambda\}$ there exists $r > 0$ such that by (2.9) the mapping $f(\lambda, \cdot) : \overline{B(0,r)} \to \mathrm{cf}(C([a,b], \mathbb{R}^k))$ is homotopic to the mapping $f_0(\lambda, \cdot) : \overline{B(0,r)} \to C([a,b], \mathbb{R}^k)$ given by

$$f_0(\lambda, u) = u - \lambda m_1 SP(u).$$

Moreover for $\lambda \in (0, \frac{\mu_1}{m_1})$ the mapping $f_0(\lambda, \cdot) : \overline{B(0,r)} \to C([a,b], \mathbb{R}^k)$ is homotopic to the identity mapping $i : \overline{B(0,r)} \to \mathrm{cf}(C([a,b], \mathbb{R}^k))$, let the homotopy be given by $h(\tau, u) = u - \lambda \tau m_1 SP(u)$. Similarly to what we showed in Step 1 of this proof we conclude that the homotopy h has no non-trivial zeros. Hence by homotopy property of topological degree we have $\deg(f_0(\lambda, \cdot), B(0,r), 0) = 1$. Assume now that $\lambda \in (\frac{\mu_N}{m_1}, +\infty)$. Choose any $i \in \{1, \dots, N\}$ and denote by u_{μ_i} a continuous non-trivial function such that $u_{\mu_i} = \mu_i S(u_{\mu_i})$ and $u_{\mu_i} \geq 0$. We will show that the mapping $f_0(\lambda, \cdot) : \overline{B(0,r)} \to C([a,b], \mathbb{R}^k)$ may be joined by homotopy with the mapping $f_1 : \overline{B(0,r)} \to C([a,b], \mathbb{R}^k)$ given by $f_1(u) = f_0(\lambda, u) - u_{\mu_i}$. A homotopy $h_1 : [0,1] \times \overline{B(0,r)} \to C([a,b], \mathbb{R}^k)$ is given by $h_1(\tau, u) = f_0(\lambda, u) - \tau u_{\mu_i}$. Assume now that $h_1(\tau, u) = 0$ for some u, $\|u\|_k \leq r$ and $\tau \in (0,1]$. Hence

$$u = \lambda m_1 SP(u) + \tau \mu_i S(u_{\mu_i}) = S(\lambda m_1 P(u) + \tau \mu_i u_{\mu_i}).$$

Since $\lambda m_1 P(u) + \tau \mu_i u_{\mu_i} \geq 0$ by (2.5) we have $u \geq 0$. So that,

$$u = S(\lambda m_1 u) + \tau u_{\mu_i},$$

and by (2.2),

$$\begin{aligned}
\langle u, u_{\mu_i} \rangle &= \langle S(\lambda m_1 u) + \tau u_{\mu_i}, u_{\mu_i} \rangle \\
&= \lambda m_1 \langle S(u), u_{\mu_i} \rangle + \tau \langle u_{\mu_i}, u_{\mu_i} \rangle \\
&= \lambda m_1 \langle u, S(u_{\mu_i}) \rangle + \tau \langle u_{\mu_i}, u_{\mu_i} \rangle \\
&= \frac{\lambda m_1}{\mu_i} \langle u, u_{\mu_i} \rangle + \tau \langle u_{\mu_i}, u_{\mu_i} \rangle.
\end{aligned}$$

Then

$$\frac{\mu_i - \lambda m_1}{\mu_i} \langle u, u_{\mu_i} \rangle = \tau \langle u_{\mu_i}, u_{\mu_i} \rangle > 0,$$

and we obtain $\mu_i > \lambda m_1$, because $u \geq 0$ and $u_{\mu_i} \geq 0$. This contradicts the assumption $\lambda > \frac{\mu_i}{m_1}$. Since $m_1 \lambda \notin \Lambda$, we have $h_1(0, \cdot) = f_0(\lambda, u) = 0 \Rightarrow u = 0$. Hence the homoptopy h_1 has no non-trivial zeroes, also $h(1, \cdot)$ has no zeroes at all that is why $\deg(f_0(\lambda, \cdot), B(0,r), 0) = 0$.

Step 3. Let us observe that by Theorem 3.1 there exists a non-compact component $\mathcal{C} \subset \mathcal{R}_f$ satisfying $\mathcal{C} \cap \mathcal{B}_f \neq \emptyset$. We are going to show that there exist $\lambda > 1$ and $u \neq 0$ such that $(\lambda, u) \in \mathcal{C}$. Since the set $\mathcal{C}$ is not compact there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that either $\lambda_n \to 0$ or $\lambda_n \to +\infty$ or else $\|u_n\|_k \to +\infty$.

First let us assume that $\lambda_n \to 0$ and $\{\|u_n\|_k\}$ is bounded. In this case, the relation $u_n \in \lambda_n S\varphi(u_n)$ holds and consequently $u_n \to 0$. As we showed in Step 1 $u_n \to 0$ and $\lambda_n \to \lambda_0$ implies $\lambda_0 \in \{\frac{\lambda}{m_1} : \lambda \in \Lambda\}$, that contradicts $\lambda_n \to 0$.

Now let us consider the case $\|u_n\|_k \to +\infty$ and $\lambda_n \to \lambda_0 \leq 1$. We can see that

$$u_n \in \lambda_n S\varphi(u_n) - m_2\lambda_n SP(u_n) + m_2\lambda_n SP(u_n),$$
$$v_n \in \lambda_n S(\frac{\varphi(u_n) - m_2 P(u_n)}{\|u_n\|_k}) + \lambda_n m_2 SP(v_n),$$

where $v_n = \frac{u_n}{\|u_n\|_k}$.

By (2.8) and (2.10) similarly to what we showed in Step 1 of this proof there exists v_0 with $\|v_0\|_k = 1$ such that

$$v_0 = \lambda_0 m_2 SP(v_0).$$

Since $P(v_0) \geq 0$ then $SP(v_0) \geq 0$ and $v_0 \geq 0$. Hence $P(v_0) = v_0$ and

$$v_0 = \lambda_0 m_2 S(v_0)$$

then by (2.6) $\lambda_0 = \frac{\mu_i}{m_2}$ for some $i \in \{1, \dots, N\}$ that contradicts $\lambda_0 \leq 1$.

Finally let us assume that $\lambda_n \to +\infty$. In this situation there exist $\lambda_n > 1$ and $u_n \neq 0$ with $(\lambda_n, u_n) \in \mathcal{C}$. Since $\mathcal{C} \cap \mathcal{B}_f \neq \emptyset$ and by our assumptions $\frac{\mu_i}{m_1} < 1$ for $i = 1, \dots, N$ then there exist $\lambda < 1$ and u such that $(\lambda, u) \in \mathcal{C}$. By connectedness of $\mathcal{C}$ there exists u with $(1, u) \in \mathcal{C}$. For such solution of inclusion $0 \in f(1, u)$ there must be $u \neq 0$, because $(1, 0) \notin \mathcal{R}_f(\ (1, 0) \notin \mathcal{B}_f)$. So the proof is complete. □

4. Examples

In the first part of this section we study a class of multi-valued mappings which admit a convex-valued weakly completely continuous selectors. The problem concerning the existence of a continuous selector and a weakly completely continuous selector have been studied by many authors for; see for example: Antosiwicz and Cellina [1], Łojasiewicz [15], Pliś [18], Pruszko [19, 20], Fryszkowski [10], Bressan and Colombo [3], Frigon and Granas [9].

In what follows we will consider integrably bounded multi-valued mappings $F : [a, b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n)$ satisfying one of the following properties:

$$\begin{aligned} &F : [a, b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) \quad \text{is } \mathcal{L} \otimes B \text{ measurable} \\ &F(t, \cdot) : \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) \quad \text{is l.s.c. for a.e. } t \in [a, b]. \end{aligned} \tag{4.1}$$

Let us recall that $A \subseteq [a, b] \times \mathbb{R}^k$ is $\mathcal{L} \otimes B$ measurable if A belongs to the σ-algebra generated by all sets of the form $N \times B$ where N is Lebesgue measurable in $[a, b]$ and B is Borel measurable in $\mathbb{R}^k$.

$$\begin{aligned} &F(\cdot, x) : [a, b] \to \mathrm{cl}(\mathbb{R}^n) \quad \text{is measurable for all } x \in \mathbb{R}^k \\ &F(t, \cdot) : \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) \quad \text{is continuous for a.e. } t \in [a, b]. \end{aligned} \tag{4.2}$$

$$F : [a, b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n) \quad \text{is l.s.c.} \tag{4.3}$$

$$\begin{aligned} &F(\cdot, x) : [a, b] \to \mathrm{cf}(\mathbb{R}^n) \quad \text{is measurable for all } x \in \mathbb{R}^k \\ &F(t, \cdot) : \mathbb{R}^k \to \mathrm{cf}(\mathbb{R}^n) \quad \text{is u.s.c. for a.e. } t \in [a, b]. \end{aligned} \tag{4.4}$$

Now we state without proof the following Proposition. Next applying Theorem 2.1, we obtain the existence of solutions of integral inclusions.

Proposition 4.1 ([1, 3, 9, 10, 15, 18, 19, 20])**.** *If $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^n)$ is an integrably bounded multi-valued mapping satisfying one of the conditions* (4.1) , (4.2), (4.3) *or* (4.4) *then the Nemytskii operator $\mathcal{F} : C([a,b],\mathbb{R}^k) \to \mathrm{cl}(L^1((a,b),\mathbb{R}^n))$, associated with F, admits a convex-valued weakly completely continuous selector.*

Theorem 4.2. *Let $K : [a,b]^2 \to \mathbb{R}$ satisfy* (2.2)–(2.6) *and let $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ be an integrably bounded multi-valued mapping such that one of the hypotheses* (4.1), (4.2), (4.3) *or* (4.4) *holds. If, moreover F satisfies* (2.9) *and* (2.10) *with constants m_1, m_2 such that $m_1 > \max\Lambda$ and $m_2 < \min\Lambda$, then there exists at least one non-trivial solution of integral inclusion* (2.1).

Now we prove an existence result for differential inclusions with Sturm–Liouville boundary conditions

$$\begin{gathered} u''(t) \in -F(t,u(t),u'(t)) \quad \text{for a.e. } t \in (a,b) \\ l(u) = 0, \end{gathered} \tag{4.5}$$

where $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathrm{cl}(\mathbb{R})$ is a multi-valued mapping and $l : C^1([a,b],\mathbb{R}) \to \mathbb{R} \times \mathbb{R}$ is given by

$$l(u) = \big(u(a)\sin\alpha - u'(a)\cos\alpha, u(b)\sin\beta + u'(b)\cos\beta\big),$$

and $\alpha, \beta \in [0, \frac{\pi}{2}], \alpha^2 + \beta^2 > 0$. It is well known (cf. [6, 13]) that with the boundary value problem

$$\begin{gathered} u''(t) = h(t) \quad \text{for a.e. } t \in (a,b) \\ l(u) = 0, \end{gathered} \tag{4.6}$$

we may associate a continuous integral operator $S : L^1((a,b),\mathbb{R}) \to C^1([a,b],\mathbb{R})$, given by

$$S(u)(t) = \int_a^b -K(t,s)u(s)ds \tag{4.7}$$

where K is Green's function for (4.6). Let us observe that $S(-h) = u$ if and only if $u \in C^1([a,b],\mathbb{R})$, $u' : [a,b] \to \mathbb{R}$ is absolutely continuous and u is a solution of (4.6). Let us recall that if $h \le 0$, $h \in C([a,b],\mathbb{R}^k)$ and $u \in C^2([a,b],\mathbb{R})$ satisfies (4.6) then $u \ge 0$ (cf. [21]). It is well known (cf. [6, 13]) that there exists exactly one eigenvalue $\mu \in \mathbb{R}$ of the linear eigenvalue problem

$$\begin{gathered} u''(t) + \lambda u(t) = 0 \quad \text{for } t \in (a,b) \\ l(u) = 0 \end{gathered} \tag{4.8}$$

an eigenvector v_μ, such that $v_\mu(t) > 0$ for $t \in (a,b)$ and then $\mu > 0$. Hence the set of eigenvalues of the integral operator S for which there exists non-negative eigenvector is equal to $\Lambda = \{\mu^{-1}\}$. We will also need the linear continuous operator $T : C^1([a,b],\mathbb{R}) \to C([a,b],\mathbb{R}\times\mathbb{R})$ given by $T(u)(t) = (u(t),u'(t))$ for $t \in [a,b]$. In what follows we will use the following existence theorem which is some modification of Theorem 2.1 for the integro-differential inclusions of weakly completely continuous type

$$u(t) \in \int_a^b -K(t,s)F(s,u(s),u'(s))ds \quad \text{for all } t \in [a,b]. \tag{4.9}$$

Theorem 4.3. *Let $K : [a,b]^2 \to \mathbb{R}$ be Green's function for* (4.6) *and let a multi-valued mapping $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathrm{cl}(\mathbb{R})$ satisfies* (2.7), (2.8) *and for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$D(F(t,x,y), \{m_1 p(x)\}) \le \varepsilon(|x| + |y|) \quad \text{for } t \in [a,b] \text{ and } |x| + |y| \le \delta; \tag{4.10}$$

for every $\varepsilon > 0$ there exists $R_0 > 0$ such that

$$D(F(t,x,y), \{m_2 p(x)\}) \le \varepsilon(|x| + |y|) \quad \text{for } t \in [a,b] \text{ and } |x| + |y| \ge R_0; \tag{4.11}$$

with constants m_1, m_2 such that $m_2 < \mu < m_1$. Then there exists at least one non-trivial solution of integral inclusion (4.9).

Proof. Let $f : (0, +\infty) \times C^1([a,b], \mathbb{R}) \to \mathrm{cf}(C^1([a,b], \mathbb{R}))$ be a multi-valued mapping defined by

$$f(\lambda, u) = u - \lambda S\varphi T(u),$$

where $\varphi : C([a,b], \mathbb{R} \times \mathbb{R}) \to \mathrm{cf}(L^1(a,b))$ is a weakly completely continuous multi-valued mapping such that

$$\varphi(u,v) \subseteq \{w \in L^1(a,b) : w(t) \in F(t, u(t), v(t)) \quad \text{a.e. on } [a,b]\}.$$

Let the mapping $S : L^1((a,b), \mathbb{R}) \to C^1([a,b], \mathbb{R})$ be as in (4.7). By Proposition 3.3 and the well known properties of Green's function, we see that the multi-valued mapping $S\varphi T : C^1([a,b], \mathbb{R}) \to \mathrm{cf}(C^1([a,b], \mathbb{R}))$ is completely continuous. Essentially the same reasoning as in Theorem 2.1 proves this theorem. $\square$

Now from Theorem 4.3 and Proposition 4.1 we obtain an existence Theorem for differential inclusions with Sturm-Liouville conditions.

Theorem 4.4. *Let $F : [a,b] \times \mathbb{R}^k \to \mathrm{cl}(\mathbb{R}^k)$ be an integrably bounded multi-valued mapping such that one of the hypotheses* (4.1), (4.2), (4.3) *or* (4.4) *holds. If, moreover F satisfies* (4.10) *and* (4.11) *with constants m_1, m_2 such that $m_2 < \mu < m_1$. Then there exists at least one non-trivial solution of boundary value problem* (4.5).

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