Electronic Journal of Differential Equations, Vol. 2008(2008), No. 133, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXPONENTIAL CONVERGENCE FOR BAM NEURAL NETWORKS WITH DISTRIBUTED DELAYS

YONGKUN LI, YAPING REN

Abstract. This paper concerns the exponential convergence of bidirectional associative memory (BAM) neural networks with unbounded distributed delays. Sufficient conditions are derived by exploiting the exponentially fading memory property of delay kernel functions. The method is based on comparison principle of delay differential equations and does not need the construction of any Lyapunov functions.

1. Introduction

The bidirectional associative memory (BAM) model, known as an extension of the unidirectional autoassociator of Hopfied [3], was first introduced by Kosko [4]. It has been used in many fields such as the pattern recognition and automatic control. Sufficient conditions have been obtained for the global asymptotic stability of delayed BAM networks; see the references in this article. Only a few results are available on the exponential stability of BAM networks with distributed delays. As well known, the exponential stability guarantees fast response in a system and therefore is a desirable performance in evaluating and designing BAM networks.

Mathematically, the effect of distributed time delays on the dynamics of BAM networks is often characterized through delay kernel functions. It is hence natural to think that certain conditions should be required on the nature of delay kernels in order to attain the exponential convergence in the system.

2000 *Mathematics Subject Classification.* 92B20, 34K20.
Key words and phrases. Bidirectional associative memory networks; delay kernels; exponential convergence.

Submitted July 14, 2008. Published September 26, 2008.
Supported by grants 10361006 from the National Natural Sciences Foundation of China, and 2003A0001M from the Natural Sciences Foundation of Yunnan Province.

The BAM networks with unbounded distributed delays under consideration are described by the following integro-differential equations

$$
\begin{aligned}
\dot{x}_i(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{m} A_{ji}(t) f_j(x(t-\tau_j)) \\
&\quad + \sum_{j=1}^{m} C_{ji}(t) \int_{-\infty}^{t} K_{ji}(t-s) g_j(x_j(s))\, ds + I_i(t), \quad i = 1, \dots, n, \\
\dot{y}_j(t) &= -b_j(t)y_j(t) + \sum_{i=1}^{n} B_{ij}(t) h_i(y(t-\tau_i)) \\
&\quad + \sum_{i=1}^{n} D_{ij}(t) \int_{-\infty}^{t} G_{ij}(t-s) l_i(x_i(s))\, ds + J_j(t), \quad j = 1, \dots, m,
\end{aligned}
\tag{1.1}
$$

where x_i and y_j are the activations of the ith neuron and the jth neuron ($i = 1, \dots, n, j = 1, \dots, m$); $a_i > 0$ and $b_j > 0$ are passive decay rates of neurons i and j; A_{ji}, C_{ji}, B_{ij} and D_{ij} are the connection weights; f_j, g_j, h_i and l_i are the activation functions of the neurons; $I_i(t)$, $J_j(t)$ denote the ith and jth component of internal input sources introduced from outside the networks to the cells i and j, respectively; K_{ji} and G_{ij} are the distributed delay kernels representing the past history effects on the neuron state dynamics. It is usually assumed that K_{ji} and G_{ij} are non-negative and continuous functions defined on $[0, +\infty)$ and satisfy the normalization conditions

$$
\int_0^\infty K_{ji}(s)\, ds = 1, \quad \int_0^\infty G_{ij}(s)\, ds = 1.
$$

We also assume that the delay kernels satisfy the conditions

$$
\int_0^\infty K_{ji}(s) e^{\sigma_0 s}\, ds < \infty, \quad \int_0^\infty G_{ij}(s) e^{\sigma_0 s}\, ds < \infty, \quad i = 1, \dots, n,\ j = 1, \dots, m,
\tag{1.2}
$$

for some scalar $\sigma_0 > 0$. That is, we assume that the neurons are of exponentially fading memory.

The initial conditions for system (1.1) are specified as continuous functions φ_{x_i}, φ_{y_j}: $(-\infty, 0] \to \mathbb{R}^n$, i.e., $x_i(s) = \varphi_{x_i}(s)$ and $y_j(s) = \varphi_{y_j}(s)$ for $s \le 0$. The existence and uniqueness of a solution to the initial value problem of system (1.1) can follow from the Lipshitz conditions on the activation functions:

$$
|f_j(a) - f_j(b)| \le \mu_{f_j}|a - b|, \quad |g_j(a) - g_j(b)| \le \mu_{g_j}|a - b|, \quad \forall a, b \in \mathbb{R}, \tag{1.3}
$$

$$
|h_i(a) - h_i(b)| \le \mu_{h_i}|a - b|, \quad |l_i(a) - l_i(b)| \le \mu_{l_i}|a - b|, \quad \forall a, b \in \mathbb{R}, \tag{1.4}
$$

where $\mu_{f_j} > 0$, $\mu_{g_j} > 0$, $\mu_{h_i} > 0$ and $\mu_{l_i} > 0$ are the Lipschitz constants, $i = 1, \dots, n$, $j = 1, \dots, m$.

Without loss of generality we may assume $f_j(0) = 0$, $g_j(0) = 0$, $h_i(0) = 0$ and $l_i(0) = 0$ for $i = 1, \dots, n$, $j = 1, \dots, m$. So the origin is a fixed point of system (1.1). The aim of this paper is to establish conditions for system (1.1) to converge to the origin in terms of

$$
|x_i(t)| \le \alpha_i e^{-\sigma t}, \quad |y_j(t)| \le \beta_j e^{-\sigma t}, \quad t \ge 0,\ i = 1, \dots, n, j = 1, \dots, m, \tag{1.5}
$$

whenever $|x_i(s)| \le \alpha_i$ and $|y_j(s)| \le \beta_j$, for $s \le 0$, where $\alpha_i > 0$ and $\beta_j > 0$, $\sigma > 0$ are real constants. (1.5) gives a componentwise exponential convergence estimate for the system (1.1). Clearly, σ provides an estimate of the exponential decay rate of the system, α_i and β_j give bounds on the states of the ith neuron and jth neuron.

It is not difficult to see that the componentwise exponential convergence property defined above is stronger than the conventional exponential stability in Lyapunov sense. Indeed, if the estimate (1.5) hold the origin of system (1.1) must be exponentially stable, but the converse is not true in general. An obvious advantage of this type of convergence is that it allows an individual monitoring of each neuron's state.

The main purpose of this paper is to find conditions which ensure the componentwise exponential convergence estimate. The organization of this paper is as follows. In Section 2, we introduce some lemmas cited from [1] which are useful in the proof of our main results of this paper. In Section 3, we use comparison principle to analysis the exponential convergence of BAM networks. In Section 4, we give an illustrative example of the effectiveness of the obtained results.

2. Preliminaries

Let $F(t,\varphi)$ be an n-vector-valued continuous functional with $t \geq 0$ and φ a continuous function from $(-\infty, 0]$ into $\mathbb{R}^n$, and F_i denotes the ith component of F.

$F(t,\varphi)$ is said to be quasi-monotone non-decreasing in φ if, for $i = 1,\dots,n$, $F_i(t,\varphi) \leq F_i(t,\psi)$ whenever $\varphi_i(0) = \psi_i(0)$ and $\varphi_i(s) \leq \psi_i(s)$ (in componentwise sense) for all $s \leq 0$.

For a continuous function $z(t)$ from $\mathbb{R}$ into $\mathbb{R}^n$, define the truncation z_t by $z_t(s) = z(t+s)$ for $s \leq 0$.

The following comparison principle is a direct extension of that cited in [1] to unbounded delay case.

Lemma 2.1. *Let $F(t,\varphi)$ be quasi-monotone non-decreasing in φ. If $p(t)$ and $q(t)$ are vector-valued continuous functions such that, for $i = 1,\dots,n$, $s \leq 0$, $t \geq 0$,*

- (i) $D^+p_i(t) \leq F_i(t,p_t)$,
- (ii) $\dot{q}_t(t) = F_i(t,q_t)$,
- (iii) $p_i(s) \leq q_i(s)$,

then $p_i(t) \leq q_i(t)$ for $t \geq 0$.

Following the same line of proof as in of [1, Lemma 2], we can derive from Lemma 2.1 the following comparison result.

Lemma 2.2. *Let $F(t,\varphi)$ be quasi-monotone non-decreasing in φ. If $e(t)$ and $q(t)$ are vector-valued continuous functions such that, for $i = 1,\dots,n$, $s \leq 0$, $t \geq 0$,*

- (i) $e_i(t) \geq F_i(t,e_t)$,
- (ii) $\dot{q}_t(t) = F_i(t,q_t)$,
- (iii) $q_i(s) \leq e_i(s)$,

then $q_i(t) \leq e_i(t)$ for $t \geq 0$.

3. Exponential convergence

In this section we discuss the componentwise exponential convergence of system (1.1). To get the estimate (1.5), we evaluate the upper right derivative $D^+|x_i(t)|$

for a solution $x_i(t)$ of system (1.1) to obtain

$$
\begin{aligned}
D^+|x_i(t)| &= \lim_{r\to 0^+}\sup\frac{1}{r}[|x_i(t+r)|-|x_i(t)|]\\
&\le -|a_i(t)||x_i(t)| + \sum_{j=1}^m |A_{ji}(t)||f_j(x(t-\tau_j))|\\
&\quad + \sum_{j=1}^m |C_{ji}(t)|\int_{-\infty}^t K_{ji}(t-s)|g_j(x_j(s))|\,ds + |I_i(t)|\\
&\le -|a_i(t)||x_i(t)| + \sum_{j=1}^m |A_{ji}(t)|\mu_{f_j}|x(t-\tau_j)|\\
&\quad + \sum_{j=1}^m \mu_{g_j}|C_{ji}(t)|\int_{-\infty}^t K_{ji}(t-s)|x_j(s)|\,ds + |I_i(t)|,
\end{aligned} \tag{3.1}
$$

for $i=1,\dots,n$.

In the context of this paper, we may take F in Lemma 2.1 with

$$
\begin{aligned}
F_i(0,\varphi) &= -|a_i(0)|\varphi_i(0) + \sum_{j=1}^m |A_{ji}(0)|\mu_{f_j}\varphi(-\tau_j)\\
&\quad + \sum_{j=1}^m \mu_{g_j}|C_{ji}(0)|\int_{-\infty}^0 K_{ji}(-s)\varphi_j(s)\,ds + |I_i(0)|,
\end{aligned}
$$

for $i=1,\dots,n$. It is clear that F is quasi-monotone non-decreasing in φ. So the right-hand side of the last inequality (3.1) is quasi-monotone non-decreasing in $|x_i(s)|$ for $s\in(-\infty,t]$ with $t\ge 0$. This implies that $|x_i(t)|$ can be dominated by the following comparison system:

$$
\begin{aligned}
\dot q_i(t) &= -|a_i(t)|q_i(t) + \sum_{j=1}^m |A_{ji}(t)|\mu_{f_j}q(t-\tau_j)\\
&\quad + \sum_{j=1}^m \mu_{g_j}|C_{ji}(t)|\int_{-\infty}^t K_{ji}(t-s)q_j(s)\,ds + |I_i(t)|,
\end{aligned} \tag{3.2}
$$

in the sense that $|x_i(t)|\le q_i(t)$ for $t\ge 0$ whenever $|x_i(s)|\le q_i(s)$ for $s\le 0$, $i=1,\dots,n$. The result is a special case of a general comparison principle for distributed delay systems, see Lemma 2.1. It enables us to derive properties of non-linear system (1.1) by examining a linear comparison system (3.2). However, it should be noted that even for a linear distributed delay system such as (3.2), there are not known general results which provide necessary and sufficient conditions for exponential convergence of the system. Therefore, in the sequel we should first proceed to find an appropriate exponential estimate for the comparison system (3.2), from which we can then yield the estimate (1.5) for system (1.1) by the above comparison principle. To do this, we make use of the following comparison result.

Suppose that there are n functions $e_i(t)$ such that

$$
\begin{aligned}
\dot{e}_i(t) \geq -|a_i(t)|e_i(t) &+ \sum_{j=1}^{m} |A_{ji}(t)|\mu_{f_j} e(t-\tau_j) \\
&+ \sum_{j=1}^{m} \mu_{g_j}|C_{ji}(t)| \int_{-\infty}^{t} K_{ji}(t-s)e_j(s)\, ds + |I_i(t)|,
\end{aligned}
\tag{3.3}
$$

for $i = 1, \dots, n$. Then $q_i(t) \leq e_i(t)$ for $t \geq 0$ provided $q_i(s) \leq e_i(s)$ for $s \leq 0$. Actually, this can be derived simply by using the above comparison principle along with the reverse transformations : $q_i \to -q_i$ and $e_i \to -e_i$ in (3.2) and (3.3), respectively. See Lemma 2.2.

From this we have $|x_i(t)| \leq e_i(t)$ for $t \geq 0$ provided $|x_i(s)| \leq e_i(s)$ for $s \leq 0$, $i = 1, \dots, n$. Now, taking $e_i(t) = \alpha_i e^{-\sigma t}$, to satisfy inequalities (3.3) it suffices to have

$$
\begin{aligned}
&(\sigma - |a_i(t)|)\alpha_i + \sum_{j=1}^{m} \alpha_j |A_{ji}(t)|\mu_{f_j} e^{\sigma\tau_j} \\
&+ \sum_{j=1}^{m} \alpha_j \mu_{g_j}|C_{ji}(t)| \int_{0}^{\infty} K_{ji}(s)e^{\sigma s}\, ds + |I_i(t)| \leq 0,
\end{aligned}
\tag{3.4}
$$

for $i = 1, \dots, n$. For the same reason we can get

$$
\begin{aligned}
&(\sigma - |b_j(t)|)\beta_j + \sum_{i=1}^{n} \beta_i |B_{ij}(t)|\mu_{h_i} e^{\sigma\tau_i} \\
&+ \sum_{i=1}^{n} \beta_i \mu_{l_i}|D_{ij}(t)| \int_{0}^{\infty} G_{ij}(s)e^{\sigma s}\, ds + |J_j(t)| \leq 0,
\end{aligned}
\tag{3.5}
$$

for $j = 1, \dots, m$. Those together with condition (1.2) lead to the following result.

Theorem 3.1. *System* (1.1) *admits the exponential convergent estimate* (1.5) *with* $0 < \sigma \leq \sigma_0$ *if conditions* (3.4) *and* (3.5) *hold.*

This result establishes an explicit relation on specific exponential convergent dynamics and system parameters including the weights, the gain of neurons, and the delay kernels.

Remark 3.2. It is obvious that criterion (3.4) depends only on the relative values of $\alpha_i (i = 1, \dots, n)$. Thus, if condition (3.4) is satisfied by a set of $\alpha_i > 0 (i = 1, \dots, n)$, it remains valid with $|a_i(t)|\alpha_i$ replacing each α_i for any $|a_i(t)| > 0$. This is essential because of the global Lipschitz conditions (1.3) and (1.4) of the nonlinear functions assumed previously. As a result, Theorem 3.1 actually provides a sufficient condition for global exponential convergence of system (1.1). In fact, for any initial function $\varphi_{x_i}(s)$, one can always pick a scalar $|a_i(t)| > 0$ large enough so that $|\varphi_{x_i}(s)| \leq |a_i(t)|\alpha_i$ for $s \leq 0$, $i = 1, \dots, n$. Hence, by the theorem it follows that:

$$
|x_i(t)| \leq |a_i(t)|\alpha_i e^{-\sigma t}, \quad i = 1, \dots, n. \tag{3.6}
$$

For the same reason, we get

$$
|y_j(t)| \leq |b_j(t)|\beta_j e^{-\sigma t}, \quad j = 1, \dots, m. \tag{3.7}
$$

So the system is globally exponentially convergent to the origin in terms of the estimates (3.6) and (3.7).

Conditions (3.4) and (3.5) are delay-dependent since they involve the delay kernels $K_{ji}(s)$ and $G_{ij}(s)$ explicitly. We can also derive the following delay-independent results. To do this, let

$$\begin{aligned} z_x(\sigma) = \max_{1\le i\le n} \Big\{ &(\sigma - |a_i(t)|)\alpha_i + \sum_{j=1}^{m} \alpha_j |A_{ji}(t)| \mu_{f_j} e^{\sigma\tau_j} \\ &+ \sum_{j=1}^{m} \alpha_j \mu_{g_j} |C_{ji}(t)| \int_{-\infty}^{t} K_{ji}(t-s) e^{\sigma s}\, ds + |I_i(t)| \Big\} \end{aligned}$$

and

$$\begin{aligned} z_y(\sigma) = \max_{1\le j\le m} \Big\{ &(\sigma - |b_j(t)|\beta_j + \sum_{i=1}^{n} \beta_i |B_{ij}(t)| \mu_{h_i} e^{\sigma\tau_i} \\ &+ \sum_{i=1}^{n} \beta_i \mu_{l_i} |D_{ij}(t)| \int_{0}^{\infty} G_{ij}(s) e^{\sigma s}\, ds + |J_j(t)| \Big\}. \end{aligned}$$

It is clear that $z_x(\sigma)$ and $z_y(\sigma)$ are continuous for $\sigma \in [0, \sigma_0]$ by condition (1.2). Therefore, if $z_x(0) < 0$ and $z_y(0) < 0$; i.e.,

$$-|a_i(t)|\alpha_i + \sum_{j=1}^{m} \alpha_j |A_{ji}(t)| \mu_{f_j} + \sum_{j=1}^{m} \alpha_j \mu_{g_j} |C_{ji}(t)| + |I_i(t)| < 0, \tag{3.8}$$

$$-|b_j(t)|\beta_j + \sum_{i=1}^{n} \beta_i |B_{ij}(t)| \mu_{h_i} + \sum_{i=1}^{n} \beta_i \mu_{l_i} |D_{ij}(t)| + |J_j(t)| < 0, \tag{3.9}$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$, then by continuity, there should be some $\sigma \in (0, \sigma_0]$ such that $z_x(\sigma) \le 0$ and $z_y(\sigma) \le 0$, i.e., conditions (3.4) and (3.5) holds, and vice versa. Thus, by noting Remark 3.2 also, we conclude the following equivalent condition to Theorem 3.1.

Theorem 3.3. *System* (1.1) *is globally exponentially convergent in terms of the estimates* (3.6) *and* (3.7) *for some* $\sigma \in (0, \sigma_0]$ *and* $\alpha_i > 0$*,* $\beta_j > 0$*,* $i = 1, \dots, n$*,* $j = 1, \dots, m$*, if conditions* (3.8) *and* (3.9) *holds.*

4. Examples

To illustrate the above results, we now consider a simple example of system (1.1) comprising n identical neurons coupled through weights A_{ji}, B_{ij}, C_{ji} and D_{ij} whose absolute values $|A_{ji}|$, $|B_{ji}|$, $|C_{ji}|$ and $|D_{ij}|$ constitute doubly stochastic matrix. Examples of such matrix are as below:

$$A = B = C = D := \frac{1}{n-1} \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}.$$

Assume the neuronal passive decay rates $a_i = a > 0$, $b_j = b > 0$ and the neuron activation satisfies conditions (1.3) and (1.4) with the gains $0 < \mu_{f_j} = \mu_{h_i} < 1$ and $\mu_{g_j} = \mu_{l_i} := \mu > 0$. The delay kernels K_{ji} and G_{ij} are taken as

$$K_{ji} = G_{ij} = \frac{r^{m+1}}{m!} t^m e^{-rt}, \quad r \in (0, \infty), m = 0, 1, 2, \dots.$$

It can be calculated that

$$\int_0^\infty K_{ji}(s)e^{\sigma s}\,ds = \int_0^\infty G_{ij}(s)e^{\sigma s}\,ds = \big(\frac{r}{r-\sigma}\big)^{m+1}.$$

For simplicity, we take all $\alpha_i = 1$ in condition (3.4). Then by Theorem 3.1 and Remark 3.2, if the neuron gain μ satisfies the bound

$$\mu \le (c-\sigma)\big(1-\frac{\sigma}{r}\big)^{m+1},$$

the system will globally converge to the origin in terms of

$$|x_i(t)| \le ke^{-\sigma t}, \quad |y_j(t)| \le ke^{-\sigma t}, \quad t \ge 0,\ i = 1,\dots,n, j = 1,\dots,m,$$

whenever $|x_i(\theta)| \le k$ and $|y_j(\theta)| \le k$ for $\theta \le 0$, with $k > 0$ a constant depending on the initial condition $x_i(\theta)$ and $y_j(\theta)$, $i = 1,\dots,n$, $j = 1,\dots,m$, and $0 < \sigma < \min\{c, r\}$. If one is merely interested in qualitatively confirming global exponential convergence of the system, it is convenient to use the criterion

$$\mu < c$$

according to condition (3.4) with all $\alpha_i = 1$. That is, the neuron activation gain should not exceed the value of the neuronal passive decay rates.

References

[1] T. Chu; *An exponential convergence estimate for analog neural networks with delays.* Phys. Lett. A. 283 (2001) 113-118.

[2] K. Gopalsamy, X. Z. He; *Delay-independent stability in bi-directional associative memory networks*, IEEE Trans. Neural Networks. 5 (1994) 998-1002.

[3] J. J. Hopfield; *Neuron with graded response have collective computational properties like those of tew-state neurons*, Proc. Natl. Acad. Sci. UAS. 81 (1984) 3088-92.

[4] B. Kosto; *Bi-directional associative mamories*, IEEE Trans. Syst. Man Cybernet. 18 (1988) 49-60.

[5] X. F. Liao, J. B. Yu; *Qualitative analysis of bidrectional associative memory with time delays*, Int. J. Circuit Theory Appl. 26 (1998) 210-229.

[6] Y. Liu, Z. Wang, X. Liu; *Global asymptotic stability of generalized bi-directional associative memory networks with discrete and distributed delays*, Chaos, Solitons and Fractals, 28 (3) 793-803.

[7] X. Lou, B. Cui; *On the global robust asymptotic stability of BAM neural networks with time-varying delays*, Neurocomputing, 70(2006) 273-279.

[8] J. H. Park; *A novel criterion for global asymptotic stability of BAM neural networks with time delays Chaos*, Solitons & Fractals, 29 (2006) 446-453.

[9] J. H. Park, O. M. Kwon; *On improved delay-dependent criterion for global stability of bidirectional associative memory neural networks with time-varying delays*, Appl. Math. Comput., 199(2008) 435-446.

[10] V. S. H. Rao, Bh. R. M. Phaneendra; *Global dynamics of bidirectional associative memory neural networks involving transmission delays and dead zones*, Neural networks. 12 (1999) 455-465.

[11] Q. Song, Z. Zhao, Y. Li; *Global exponential stability of BAM neural networks with distributed delays and reaction-diffusion terms*, Physics Letters A, 335 (2005) 213-225.

[12] B. Wang, J. Jian, C. Guo; *Global exponential stability of a class of BAM networks with time-varying delays and continuously distributed delays*, Neurocomputing, 71(2008)495-501.

[13] B. Wang, J. Jian, C. Guo, Global exponential stability of a class of BAM networks with time-varying delays and continuously distributed delays, Neurocomputing, 71(2008) 495-501.

[14] Y. Xia, J. Cao, M. Lin; *New results on the existence and uniqueness of almost periodic solution for BAM neural networks with continuously distributed delays*, Chaos, Solitons and Fractals, 31 (2007) 928-936.

[15] H. Zhao; *Global stability of bidirectional associative mamory neural networks with distributed delays*, Phys. Lett. A. 297 (2002) 182-190.

[16] H. Zhao; *Global stability of bidirectional associative memory neural networks with distributed delays*, Physics Letters A, 297 (2002) 182-190.

Yongkun Li
Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China
E-mail address: yklie@ynu.edu.cn

Yaping Ren
School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming 650221, China
E-mail address: ren_yaping5@yahoo.com.cn