AMBROSETTI-PRODI TYPE RESULTS IN A SYSTEM OF SECOND AND FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, by the variational method, we study the existence, nonexistence, and multiplicity of solutions of an Ambrosetti-Prodi type problem for a system of second and fourth order ordinary differential equations.

1. Introduction

Lazer and McKenna [1] presented the following (one-dimensional) mathematical model for the suspension bridge:

\[ \begin{align*}
    & y_{tt} + y_{xxxx} + \delta_1 y_t + k (y - z)^+ = W(x), \quad \text{in} \ (0, L) \times \mathbb{R}, \\
    & z_{tt} - z_{xx} + \delta_2 z_t - k (y - z)^+ = h(x, t), \quad \text{in} \ (0, L) \times \mathbb{R}, \\
    & y(0, t) = y(L, t) = y_{xx}(0, t) = y_{xx}(L, t) = 0, \quad t \in \mathbb{R}, \\
    & z(0, t) = z(L, t) = 0, \quad t \in \mathbb{R},
\end{align*} \tag{1.1} \]

Where the variable \( z \) measures the displacement from equilibrium of the cable and the variable \( y \) measures the displacement of the road bed. The constant \( k \) is spring constant of the ties.

When the motion of the cable is ignored, the coupled system (1.1) can be simplified into a single equation which describes the motion of the road bed of suspension bridge, as follows

\[ \begin{align*}
    & y_{tt} + y_{xxxx} + \delta_1 y_t + ky^+ = W(x, t), \quad \text{in} \ (0, L) \times \mathbb{R}, \\
    & y(0, t) = y(L, t) = y_{xx}(0, t) = y_{xx}(L, t) = 0, \quad t \in \mathbb{R}.
\end{align*} \tag{1.2} \]

This Problem have been studied by many authors. In [2, 3, 4], the authors, using degree theory and the variational method, investigated the multiplicity of some symmetrical periodic solutions when \( \delta = 0 \) and \( W(x, t) = 1 + \epsilon h(x, t) \) or \( W(x, t) = \alpha \cos x + \beta \cos 2t \cos x \). In [5], the similar results for (1.2) are obtained in case of \( \delta \neq 0 \) and \( W(x, t) = h(x, t) = \alpha \cos x + \beta \cos 2t \cos x + \gamma \sin 2t \cos x \). Those results give the conditions impose on the spring constant \( k \) which guarantees the existence of multiple periodic solutions, especially the sign-changing periodic
solutions in the case of $W(x,t)$ is single-sign. It is notable that the functions \( \cos x, \cos 2t \cos x, \sin 2t \cos x \) are the eigenfunctions of linear principal operator of (1.2) in some function spaces.

When we consider only the steady state solutions of problem (1.1), we arrive at the system

\[
\begin{align*}
    y_{xxxx} + k(y-z)^+ &= h_1(x), \quad \text{in } (0, \pi), \\
    -z_{xx} - k(y-z)^+ &= h_2(x), \quad \text{in } (0, \pi), \\
    y(0) = y(\pi) = y_{xx}(0) = y_{xx}(\pi) &= 0, \\
    z(0) = z(\pi) &= 0.
\end{align*}
\] (1.3)

This problem has little been studied in [12, 13]. In [6, 15], the analogous partial differential systems have been considered when the nonlinearities $k(y-z)^+, -k(y-z)^+$ are replaced by general $f_1(y,z), f_2(y,z)$. And also, in recently, literature [16] studied the system

\[
\begin{align*}
    y_{xx} + k_1 y^+ + \epsilon z^+ &= \sin x, \quad \text{in } (0, \pi), \\
    z_{xx} + \epsilon y^+ + k_2 z^+ &= \sin x, \quad \text{in } (0, \pi), \\
    y(0) = y(\pi) &= 0, \\
    z(0) = z(\pi) &= 0.
\end{align*}
\] (1.4)

Where $u^+ = \max\{u,0\}$, the constant $\epsilon$ is small enough such that the matrix

\[
\begin{pmatrix}
    k_1 & \epsilon \\
    \epsilon & k_2
\end{pmatrix}
\]

is a near-diagonal matrix and the positive numbers $k_1, k_2$ satisfy

\[m_1^2 < k_1 < (m_1 + 1)^2, \quad m_2^2 < k_2 < (m_2 + 1)^2\] for some $m_1, m_2 \in \mathbb{N}$.

This is a first work in the direction of extending to systems some of well-known results established on nonlinear equation with an asymmetric nonlinearity. Meanwhile in [16] there are two open questions to be interesting:

**Question 1.** Can one obtain corresponding results if the second-order differential operator is replaced with a fourth-order differential operator with corresponding boundary conditions?

**Question 2.** Can one replace the near-diagonal matrix with something more general and use information on the eigenvalues of matrix?

Following the above works and questions, we consider the system

\[
\begin{align*}
    -u'' &= f_1(x, u, v) + t_1 \sin x + h_1(x), \quad \text{in } (0, \pi) \\
    v''' &= f_2(x, u, v) + t_2 \sin x + h_2(x), \quad \text{in } (0, \pi) \\
    u(0) &= u(\pi) = 0, \\
    v(0) &= v(\pi) = v''(0) = v'''(\pi) = 0.
\end{align*}
\] (1.5)

where $t_1, t_2$ are parameters and $(f_1, f_2): [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ is asymptotically linear.

On the other hand, the second order elliptic systems as follows

\[
\begin{align*}
    -\Delta u &= f_1(u, v) + t_1 \varphi_1 + h_1(x), \quad \text{in } \Omega, \\
    -\Delta v &= f_2(u, v) + t_2 \varphi_1 + h_2(x), \quad \text{in } \Omega, \\
    u &= v = 0, \quad \text{on } \partial \Omega
\end{align*}
\] (1.6)
have been widely studied. Here we mention the papers [7,8,9,10] and the references therein. If $(f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is asymptotically linear and the asymptotic matrices at $-\infty$ and $+\infty$ are

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix},
\begin{pmatrix}
\tilde{a} & \tilde{b} \\
\tilde{c} & \tilde{d}
\end{pmatrix}
$$

Under some growth conditions on $(f_1, f_2)$, in those papers, the Ambrosetti-Prodi type results for (1.6) have been given respectively.

We remind that let $g \in C^{\alpha}(\Omega \times \mathbb{R})$ be a given function such that

$$
\limsup_{s \to -\infty} \frac{g(x,s)}{s} < \lambda_1 < \liminf_{s \to +\infty} \frac{g(x,s)}{s}
$$

uniformly in $x \in \Omega$, where $\lambda_1$ is the first eigenvalue of the Laplacian on a bounded domain $\Omega$ under the Dirichlet condition and $\varphi_1$ is the associated eigenfunction.

The Ambrosetti-Prodi type result in a Cartesian version states that for a given $h \in C^{\alpha}(\Omega)$ there exists a real number $t_0$ such that the problem

$$
-\Delta u = g(x,u) + t\varphi_1 + h, \quad\text{in } \Omega
$$

$$
u = 0, \quad\text{on } \partial\Omega
$$

(i) has no solution if $t > t_0$;

(ii) has at least two solutions if $t < t_0$.

With different variants and formulations this problem has been extensively studied.

Inspired, we consider the Ambrosetti-Prodi type problem for system (1.5). This paper is organized as follows: in Section 2, we prepare the proper variational framework and prove (PS) condition to the Euler-Lagrange functional associated to our problem. In Section 3, we prove the main theorem. Finally, a piecewise linear problem is considered as an example in Section 4.

2. PRELIMINARIES

In this section, we prepare the proper variational frame work for (1.5), that is

$$
-u'' = f_1(x,u,v) + t_1 \sin x + h_1(x), \quad\text{in } (0,\pi)
$$

$$
v''' = f_2(x,u,v) + t_2 \sin x + h_2(x), \quad\text{in } (0,\pi)
$$

$$
u(0) = u(\pi) = 0,
$$

$$
v(0) = v''(0) = v''(\pi) = 0.
$$

Where $t_1, t_2$ are parameters, $h_1, h_2 \in C[0,\pi]$ are fixed functions with $\int_0^\pi h_1 \sin x = \int_0^\pi h_2 \sin x = 0$.

We shall need some assumptions on the nonlinearities, which are necessary to settle the existence or not of solutions in the case of the Ambrosetti-Prodi type problem and to establish (PS) condition.

Let us order $\mathbb{R}^2$ with the order defined by

$$
\xi = (\xi_1, \xi_2) \geq 0 \iff \xi_1, \xi_2 \geq 0.
$$

and denote $W = (u,v)$ and $F(x,W) = (f_1(x,u,v), f_2(x,u,v))$.

We will use the following hypotheses in this article.
(H1) \( F = (f_1, f_2) : [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is locally Lipschitzian function respect to \( u, v \), and there exists a function \( H : [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) such that
\[
\nabla H(x, u, v) = \left( \frac{\partial H}{\partial u}, \frac{\partial H}{\partial v} \right) = (f_1(x, u, v), f_2(x, u, v)).
\]

(H2) For \( \xi = (\xi_1, \xi_2) > 0 \) large enough,
\[
F(x, \xi) \geq 0.
\]

(H3) \( F \) satisfies
\[
|F(x, \xi)| \leq c(|\xi_1| + |\xi_2| + 1), \quad \forall \xi \in \mathbb{R}^2, \ x \in (0, \pi)
\]
where \( c > 0 \) is constant.

(H4) For \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) and \( x \in (0, \pi) \) there holds
\[
F(x, \xi) \geq A\xi - ce,
\]
for some constant \( c > 0 \). Where \( e = (1, 1) \) and the matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) satisfies
\[
\begin{align*}
&b, c \geq 0, \\
&(A\xi, \xi) \leq \mu|\xi|^2, \quad \text{for some } 0 < \mu < 1.
\end{align*}
\]

(H5) For \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) and \( x \in (0, \pi) \) there holds
\[
F(x, \xi) \geq \overline{A}\xi - ce,
\]
for some constant \( c > 0 \). Where \( e = (1, 1) \) and the matrix \( \overline{A} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \) satisfies
\[
\begin{align*}
&\overline{b}, \overline{c} \leq 0, \\
&(\overline{A}\xi, \xi) \geq \overline{\mu}|\xi|^2, \quad \text{for some } \overline{\mu} > 1.
\end{align*}
\]

(If not mentioned, \( c \) will always denote a generic positive constant.)

**Remark 2.1.** With a simple computation it is easy to show that (2.4)-(2.5) and (2.7)-(2.8) imply, respectively,
\[
(1 - a)(1 - d) - bc > 0, \quad a, d < 1,
\]
\[
(\overline{A} - I)^{-1}\xi \leq 0, \quad \forall \xi \in \mathbb{R}^2, \ \xi \geq 0,
\]
and
\[
(1 - \overline{a})(1 - \overline{d}) - \overline{b}\overline{c} > 0, \quad \overline{a}, \overline{d} > 1,
\]
\[
(\overline{A} - I)^{-1}\xi \geq 0, \quad \forall \xi \in \mathbb{R}^2, \ \xi \geq 0,
\]
where \( I \) is the identity matrix.

Let \( X = H^1_0(0, \pi) \times (H^1_0(0, \pi) \cap H^2(0, \pi)) \) be Hilbert space with the inner product
\[
\langle W, \Psi \rangle = \int_0^\pi (u' \psi'_1 + v' \psi''_2), \quad \forall W = (u, v), \ \Psi = (\psi_1, \psi_2) \in X,
\]
and the corresponding norm
\[
\|W\|^2_X = \int_0^\pi (u'^2 + v''^2).
\]
Consider the second-order ordinary differential eigenvalue problem
\[-u'' = \lambda u, \quad \text{in } (0, \pi),\]
\[u(0) = u(\pi) = 0,\]
and the fourth-order ordinary differential eigenvalue problem
\[v'''' = \lambda v, \quad \text{in } (0, \pi),\]
\[v(0) = v(\pi) = v''(0) = v''''(\pi) = 0.\]

It is well known that \(\lambda_1 = 1\) and \(\varphi_1 = \sin x\) are the positive first eigenvalue and the associated eigenfunction, respectively. Hence, it follows from the Poincare inequality that, for all \(W \in X\),
\[
\int_0^\pi |W|^2 \leq \|W\|_X^2. \quad (\text{2.11})
\]

A vector \(W \in X\) is a weak solution of (1.5) if, and only if, it is a critical point of the associated Euler-Lagrange functional
\[
J(W) = \frac{1}{2} \int_0^\pi (u'^2 + v''^2) - \int_0^\pi H(x, u, v) - \int_0^\pi [(t_1 \sin x + h_1)u + (t_2 \sin x + h_2)v] \quad (\text{2.12})
\]

It is standard to show that the functional \(J(W)\) is well defined, \(J(W) \in C^1(X, \mathbb{R})\) and \(X \to \mathbb{R}; \; W \to \int_0^\pi H(x, u, v) + \int_0^\pi [(t_1 \sin x + h_1)u + (t_2 \sin x + h_2)v]\) has compact derivative under the assumptions (H1) and (H3).

**Lemma 2.2.** Assume that (H1)-(H5) hold. Then \(J\) satisfies the (PS) condition.

**Proof.** Let \(\{W_n = (u_n, v_n)\} \subset X\) be a sequence such that \(|J(W_n)| \leq c\) and \(J'(W_n) \to 0\). This implies
\[
\left| \int_0^\pi (u_n' \psi_1' + v_n'' \psi_2'') - \int_0^\pi [(f_1 \psi_1 + f_2 \psi_2) + (t_1 \sin x + h_1)\psi_1 + (t_2 \sin x + h_2)\psi_2] \right| \\
\leq \varepsilon_n \|\Psi\|_X \quad (\text{2.13})
\]
for all \(\Psi = (\psi_1, \psi_2) \in X\), where \(\varepsilon_n \to 0(n \to \infty)\). Then by the above discussion it suffices to prove that \(\{W_n\}\) is bounded.

**Step 1:** Show the boundedness of \(\{W_n^-\}\). Let \(W_n^- = (u_n^-, v_n^-)\), \(w^- = \max\{0, -w\}\). Since \(h_1, h_2\) are bounded, there exists \(M_1, M_2 \geq 0\) such that
\[
|t_1 \sin x + h_1| \leq M_1, \quad |t_2 \sin x + h_2| \leq M_2. \quad (\text{2.14})
\]

Moreover, from (2.3) and (2.4), we have
\[
f_1(x, u_n, v_n)(-u_n^-) \leq \alpha (u_n^-)^2 + b u_n^- v_n^- + c u_n^-,
\]
\[
f_2(x, u_n, v_n)(-v_n^-) \leq \alpha (v_n^-)^2 + b u_n^- v_n^- + c v_n^-.
\]

Choosing \(c > \max\{M_1, M_2\}\) and taking \(\psi_1 = -u_n^-, \psi_2 = -v_n^-\) in (2.13), then using the above inequalities and (2.5), we obtain
\[
\|W_n^-\|_X^2 \leq \int_0^\pi (AW_n^-, W_n^-) + \int_0^\pi (c u_n^- - M_1 u_n + c v_n^- - M_2 v_n^-) + c\|W_n^-\|_X \\
\leq \mu \int_0^\pi (u_n^- + v_n^-) + d \int_0^\pi (u_n^- + v_n^-) + c\|W_n^-\|_X.
\]
Moreover, from (2.14) we have
\[ G \text{ characteristic function, then using the Lebesgue Dominated Convergence Theorem, we get} \]
\[ \int_0^\pi |u_n| \leq c \int_0^\pi |u_n|^2 \leq c \int_0^\pi |u_n'|^2, \]
\[ \int_0^\pi |v_n| \leq c \int_0^\pi |v_n|^2 \leq c \int_0^\pi |v_n''|^2. \]

Then from these two inequalities and (2.11) we have
\[ (1 - \mu)\|W_n\|^2_X \leq c\|W_n\|^2_X, \]
since \(0 < \mu < 1, \|W_n\|^2_X\) is bounded.

**Step 2:** Show the boundedness of \(\{W_n\}\). Suppose by contradiction that \(\{W_n\}\) is unbounded, then there exists a subsequence (still denote \(\{W_n\}\)) such that \(\|W_n\|^2_X \to \infty\) as \(n \to \infty\). Setting \(V_n = (x_n, y_n) = W_n/\|W_n\|^2_X\), then \(\|V_n\|_X = 1\) and there exists a subsequence such that
\[ V_n \rightharpoonup V_0 = (x_0, y_0), \quad \text{in } X, \]
\[ V_n \to V_0, \quad \text{in } L^2(0, \pi) \times L^2(0, \pi), \]
\[ V_n \to V_0, \quad \text{a.e. in } (0, \pi), \]
with \(|x_n(x)|, |y_n(x)| \leq h(x) \in L^2, x \in (0, \pi)\).

By step 1 we may assume that \(V_n^- \to 0\) in \(L^2 \times L^2\) and \(V_n^- \to 0\) a.e.in \((0, \pi)\).

Clearly, \(V_0 \geq 0\). Denote
\[ G_n(x) = (g_n^1(x), g_n^2(x)) \]
\[ = \left( f_1(x, W_n(x)) + t_1 \sin x + h_1, f_2(x, W_n(x)) + t_2 \sin x + h_2 \right) / \|W_n\|^2_X. \]

We claim that
\[ G_n \to \gamma = (\gamma_1, \gamma_2) \geq 0 \quad \text{in } L^2 \times L^2. \]

In fact, let \(A_n = \{x \in (0, \pi): u_n(x) \leq 0 \text{ and } v_n(x) \leq 0\}\) and let \(\chi_n\) denotes its characteristic function, then \(G_n = \chi_nG_n + (1 - \chi_n)G_n\). By (H3), (2.19), (2.17) and using the Lebesgue Dominated Convergence Theorem, we get
\[ \chi_n F(x, W_n) / \|W_n\|^2_X \to 0 \quad \text{in } L^2 \times L^2. \]

Moreover, from (2.14) we have
\[ \chi_n(t_1 \sin x + h_1, t_2 \sin x + h_2) / \|W_n\|^2_X \to 0 \quad \text{in } L^2 \times L^2. \]

Hence \(\chi_nG_n \to 0\) in \(L^2 \times L^2\). With the same reasoning \((1 - \chi_n)G_n \to \gamma' = (\gamma_1', \gamma_2')\) in \(L^2 \times L^2\). Therefore, we only need to prove that \(\gamma' \geq 0\).

(i) If \(u_n(x) \geq 0\) and \(v_n(x) \leq 0\), since \(\pi > 1\), from (2.6) we have
\[ (1 - \chi_n)g_n^1(x) + b(y_n^-(x)) + c \|W_n\|^2_X - (1 - \chi_n) \frac{t_1 \sin x + h_1}{\|W_n\|^2_X} \geq \pi x_n^+(x) \geq 0 \]
and from (2.3) and (2.4), we obtain
\[ (1 - \chi_n)g_n^2(x) + d(y_n^+(x)) + c \|W_n\|^2_X - (1 - \chi_n) \frac{t_2 \sin x + h_2}{\|W_n\|^2_X} \geq \xi x_n^+(x) \geq 0. \]
Since $V_n^c \to 0$ in $L^2 \times L^2$ and
\[ (1 - \chi_n)g_1^n(x) + \mathcal{U}(y_n^c(x)) + \frac{c}{\|W_n\|_X} - (1 - \chi_n)\frac{t_1 \sin x + h_1}{\|W_n\|_X} \to \gamma', \]
\[ (1 - \chi_n)g_2^n(x) + \mathcal{G}(y_n^c(x)) + \frac{c}{\|W_n\|_X} - (1 - \chi_n)\frac{t_2 \sin x + h_2}{\|W_n\|_X} \to \gamma'' \]
we get $\gamma' \geq 0$.

(ii) If $u_n(x) \leq 0$ and $v_n(x) \geq 0$, we can handle in the same way to obtain that $\gamma' \geq 0$.

(iii) If $u_n(x) \geq 0$ and $v_n(x) \geq 0$, the assertion $\gamma' \geq 0$ can be inferred from (H2).

Now dividing (2.13) by $\|W_n\|_X$, using (2.15), (2.18) and passing to the limit we obtain
\[ \int_0^\pi (\psi_1' + \psi_2') = \int_0^\pi (\gamma_1 \psi_1 + \gamma_2 \psi_2), \quad \forall \Psi = (\psi_1, \psi_2) \in X. \quad (2.19) \]
From (2.6) we have
\[ \frac{(f_1(x, W_n(x)) + t_1 \sin x + h_1, f_2(x, W_n(x)) + t_2 \sin x + h_2)}{\|W_n\|_X} \geq \mathcal{A}V_0 - \frac{ce}{\|W_n\|_X}. \]
Passing to the limit in this inequality we get
\[ \gamma \geq \mathcal{A}V_0. \quad (2.20) \]
Taking $\psi_1 = \sin x$, $\psi_2 = 0$ and then $\psi_1 = 0$, $\psi_2 = \sin x$ in (2.19) and using (2.20), it is achieved that
\[ (\mathcal{A} - I) \left( \int_0^\pi x \sin x \right) \leq 0. \quad (2.21) \]
From Remark 2.1, applying $(\mathcal{A} - I)^{-1}$ to (2.21) we get $(\int_0^\pi x \sin x, \int_0^\pi y \sin x) \leq 0$. Hence $x = y = 0$ a.e. So, from (2.19), $\int_0^\pi (\gamma, \Psi) = 0$ and taking $\Psi > 0$ we have $\gamma = 0$.

Finally, consider $\psi_1 = x_n, \psi_2 = y_n$ in (2.13). Dividing the resulting expression by $\|W_n\|_X$, and passing to the limit we obtain $1 \leq 0$, that is impossible. \hfill \Box

**Lemma 2.3.** Suppose (H5) hold. Then
\[ \lim_{s \to +\infty} J(s \sin x, s \sin x) = -\infty. \quad (2.22) \]

**Proof.** From (2.6) we have
\[ H(x, u, v) \geq \frac{\pi}{2} u^2 + \bar{b}uv - cu + H(x, 0, 0) \quad \text{as } u \geq 0, \forall v, \quad (2.23) \]
\[ H(x, u, v) \geq \frac{\pi}{2} v^2 + \bar{c}uv - cv + H(x, 0, 0) \quad \text{as } v \geq 0, \forall u. \quad (2.24) \]

Adding (2.23), (2.24) and using them again,
\[ 2H(x, u, v) \geq \frac{\pi}{2} u^2 + (\bar{b} + \bar{c})uv + \bar{d}v^2 - cu - cv + H(x, 0, 0) + H(x, u, 0) \]
\[ \geq \pi u^2 + (\bar{b} + \bar{c})uv + dv^2 - 2cu - 2cv + 2H(x, 0, 0) \]
\[ \geq \pi v^2 + (\bar{b} + \bar{c})uv + dv^2 - 2cu - 2cv + 2c, \quad \text{for } u, v \geq 0. \]

Then by (2.8) we have
\[ H(x, W) \geq \frac{\pi}{2} |W|^2 - cv - c. \quad (2.25) \]
Taking $W = (s\sin x, s\sin x)$, where $s > 0$, from (2.14) and (2.25) we get

$$J(s\sin x, s\sin x) \leq \frac{\pi s^2}{2}(1 - \overline{\mu}) + (c + M_1)\int_0^\pi s\sin x + (c + M_2)\int_0^\pi s\sin x - c$$

$$\leq \frac{\pi s^2}{2}(1 - \overline{\mu}) + cs - c$$

since $\overline{\mu} > 1$, (2.22) holds. □

3. The Ambrosetti-Prodi type result

In this section, we state and prove the Ambrosetti-Prodi type result for system (1.5). We need the following concepts.

**Definition 3.1.** (1) We say that a vector function $W \in X$ is a weak subsolution of (1.5) if

$$J'(W)(\Psi) \leq 0, \quad \forall \Psi \in X, \quad \Psi \geq 0.$$  

(2) $W = (u, v) \in C^2 \times C^4$ is a subsolution (classical) of (1.5) if

$$-u'' \leq f_1(x, u, v) + t_1 \sin x + h_1, \quad \text{in } (0, \pi),$$

$$v''' \leq f_2(x, u, v) + t_2 \sin x + h_2, \quad \text{in } (0, \pi),$$

$$u(0) = u(\pi) = 0,$$

$$v(0) = v'(0) = v''(0) = v'''(0) = 0.$$ 

(3) Weak supersolutions and supersolutions (classical) are defined likewise by reversing the above inequalities.

We can easily show that each a subsolution or a supersolution of (1.5) is indeed also a weak subsolution or a weak supersolution, respectively.

For to present the subsolution and supersolution for (1.5), we firstly show a maximum principle as follows.

**Lemma 3.2.** Let $A$ be a matrix-function with entries in $C[0, \pi]$ satisfy (2.4) and (2.5). If $W = (u, v) \in X$ is such that

$$\int_0^\pi (u'\psi'_1 + v''\psi''_2) \geq \int_0^\pi (AW, \Psi), \quad \forall \Psi = (\psi_1, \psi_2) \in X, \quad (3.1)$$

then $W \geq 0$.

**Proof.** Let $\Psi = W^- = (u^-, v^-)$ in (3.1), by (2.4) and (2.5), we obtain

$$\int_0^\pi (|u^-|^2 + |v^-|^2) \leq \int_0^{\pi} (AW^-, W^-) - \int_0^{\pi} (AW^+, W^-)$$

$$\leq \mu \int_0^{\pi} |W^-|^2 \leq \mu \|W^-\|_X^2.$$ 

Therefore, $W^- = 0$, i.e. $W \geq 0$. □

**Remark 3.3.** In the classical sense, (2.4) and (2.5) are also sufficient conditions for having a maximum principle for the problem

$$-u'' = au + bv + g_1(x), \quad \text{in } (0, \pi),$$

$$v''' = cu + dv + g_2(x), \quad \text{in } (0, \pi),$$

$$u(0) = u(\pi) = 0,$$
\[ v(0) = v(\pi) = v''(0) = v''(\pi) = 0. \]

This is, \( W = (u, v) \geq 0 \) if \( g_1 \geq 0, g_2 \geq 0. \)

**Lemma 3.4.** Assume condition (H4), i.e. \( (2.3), (2.4) \) and \( (2.5) \) hold. Then, for all \( t = (t_1, t_2) \in \mathbb{R}^2, \) system \( (1.5) \) has a subsolution \( W_t \) such that, if \( W^t \) is any supersolution we have

\[ W_t \leq W^t \quad \text{in} \ (0, \pi). \quad (3.2) \]

**Proof.** We consider the system

\[
\begin{align*}
-u'' &= au + bv - c + t_1 \sin x + h_1, \quad \text{in} \ (0, \pi), \\
v''' &= cu + dv - c + t_2 \sin x + h_2, \quad \text{in} \ (0, \pi), \\
u(0) &= u(\pi) = 0, \\
v(0) &= v(\pi) = v''(0) = v''(\pi) = 0,
\end{align*}
\]

where \( c \) is the constant in \( (2.3) \) and \( (2.6). \) From the hypotheses on \( A \) and \( h_1, h_2, \) \( (3.3) \) has a unique solution \( W_t \in C^2 \times C^2. \) Then, using \( (2.3) \) we conclude that \( W_t \) is in fact a subsolution of \( (1.5). \)

Finally, suppose that \( W^t \) is any supersolution of \( (1.5), \) from \( (2.3) \) and applying Lemma 3.2 directly we can get the assertion \( (3.2). \) \( \Box \)

**Lemma 3.5.** Suppose (H1) holds and \( (h_1, h_2) \in C[0, \pi] \times C[0, \pi]. \) Then there exists \( t^0 \in \mathbb{R}^2 \) such that, for all \( t \leq t^0, \) system \( (1.5) \) has a supersolution \( W^t. \)

**Proof.** Let \( \pi, \bar{\pi} \) be the solution of the system

\[
\begin{align*}
-\bar{\pi}'' &= f_1(x, 0, 0) + h_1(x), \quad \text{in} \ (0, \pi), \\
\bar{\pi}''' &= f_2(x, 0, 0) + h_2(x), \quad \text{in} \ (0, \pi), \\
u(0) &= u(\pi) = 0, \\
v(0) &= v(\pi) = v''(0) = v''(\pi) = 0.
\end{align*}
\]

Due to the locally Lipschitzian condition on \( f_1, f_2, \) it is possible to choose \( t^0 = (t_1^0, t_2^0) < 0 \) such that

\[
\begin{align*}
f_1(x, \pi, \bar{\pi}) - f_1(x, 0, 0) + t^0 \sin x &\leq 0, \\
f_2(x, \pi, \bar{\pi}) - f_2(x, 0, 0) + t^0 \sin x &\leq 0.
\end{align*}
\]

Hence, from these inequalities and the system \( (3.4), \) for all \( t \leq t^0, \) \( W^{t^0} = (\pi, \bar{\pi}) \) is a supersolution for \( (1.5). \) \( \Box \)

**Lemma 3.6.** Let (H4), (H5) hold. Then for a given \( h_1, h_2, \) there exists an unbounded domain \( \mathbb{R} \) in the plane such that if \( t \in \mathbb{R}, \) system \( (1.5) \) has no supersolution.

**Proof.** Suppose \( W = (u, v) \) is a supersolution for \( (1.5). \) Multiplying both equations of this system by \( \sin x, \) integration them by parts and using \( (2.3), (2.6) \) we deduce that

\[
\begin{align*}
(A - I) \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} &\leq \pi \begin{pmatrix} -s_1 \\ -s_2 \end{pmatrix}, \\
(A - I) \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} &\leq \pi \begin{pmatrix} -s_1 \\ -s_2 \end{pmatrix}. \quad (3.5)
\end{align*}
\]
Where $\rho_1 = \int_0^\pi u \sin x, \rho_2 = \int_0^\pi v \sin x, s_1 = t_1 - c, s_2 = t_2 - c$ and $c$ is the constant in (2.3) and (2.6). From remark 2.1 applying $(A - I)^{-1}$ and $(\overline{A} - I)^{-1}$ to (3.5) and (3.6), respectively, we obtain that

1. If $\rho_1 \leq 0$, then $s_2 \leq \frac{d-1}{b} s_1$ when $b \neq 0$, or $s_1 \leq 0$ when $b = 0$.

2. If $\rho_1 \geq 0$, then $s_2 \leq \frac{d-1}{b} s_1$ when $b \neq 0$, or $s_1 \leq 0$ when $b = 0$.

Therefore, independently of the sign of $\rho_1$, the pair $(s_1, s_2)$ is in a region composed of the union of two half-planes passing through the origin, each of them bounded above by a straight-line of negative or infinity slope. $\mathcal{R}$ is the complement of this region in the original variables $t_1$ and $t_2$. \hfill \square

Now, we are at a position to prove the Ambrosetti-Prodi type result for system (1.5).

**Theorem 3.7.** Suppose that conditions (H1)–(H5) are satisfied and that there exists a matrix

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

with $b(x), c(x) \geq 0$ (cooperativeness condition on $A(x)$) satisfies (2.5) such that

$$F(x, \xi) - F(x, \eta) \geq A(x)(\xi - \eta), \quad \text{for } \xi, \eta \in \mathbb{R}^2, \xi \geq \eta. \quad (3.7)$$

Then there exists a continuous curve $\Gamma$ splitting $\mathbb{R}^2$ into two unbounded components $N$ and $E$ such that:

1. for each $t = (t_1, t_2) \in N$, (1.5) has no solution;
2. for each $t = (t_1, t_2) \in E$, (1.5) has at least two solutions.

**Proof.** For each $\theta \in \mathbb{R}$, define

$$L_\theta = \{(t_1, t_2) \in \mathbb{R}^2; t_2 + \theta = t_1\},$$

and $R(\theta) = \{t_2 \in \mathbb{R}; (t_1, t_2) \in (1.5)\}$ has a supersolution with $t \in L_\theta$ for some $t_2 \in \mathbb{R}$.

Lemmas 3.5 and 3.6 allows us to define the continuous curve

$$\Gamma(\theta) = (\sup R(\theta), \sup R(\theta) - \theta),$$

which splits the plane into two disjoint unbounded domains $N$ and $E$ such that

- for all $t \in N$ no supersolution exists for (1.5), while for all $t \in E$ (1.5) has a supersolution.

Obviously, for all $t \in N$, no solution exists for (1.5), result (1) is proved.

To prove result (2), now we use the abstract variational theorems to find the solutions of (1.5) when $t \in E$. We write

$$\langle J'(W), \Psi \rangle = \langle W, \Psi \rangle - \int_0^\pi \left[(f_1(x, u, v) + t_1 \sin x + h_1)\psi_1 + (f_2(x, u, v) + t_2 \sin x + h_2)\psi_2\right].$$

Given $t \in E$ there exists a supersolution $W^t = (u^t, v^t)$ and a subsolution $W_t = (u_t, v_t)$ of (1.5) such that $W_t \leq W^t$ in $(0, \pi)$. Let

$$M = [W_t, W^t] = \{W \in X; W_t \leq W \leq W^t\},$$

since $W_t, W^t \in L^\infty$ by assumption, also $M \subset L^\infty$ and $H(x, W(x)) + (t_1 \sin x + h_1) + (t_2 \sin x + h_2)v \leq c$ for all $W \in M$ and almost every $x \in (0, \pi)$.

Clearly, $M$ is a closed and convex subset of $X$, hence weakly closed. Since $M$ is essentially bounded, $J(W) \geq \frac{1}{2}\|W\|_X^2 - c$ is coercive on $M$. On the other hand, if
$W_n \to W$ weakly in $X$, where $W_n, W \in M$, we may assume that $W_n \to W$ pointwise almost everywhere; moreover, $|H(x, W_n) + (t_1 \sin x + h_1)u_n + (t_2 \sin x + h_2)v_n| \leq c$ uniformly, using Lebesgue Dominated Convergence Theorem, we have

$$\int_0^\pi H(x, W_n) + \int_0^\pi [(t_1 \sin x + h_1)u_n + (t_2 \sin x + h_2)v_n] \\
\to \int_0^\pi H(x, W) + \int_0^\pi [(t_1 \sin x + h_1)u + (t_2 \sin x + h_2)v].$$

Hence $J$ is weakly lower semi-continuous on $M$. Then we can use \cite{17} Theorem 1.2 to find a vector function $W_0 = (u_0, v_0) \in X$ such that $W_0 \in M$ is the infimum of the functional $J$ restricted to $M$.

To see that $W_0$ is a weak solution of \eqref{1.5}, for $\varphi = (\varphi_1, \varphi_2) \in C_0^\infty(0, \pi)$ and $\varepsilon > 0$ let

$$u_\varepsilon = \min\{u^t, \max\{u_t, u_0 + \varepsilon \varphi_1\}\} = u_0 + \varepsilon \varphi_1 - \varphi_1^\varepsilon + \varphi_1 \varepsilon$$
$$v_\varepsilon = \min\{v^t, \max\{v_t, v_0 + \varepsilon \varphi_2\}\} = v_0 + \varepsilon \varphi_2 - \varphi_2^\varepsilon + \varphi_2 \varepsilon$$

with

$$\varphi_1^\varepsilon = \max\{0, u_0 + \varepsilon \varphi_1 - u^t\} \geq 0,$$
$$\varphi_2^\varepsilon = \max\{0, v_0 + \varepsilon \varphi_2 - v^t\} \geq 0,$$

and

$$\varphi_1 \varepsilon = -\min\{0, u_0 + \varepsilon \varphi_1 - u_t\} \geq 0,$$
$$\varphi_2 \varepsilon = -\min\{0, v_0 + \varepsilon \varphi_2 - v_t\} \geq 0.$$

Note that $W_\varepsilon = (u_\varepsilon, v_\varepsilon) \in M$ and $\varphi^\varepsilon = (\varphi_1^\varepsilon, \varphi_2^\varepsilon), \varphi_\varepsilon = (\varphi_1 \varepsilon, \varphi_2 \varepsilon) \in X \cap L^\infty(0, \pi)$.

The functional $J$ is differentiable in direction $W_\varepsilon - W_0$. Since $W_0$ minimizes $J$ in $M$ we have

$$0 \leq \langle W_\varepsilon - W_0, J'(W_0) \rangle = \varepsilon \langle \varphi, J'(W_0) \rangle - \langle \varphi^\varepsilon, J'(W_0) \rangle + \langle \varphi_\varepsilon, J'(W_0) \rangle,$$

so that

$$\langle \varphi, J'(W_0) \rangle \geq \frac{1}{\varepsilon} \left[\langle \varphi^\varepsilon, J'(W_0) \rangle - \langle \varphi_\varepsilon, J'(W_0) \rangle\right].$$

Now, from $W_\varepsilon$ is a supersolution to \eqref{1.5}, we get

$$\langle \varphi^\varepsilon, J'(W_0) \rangle = \langle \varphi^\varepsilon, J'(W_\varepsilon) \rangle + \langle \varphi^\varepsilon, J'(W_0) - J'(W_\varepsilon) \rangle$$
$$\geq \langle \varphi^\varepsilon, J'(W_0) \rangle - J'(W_\varepsilon)$$
$$= \int_\Omega [(u_0 - u^t)'(u_0 + \varepsilon \varphi_1 - u^t)' + (v_0 - v^t)''(v_0 + \varepsilon \varphi_2 - v^t)']$$
$$- \int_\Omega [f_1(x, W_0) - f_1(x, W_\varepsilon)](u_0 + \varepsilon \varphi_1 - u^t)$$
$$- \int_\Omega [f_2(x, W_0) - f_2(x, W_\varepsilon)](v_0 + \varepsilon \varphi_2 - v^t)$$
$$\geq \varepsilon \int_\Omega [(u_0 - u^t)' \varphi_1' + (v_0 - v^t)'' \varphi_2'']$$
$$- \varepsilon \int_\Omega |f_1(x, W_0) - f_1(x, W_\varepsilon)||\varphi_1| - \varepsilon \int_\Omega |f_2(x, W_0) - f_2(x, W_\varepsilon)||\varphi_2|$$
where $\Omega = \{x \in (0, \pi); W_0(x) + \varepsilon \varphi(x) \geq W^t(x) > W_0(x)\}$. Note that $\text{meas}(\Omega) \to 0$ as $\varepsilon \to 0$. Hence by absolute continuity of the Lebesgue integral we obtain that
\[
\langle \varphi^\varepsilon, J'(W_0) \rangle \geq o(\varepsilon)
\]
where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. Similarly, we conclude that $\langle \varphi, J'(W_0) \rangle \leq o(\varepsilon)$; thus
\[
\langle \varphi, J'(W_0) \rangle \geq 0
\]
for all $\varphi \in C_0^\infty(0, \pi)$. Reversing the sign of $\varphi$ and since $C_0^\infty(0, \pi)$ is dense in $X$ we finally get that $J'(W_0) = 0$, i.e. $W_0$ is a weak solution to (1.5). Then using (3.7) and a Maximum Principle Lemma 3.2 we claim that $W_0$ is a local minimum of $J$.

Suppose by contradiction that $W_0$ is not a local minimum, then for every $\varepsilon > 0$ there is $\tilde{W}_\varepsilon \in B_\varepsilon(W_0)$ (a ball of radius $\varepsilon$ around $W_0 \in X$) such that $J(\tilde{W}_\varepsilon) < J(W_0)$. We know that $B_\varepsilon(W_0)$ is weaker sequentially compact in $X$ and $J$ is weakly lower semi-continuous, therefore there is $\tilde{W}_\varepsilon \in B_\varepsilon(W_0)$ such that
\[
J(\tilde{W}_\varepsilon) = \inf_{B_\varepsilon(W_0)} J \leq J(\tilde{W}_\varepsilon) < J(W_0),
\]
and $\langle J'(\tilde{W}_\varepsilon), \tilde{W}_\varepsilon - W_0 \rangle \leq 0$, or
\[
J'(\tilde{W}_\varepsilon) = \lambda_\varepsilon(\tilde{W}_\varepsilon - W_0) \quad \text{with } \lambda_\varepsilon \leq 0,
\]
namely
\[
\int_0^\pi (\tilde{u}_\varepsilon \psi_1 + \tilde{v}_\varepsilon \psi_2') - \int_0^\pi [ f_1(x, \tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \psi_1 + f_2(x, \tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \psi_2 ]
\]
\[
- \int_0^\pi [(t_1 \sin x + h_1) \psi_1 + (t_2 \sin x + h_2) \psi_2] = \lambda_\varepsilon[(\tilde{u}_\varepsilon - u_0) \psi_1 + (\tilde{v}_\varepsilon - v_0) \psi_2].
\]

On the other hand, from Definition 3.1 we have
\[
\int_0^\pi (u_\varepsilon \psi_1 + v_\varepsilon \psi_2') - \int_0^\pi [ f_1(x, u_\varepsilon, v_\varepsilon) \psi_1 + f_2(x, u_\varepsilon, v_\varepsilon) \psi_2 ]
\]
\[
- \int_0^\pi [(t_1 \sin x + h_1) \psi_1 + (t_2 \sin x + h_2) \psi_2] \leq 0,
\]
and
\[
\int_0^\pi (u_\varepsilon \psi_1 + v_\varepsilon \psi_2') - \int_0^\pi [ f_1(x, u_\varepsilon, v_\varepsilon) \psi_1 + f_2(x, u_\varepsilon, v_\varepsilon) \psi_2 ]
\]
\[
- \int_0^\pi [(t_1 \sin x + h_1) \psi_1 + (t_2 \sin x + h_2) \psi_2] \geq 0.
\]

From (3.8)–(3.9), we obtain
\[
\int_0^\pi [(\tilde{u}_\varepsilon - u_\varepsilon) \psi_1 + (\tilde{v}_\varepsilon - v_\varepsilon) \psi_2']
\]
\[
- \int_0^\pi [(f_1(x, \tilde{W}_\varepsilon) - f_1(x, W_\varepsilon)) \psi_1 + (f_2(x, \tilde{W}_\varepsilon) - f_2(x, W_\varepsilon)) \psi_2]
\]
\[
\geq \lambda_\varepsilon[(\tilde{u}_\varepsilon - u_\varepsilon + u_\varepsilon - u_0) \psi_1 + (\tilde{v}_\varepsilon - v_\varepsilon + v_\varepsilon - v_0) \psi_2].
\]

This implies
\[
-(\tilde{u}_\varepsilon - u_\varepsilon)'' \geq f_1(x, \tilde{W}_\varepsilon) - f_1(x, W_\varepsilon) + \lambda_\varepsilon(\tilde{u}_\varepsilon - u_\varepsilon) + \lambda_\varepsilon(u_\varepsilon - u_0),
\]
\[
(\tilde{v}_\varepsilon - v_\varepsilon)' \geq f_2(x, \tilde{W}_\varepsilon) - f_2(x, W_\varepsilon) + \lambda_\varepsilon(\tilde{v}_\varepsilon - v_\varepsilon) + \lambda_\varepsilon(v_\varepsilon - v_0).
\]
Then from (3.7) we obtain
\[
\left( -\left( \hat{u}_\varepsilon - u_t \right)' \right)'' \geq A(x) \left( \hat{W}_\varepsilon - W_t \right) + \lambda \varepsilon \left( \hat{W}_\varepsilon - W_t \right),
\]

note that \(\lambda \varepsilon \leq 0\), and by using Lemma 3.2 we obtain
\[
\hat{W}_\varepsilon - W_t \geq 0, \quad \text{or} \quad W_t \leq \hat{W}_\varepsilon.
\]

Similarly, from (3.10)–(3.8), we can obtain
\[
\hat{W}_\varepsilon \leq W_t.
\]

Which contradicts \(J(W_0) = \inf_M J(W)\).

Finally, since \(J\) is not bounded from below, a weaker form of the Mountain Pass Theorem can be used to find another solution \(W_1 \neq W_0\) of (1.5). Then result (2) is proved. \(\square\)

4. Example: A piecewise linear problem

Consider the system
\[
\begin{align*}
-u'' &= k_1 u^+ + \epsilon v^+ + t_1 \sin x + h_1(x), \quad \text{in } (0, \pi), \\
v^{(4)} &= \epsilon u^+ + k_2 v^+ + t_2 \sin x + h_2(x), \quad \text{in } (0, \pi), \\
\epsilon u(0) = u(\pi) &= 0, \\
v(0) = v(\pi) = v''(0) = v''(\pi) &= 0.
\end{align*}
\]

Where \(\epsilon\) and \(k_1, k_2\) are constants, \(t_1, t_2\) are parameters and \(h_1, h_2 \in C[0, \pi]\) are fixed functions with \(\int_0^\pi h_1 \sin x = \int_0^\pi h_2 \sin x = 0\). This problem is similar to system (1.4).

**Theorem 4.1.** Suppose that \(k_1 > 1, k_2 > 1\) and \(\epsilon \geq 0\). Then there exists a curve \(\Gamma\) splitting \(\mathbb{R}^2\) into two unbounded components \(N\) and \(E\) such that:

1. for each \(t = (t_1, t_2) \in N\), (4.1) has no solution;
2. for each \(t = (t_1, t_2) \in E\), (4.1) has at least two solutions.

**Proof.** Let
\[
\bar{A} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then we can easily verify that the conditions of Theorem 3.7 hold and therefore the results are follow. \(\square\)

**Remark 4.2.** (1) Denote by \(\mu_i (i = 1, 2)\) the eigenvalues of matrix
\[
A = \begin{pmatrix} k_1 & \epsilon \\ \epsilon & k_2 \end{pmatrix}
\]

and let \(\mu_1 \leq \mu_2\). It can be shown that \(\mu_2 > 1\) since \(k_1 > 1\) and \(k_2 > 1\).

(2) This result gives a partial answer to Question 1 and Question 2 that were posted in [16] and stated in Section 1.
References


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