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PARABOLIC EQUATIONS RELATIVE TO VECTOR FIELDS

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Abstract. We define two notions of viscosity solutions to parabolic equations defined using vector fields, depending on whether the test functions concern only the past or both the past and the future. Using the parabolic maximum principle for vector fields, we then prove a comparison principle for a class of parabolic equations and show the sufficiency of considering the test functions that concern only the past.

1. Vector Fields

In [1], a maximum principle for vector fields is proved and consequently, a comparison principle for subelliptic equations is established. Using this point of view, we prove an analogous comparison principle for a class of parabolic equations in vector fields. These results are a generalization of the Euclidean results of Juutinen [4].

To create our vector field environment, we replace the Euclidean vector fields $\{\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}\}$ in $\mathbb{R}^n$ with an arbitrary collection of vector fields or *frame*

$$\mathfrak{X} = \{X_1, X_2, \dots, X_n\}$$

consisting of n linearly independent smooth vector fields with the relation

$$X_i(x) = \sum_{j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_j}$$

for some choice of smooth functions $a_{ij}(x)$. Denote by $\mathbb{A}(x)$ the matrix whose (i,j)-entry is $a_{ij}(x)$. We always assume that $\det(\mathbb{A}(x)) \neq 0$ in $\mathbb{R}^n$. We note that if $\mathbb{A}$ is the identity matrix, we recover the Euclidean environment, which was considered in [4]. We choose a Riemannian metric, denoted $\langle\cdot,\cdot\rangle$, and related norm $\|\cdot\|$ so that the frame is orthonormal. The natural gradient is the vector

$$\mathfrak{X}u = (X_1(u), X_2(u), \dots, X_n(u))$$

and the natural second derivative is the $n \times n$ *not necessarily symmetric* matrix $\mathfrak{X}^2 u$ with entries $X_i(X_j(u))$. Because of the lack of symmetry, we introduce the symmetrized second-order derivative matrix with respect to this frame, given by

$$(\mathfrak{X}^2 u)^\star = \frac{1}{2}(\mathfrak{X}^2 u + (\mathfrak{X}^2 u)^t).$$

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Fix a point $x \in \mathbb{R}^n$ and let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ denote a vector close to zero. We define the exponential based at x of ξ, denoted by $\Theta_x(\xi)$, as follows: Let γ be the unique solution to the system of ordinary differential equations

$$\gamma'(s) = \sum_{i=1}^{n} \xi_i X_i(\gamma(s))$$

satisfying the initial condition $\gamma(0) = x$. We set $\Theta_x(\xi) = \gamma(1)$ and note this is defined in a neighborhood of zero.

2. Parabolic Jets and Viscosity Solutions to Parabolic Equations

In this section, we define and compare various notions of solutions to parabolic equations. We begin by letting $u(x,t)$ be a function in $\mathbb{R}^n \times [0,T]$ for some $T > 0$ and by denoting the set of $n \times n$ symmetric matrices by S^n. We consider parabolic equations of the form

$$u_t + F(t, x, u, \mathfrak{X}u, (\mathfrak{X}^2 u)^\star) = 0 \tag{2.1}$$

for continuous and proper $F : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$. Recall that F is proper means

$$F(t, x, r, \eta, X) \le F(t, x, s, \eta, Y)$$

when $r \le s$ and $Y \le X$ in the usual ordering of symmetric matrices [3]. We note that the derivatives $\mathfrak{X}u$ and $(\mathfrak{X}^2 u)^\star$ are taken in the space variable x. Examples of parabolic equations include the parabolic infinite Laplace equation

$$u_t + \Delta_\infty u = u_t - \langle (\mathfrak{X}^2 u)^\star \mathfrak{X}u, \mathfrak{X}u \rangle = 0$$

and the parabolic p-Laplace equation for $2 \le p < \infty$ given by

$$u_t + \Delta_p u = u_t - \operatorname{div}_{\mathfrak{X}}(\|\mathfrak{X}u\|^{p-2}\mathfrak{X}u) = u_t - \sum_{i=1}^{n} X_i(\|\mathfrak{X}u\|^{p-2} X_i u) = 0$$

where we observe that for a smooth function f,

$$\operatorname{div}_{\mathfrak{X}} f = \sum_{i=1}^{n} X_i f.$$

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set containing the point x_0. We define the set $\mathcal{O}_T \equiv \mathcal{O} \times (0,T)$. Following the definition of jets in [1], we can define the parabolic jets of $u(x,t)$ at the point $(x_0,t_0) \in \mathcal{O}_T$ by using the appropriate test functions. Namely, we consider the set $\mathcal{A}u(x_0,t_0)$ by

$$\mathcal{A}u(x_0,t_0) = \{\phi \in C^2(\mathcal{O}_T) : u(x,t) - \phi(x,t) \le u(x_0,t_0) - \phi(x_0,t_0)\}$$

consisting of all test functions that touch from above. We define the set of all test functions that touch from below, denoted $\mathcal{B}u(x_0,t_0)$, by

$$\mathcal{B}u(x_0,t_0) = \{\phi \in C^2(\mathcal{O}_T) : u(x,t) - \phi(x,t) \ge u(x_0,t_0) - \phi(x_0,t_0)\}.$$

We then have

$$P^{2,+}u(x_0,t_0) = \{(\phi_t(x_0,t_0), \mathfrak{X}\phi(x_0,t_0), (\mathfrak{X}^2\phi(x_0,t_0))^\star) : \phi \in \mathcal{A}u(x_0,t_0)\},$$
$$P^{2,-}u(x_0,t_0) = \{(\phi_t(x_0,t_0), \mathfrak{X}\phi(x_0,t_0), (\mathfrak{X}^2\phi(x_0,t_0))^\star) : \phi \in \mathcal{B}u(x_0,t_0)\}.$$

We call $P^{2,+}u(x_0,t_0)$ the parabolic superjet of u at (x_0,t_0) and $P^{2,-}u(x_0,t_0)$ the parabolic subjet of u at (x_0,t_0).

Extending [1, Lemma 5], we have the following lemma.

Lemma 2.1. *Let $\Theta_{x_0}(\xi)$ be the exponential map based on the point x_0 and let ξ be an n-dimensional vector. Then $P^{2,+}u(x_0,t_0) = (a,\eta,X) \in \mathbb{R}\times\mathbb{R}^n\times S^n$ such that*

$$u(\Theta_{x_0}(\xi),t) \le u(x_0,t_0) + a(t-t_0) + \langle \eta,\xi\rangle + \frac{1}{2}\langle X\xi,\xi\rangle + o(\|\xi\|^2)$$

as $\xi \to 0$. Additionally,

$$P^{2,-}u(x_0,t_0) = -P^{2,+}(-u)(x_0,t_0)$$

or, alternatively, $P^{2,-}u(x_0,t_0) = (b,\nu,Y) \in \mathbb{R}\times\mathbb{R}^n\times S^n$ such that

$$u(\Theta_{x_0}(\xi),t) \ge u(x_0,t_0) + b(t-t_0) + \langle \nu,\xi\rangle + \frac{1}{2}\langle Y\xi,\xi\rangle + o(\|\xi\|^2)$$

as $\xi \to 0$.

We also define the set theoretic closure of the superjet, denoted $\overline{P}^{2,+}u(x_0,t_0)$, by requiring that (a,η,X) be in $\overline{P}^{2,+}u(x_0,t_0)$ exactly when there is a sequence $(a_n,x_n,t_n,u(x_n,t_n),\eta_n,X_n) \to (a,x_0,t_0,u(x_0,t_0),\eta,X)$ with the triple (a_n,η_n,X_n) in $P^{2,+}u(x_n,t_n)$. A similar definition holds for the closure of the subjet.

We next recall the relationship between these jets and the usual Euclidean jets, given by the following Lemma.

Lemma 2.2 ([1, Lemma 3])**.** *For smooth functions u we have*

$$\mathfrak{X}u(x) = \mathbb{A}(x)\cdot\nabla u(x),$$

and for all $s \in \mathbb{R}^n$

$$\langle \left(\mathfrak{X}^2 u(x)\right)^{*}\cdot s, s\rangle = \langle \mathbb{A}(x)\cdot D^2u(x)\cdot\mathbb{A}^t(x)\cdot s, s\rangle + \sum_{k=1}^{n}\langle \mathbb{A}^t(x)\cdot s, D\left(\mathbb{A}^t(x)\cdot s\right)_k\rangle \frac{\partial u}{\partial x_k}(x).$$

Here ∇u is the usual Euclidean gradient of u, D^2u is the Euclidean second-order derivative matrix of u and D signifies Euclidean differentiation.

We then use these jets to define subsolutions and supersolutions to Equation (2.1).

Definition 2.3. Let $(x_0,t_0) \in \mathcal{O}_T$ be as above. The upper semicontinuous function u is a *viscosity subsolution* in $\mathcal{O}_T$ if for all $(x_0,t_0) \in \mathcal{O}_T$ we have $(a,\eta,X) \in P^{2,+}u(x_0,t_0)$ produces

$$a + F(t_0,x_0,u(x_0,t_0),\eta,X) \le 0. \tag{2.2}$$

A lower semicontinuous function u is a *viscosity supersolution* in $\mathcal{O}_T$ if for all $(x_0,t_0) \in \mathcal{O}_T$ we have $(b,\nu,Y) \in P^{2,-}u(x_0,t_0)$ produces

$$b + F(t_0,x_0,u(x_0,t_0),\nu,Y) \ge 0. \tag{2.3}$$

A continuous function u is a *viscosity solution* in $\mathcal{O}_T$ if it is both a viscosity subsolution and viscosity supersolution.

We observe that the continuity of the function F allows Equations (2.2) and (2.3) to hold when $(a,\eta,X) \in \overline{P}^{2,+}u(x_0,t_0)$ and $(b,\nu,Y) \in \overline{P}^{2,-}u(x_0,t_0)$, respectively.

We also wish to define what [4] refers to as parabolic viscosity solutions. We first need to consider the sets

$$\mathcal{A}^- u(x_0,t_0) = \{\phi \in C^2(\mathcal{O}_T) : u(x,t) - \phi(x,t) \le u(x_0,t_0) - \phi(x_0,t_0) \text{ for } t < t_0\}$$

consisting of all functions that touch from above only when $t < t_0$ and the set

$$\mathcal{B}^- u(x_0,t_0) = \{\phi \in C^2(\mathcal{O}_T) : u(x,t) - \phi(x,t) \geq u(x_0,t_0) - \phi(x_0,t_0) \text{ for } t < t_0\}$$

consisting of all functions that touch from below only when $t < t_0$. Note that $\mathcal{A}^- u$ is larger than $\mathcal{A}u$ and $\mathcal{B}^- u$ is larger than $\mathcal{B}u$. These larger sets correspond physically to the past alone playing a role in determining the present.

We then have the following definition.

Definition 2.4. An upper semicontinuous function u on $\mathcal{O}_T$ is a *parabolic viscosity subsolution* in $\mathcal{O}_T$ if $\phi \in \mathcal{A}^- u(x_0,t_0)$ produces

$$\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),\mathfrak{X}\phi(x_0,t_0),(\mathfrak{X}^2\phi(x_0,t_0))^\star) \leq 0.$$

A lower semicontinuous function u on $\mathcal{O}_T$ is a *parabolic viscosity supersolution* in $\mathcal{O}_T$ if $\phi \in \mathcal{B}^- u(x_0,t_0)$ produces

$$\phi_t(x_0,t_0) + F(t_0,x_0,u(x_0,t_0),\mathfrak{X}\phi(x_0,t_0),(\mathfrak{X}^2\phi(x_0,t_0))^\star) \geq 0.$$

A continuous function is a *parabolic viscosity solution* if it is both a parabolic viscosity supersolution and subsolution.

It is easy to see that parabolic viscosity sub (super-) solutions are viscosity sub (super-) solutions. The reverse implication will be a consequence of the comparison principle proved in the next section.

3. Comparison Principle

In order to prove our comparison principle, we will need a parabolic maximum principle in vector fields, analogous to the maximum principle for subelliptic equations in [1]. The theorem we will prove is based on [3, Thm. 8.2], which details the Euclidean case. We will denote the Euclidean distance between the points x and y by $|x-y|$.

Theorem 3.1. *Let u be a viscosity subsolution to Equation* (2.1) *and v be a viscosity supersolution to Equation* (2.1) *in the bounded set $\Omega \times (0,T)$ where Ω is a bounded domain. Let τ be a positive real parameter and let $\psi(x,y) = |x-y|^\alpha$ for $\alpha > 2$ and $x,y \in \Omega$. Suppose the local maximum of*

$$M_\tau(x,y,t) \equiv u(x,t) - v(y,t) - \tau\psi(x,y)$$

occurs at the interior point (x_τ,y_τ,t_τ) of the set $\Omega\times\Omega\times(0,T)$. Then, for each $\tau > 0$, there are elements $(a,\Upsilon_\tau^+,\mathcal{X}^\tau) \in \overline{P}^{2,+}u(x_\tau,t_\tau)$ and $(a,\Upsilon_\tau^-,\mathcal{Y}^\tau) \in \overline{P}^{2,-}v(y_\tau,t_\tau)$ so that if

$$\lim_{\tau\to\infty} \tau\psi(x_\tau,y_\tau) = 0$$

then

$$\Upsilon_\tau^+ - \Upsilon_\tau^- = o(1), \tag{3.1}$$

$$\mathcal{X}^\tau - \mathcal{Y}^\tau \leq o(1) \tag{3.2}$$

as $\tau \to \infty$.

Proof. We first need to check that [3, condition 8.5] is satisfied, namely that there exists an $r > 0$ so that for each M, there exists a C so that $a \leq C$ when $(a,\eta,X) \in P^{2,+}_{\text{eucl}}u(x,t)$, $|x-x_\tau| + |t-t_\tau| < r$, and $|u(x,t)| + \|\eta\| + \|X\| \leq M$ with a similar statement holding for $-v$. If this condition is not met, then for each $r > 0$, we have an M so that for all C, $a > C$ when $(a, \eta, X) \in P^{2,+}_{\text{eucl}} u(x,t)$. By Lemma 2.2, we would have jet elements

$$(a, \mathbb{A}(x) \cdot \eta, \mathcal{X}) \in P^{2,+} u(x,t)$$

contradicting the fact that u is a subsolution. A similar conclusion is reached for $-v$ and so we conclude that this condition holds. The result follows by applying Theorem 8.3 of [3] and proceeding as in the proof of the maximum principle [1]. $\square$

Using this theorem, we now define a class of parabolic equations to which we shall prove a comparison principle.

Definition 3.2. We say the continuous, proper function

$$F : [0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$$

is *admissible* if for each $t \in [0,T]$, there is the same function $\omega : [0,\infty] \to [0,\infty]$ with $\omega(0+) = 0$ so that F satisfies

$$F(t,y,r,\nu,\mathcal{Y}) - F(t,x,r,\eta,\mathcal{X}) \le \omega\big(|x-y| + \|\nu - \eta\| + \|\mathcal{Y} - \mathcal{X}\|\big). \tag{3.3}$$

We now formulate the comparison principle for the problem.

$$\begin{gathered} u_t + F(t,x,u,\mathfrak{X}u,(\mathfrak{X}^2 u)^\star) = 0 \quad \text{in } (0,T) \times \Omega \quad \text{(E)} \\ u(x,t) = h(x,t) \quad x \in \partial\Omega,\ t \in [0,T) \quad \text{(BC)} \\ u(x,0) = \varphi(x) \quad x \in \overline{\Omega} \quad \text{(IC)} \end{gathered} \tag{3.4}$$

Here, $\varphi \in C(\overline{\Omega})$ and $h \in C(\overline{\Omega} \times [0,T))$. We also adopt the convention in [3] that a subsolution $u(x,t)$ to Problem (3.4) is a viscosity subsolution to (E), $u(x,t) \le h(x,t)$ on $\partial\Omega$ with $0 \le t < T$ and $u(x,0) \le \varphi(x)$ on $\overline{\Omega}$. Supersolutions and solutions are defined in an analogous matter.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let F be admissible. If u is a viscosity subsolution and v a viscosity supersolution to Problem (3.4) then $u \le v$ on $\Omega \times [0,T)$.*

Proof. Our proof follows [3, Theorem 8.2] and so we discuss only the main parts. For $\epsilon > 0$, we substitute $\tilde{u} = u - \frac{\varepsilon}{T-t}$ for u and prove the theorem for

$$u_t + F(t,x,u,\mathfrak{X}u,(\mathfrak{X}^2 u)^\star) \le -\frac{\varepsilon}{T^2} < 0 \tag{3.5}$$

$$\lim_{t \uparrow T} u(x,t) = -\infty \quad \text{uniformly on } \overline{\Omega} \tag{3.6}$$

and take limits to obtain the desired result. Assume the maximum occurs at $(x_0,t_0) \in \Omega \times (0,T)$ with

$$u(x_0,t_0) - v(x_0,t_0) = \delta > 0.$$

Let

$$M_\tau = u(x_\tau,t_\tau) - v(y_\tau,t_\tau) - \tau\psi(x_\tau,y_\tau)$$

with (x_τ,y_τ,t_τ) the maximum point in $\overline{\Omega} \times \overline{\Omega} \times [0,T)$ of $u(x,t) - v(y,t) - \tau\psi(x,y)$. Using the same proof as in [1, Theorem 1] we conclude that

$$\lim_{\tau \to \infty} \tau\psi(x_\tau,y_\tau) = 0.$$

If $t_\tau = 0$, we have

$$0 < \delta \le M_\tau \le \sup_{\overline{\Omega} \times \overline{\Omega}} (\varphi(x) - \varphi(y) - \tau\psi(x,y))$$

leading to a contradiction for large τ. We therefore conclude $t_\tau > 0$ for large τ. Since $u \le v$ on $\partial\Omega \times [0,T)$ by Equation (BC) of Problem (3.4), we conclude that for large τ, we have (x_τ, y_τ, t_τ) is an interior point. That is, $(x_\tau, y_\tau, t_\tau) \in \Omega\times\Omega\times(0,T)$. Using Lemma 3.1, we obtain

$$(a, \Upsilon_\tau^+, \mathcal{X}^\tau) \in \overline{P}^{2,+} u(x_\tau, t_\tau)$$
$$(a, \Upsilon_\tau^-, \mathcal{Y}^\tau) \in \overline{P}^{2,-} v(y_\tau, t_\tau)$$

satisfying the equations

$$a + F(t_\tau, x_\tau, u(x_\tau, t_\tau), \Upsilon_\tau^+, \mathcal{X}^\tau) \le -\frac{\varepsilon}{T^2}$$
$$a + F(t_\tau, y_\tau, v(y_\tau, t_\tau), \Upsilon_\tau^-, \mathcal{Y}^\tau) \ge 0.$$

Using the fact that F is proper, the fact that $u(x_\tau, t_\tau) \ge v(y_\tau, t_\tau)$ (otherwise $M_\tau < 0$), we have

$$\begin{aligned} 0 < \frac{\varepsilon}{T^2} &\le F(t_\tau, y_\tau, v(y_\tau, t_\tau), \Upsilon_\tau^-, \mathcal{Y}^\tau) - F(t_\tau, x_\tau, u(x_\tau, t_\tau), \Upsilon_\tau^+, \mathcal{X}^\tau) \\ &\le \omega(|x_\tau - y_\tau| + \|\Upsilon_\tau^- - \Upsilon_\tau^+\| + \|\mathcal{Y}^\tau - \mathcal{X}^\tau\|). \end{aligned}$$

We arrive at a contradiction as $\tau \to \infty$ by invoking Equations (3.1) and (3.2). □

We then have the following corollary, showing the equivalence of parabolic viscosity solutions and viscosity solutions.

Corollary 3.4. *For admissible F, we have the parabolic viscosity solutions are exactly the viscosity solutions.*

Proof. We showed above that parabolic viscosity sub(super-)solutions are viscosity sub(super-)solutions. To prove the converse, we will follow the proof of the subsolution case found in [4], highlighting the main details. Assume that u is not a parabolic viscosity subsolution. Let $\phi \in \mathcal{A}^- u(x_0, t_0)$ have the property that

$$\phi_t(x_0, t_0) + F(t_0, x_0, \phi(x_0, t_0), \mathfrak{X}\phi(x_0, t_0), (\mathfrak{X}^2\phi(x_0, t_0))^\star) \ge \epsilon > 0$$

for a small parameter ϵ. For $r > 0$ let $S_r = B_{x_0}(r) \times (t_0 - r, t_0)$ be the parabolic ball and let ∂S_r be its parabolic boundary. Here $B_{x_0}(r)$ is the Euclidean ball of radius r centered at x_0. Then the function

$$\tilde{\phi}_r(x, t) = \phi(x, t) + |t_0 - t|^8 - r^8 + |x - x_0|^8$$

is a classical supersolution for sufficiently small r. We then observe that $u \le \tilde{\phi}_r$ on ∂S_r but $u(x_0, t_0) > \tilde{\phi}(x_0, t_0)$. Thus, the comparison prinicple, Theorem 3.3, does not hold. Thus, u is not a viscosity subsolution. The supersolution case is identical and omitted. □

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