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EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish the existence of a positive solution to a singular boundary-value problem of nonlinear fractional differential equation. Our analysis rely on nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorem in a cone.

1. Introduction

Many papers and books on fractional calculus differential equation have appeared recently. Most of them are devoted to the solvability of the linear fractional equation in terms of a special function and to problems of analyticity in the complex domain(see, for example [2, 8]). Moreover, Delbosco and Rodino [3] considered the existence of a solution for the nonlinear fractional differential equation $D^{\alpha}_{0+}u = f(t, u)$, where $0 < \alpha < 1$, and $f : [0, a] \times \mathbb{R} \to \mathbb{R}, 0 < a \leq +\infty$ is a given function,continuous in $(0, a)\times\mathbb{R}$. They obtained results for solutions by using the Schauder fixed point theorem and the Banach contraction principle. Recently, Zhang [11] considered the existence of positive solution for equation $D^{\alpha}_{0+}u = f(t, u)$, where $0 < \alpha < 1$, and $f : [0, 1]\times[0, +\infty) \to [0, +\infty)$, is a given continuous function, by using the sub-and super-solution method.

In this article, we discuss the existence of a positive solution to boundary-value problems of the nonlinear fractional differential equation

$$\begin{gathered} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0\,, \end{gathered} \tag{1.1}$$

where $2 < \alpha \leq 3$, D^{α}_{0+} is the Caputo's differentiation, and $f : (0, 1]\times[0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} f(t, \cdot) = +\infty$ (that is f is singular at $t = 0$). We obtain two results about this boundary-value problem, by using Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type in a cone.

For existence theorems for fractional differential equation and applications, we refer the reader to the survey by Kilbas and Trujillo [6]. Concerning the definitions

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of fractional integral and derivative and related basic properties, we refer the reader to Samko, Kilbas, and Marichev [5] and Delbosco and Rodino [3].

2. Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the literature.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.3 ([10])**.** *Let $n - 1 < \alpha \le n$, $u \in C^n[0, 1]$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - C_1 - C_2 t - \cdots - C_n t^{n-1},$$

where $C_i \in \mathbb{R}$, $i = 1, 2, \ldots n$.

Lemma 2.4 ([10])**.** *The relation*

$$I_{a+}^{\alpha} I_{a+}^{\beta} \varphi = I_{a+}^{\alpha+\beta} \varphi$$

is valid in following case

$$Re\beta > 0, \ Re(\alpha + \beta) > 0, \ \varphi(x) \in L_1(a, b).$$

Lemma 2.5. *Given $f \in C[0, 1]$, and $2 < \alpha \le 3$, the unique solution of*

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + f(t) = 0, \qquad 0 < t < 1, \\ &u(0) = u'(1) = u''(0) = 0. \end{aligned} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t, s) f(s) ds$$

where

$$G(t, s) = \begin{cases} \frac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Proof. We may apply Lemma2.3 to reduce Eq.(2.1) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} f(t) + C_1 + C_2 t + C_3 t^2$$

for some $C_i \in R$, $i = 1, 2, 3$. By Lemma2.4 we have

$$\begin{aligned} u'(t) &= -D_{0+}^{1} I_{0+}^{\alpha} f(t) + C_2 + 2C_3 t = -D_{0+}^{1} I_{0+}^{1} I_{0+}^{\alpha-1} f(t) + C_2 + 2C_3 t \\ &= -I_{0+}^{\alpha-1} f(t) + C_2 + 2C_3 t \end{aligned}$$

$$u''(t) = -D_{0+}^1 I_{0+}^{\alpha-1} f(t) + 2C_3 = -D_{0+}^1 I_{0+}^1 I_{0+}^{\alpha-2} f(t) + 2C_3 = -I_{0+}^{\alpha-2} f(t) + 2C_3.$$

From $u(0) = u'(1) = u''(0) = 0$, one has

$$C_1 = 0,\ C_2 = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s)ds,\ C_3 = 0.$$

Therefore, the unique solution of problem (2.1) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 t(1-s)^{\alpha-2} f(s)ds \\ &= \int_0^t \Big[\frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big] f(s)ds + \int_t^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)ds \\ &= \int_0^1 G(t,s) f(s)ds \end{aligned}$$

For $G(t,s)$, since $2 < \alpha \leq 3, 0 \leq s \leq t \leq 1$ we can obtain

$$(\alpha-1)t(1-s)^{\alpha-2} \geq t(1-s)^{\alpha-2} \geq t(t-s)^{\alpha-2} \geq (t-s)^{\alpha-1}$$

obviously,we get $G(t,s) > 0$. The proof is complete. □

Lemma 2.6 ([7]). *Let E be a Banach space, $P \subseteq E$ a cone, and Ω_1, Ω_2 are two bounded open balls of E centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$

holds. Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.7 ([4]). *Let E be a Banach space with $C \subseteq E$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a continuous compact map. Then either*

(1) *A has a fixed point in $\overline{U}$; or*
(2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda Au$.*

3. Main Results

For our construction, we let $E = C[0,1]$ and $\|u\| = \max_{0\leq t\leq 1} |u(t)|$ which is a Banach space. We seek solutions of (1.1) that lie in the cone

$$P = \{u \in E : u(t) \geq 0,\ 0 \leq t \leq 1\}.$$

Define operator $T : P \to P$, by

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))ds.$$

Lemma 3.1. *Let $0 < \sigma < 1$, $2 < \alpha \leq 3$, $F : (0,1] \to \mathbb{R}$ is continuous and $\lim_{t\to 0^+} F(t) = \infty$. Suppose that $t^\sigma F(t)$ is continuous function on $[0,1]$. Then the function*

$$H(t) = \int_0^t G(t,s) F(s) ds$$

is continuous on $[0,1]$.

Proof. By the continuity of $t^\sigma F(t)$ and $H(t)=\int_0^t G(t,s)s^{-\sigma}s^\sigma F(s)ds$ It is easily to check that $H(0)=0$. The proof is divided into three cases:
Case 1: $t_0=0, \forall t\in(0,1]$. Since $t^\sigma F(t)$ is continuous in $[0,1]$, there exists a constant $M>0$, such that $|t^\sigma F(t)|\le M$, for $t\in[0,1]$. Hence

$$\begin{aligned}
|H(t)-H(0)| &= \Big|\int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds \\
&\quad + \int_t^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds\Big| \\
&= \Big|\int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds\Big| \\
&\le \Big|\int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds\Big| + \Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds\Big| \\
&\le M\int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}ds + M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds \\
&= \frac{Mt}{\Gamma(\alpha-1)}B(1-\sigma,\alpha-1) + \frac{M}{\Gamma(\alpha)}t^{\alpha-\sigma}B(1-\sigma,\alpha) \\
&= \frac{\Gamma(1-\sigma)Mt}{\Gamma(\alpha-\sigma)} + \frac{\Gamma(1-\sigma)Mt^{\alpha-\sigma}}{\Gamma(1+\alpha-\sigma)} \to 0 \quad (\text{as } t\to 0)
\end{aligned}$$

where B denotes the beta function.
Case 2: $t_0\in(0,1)$, for all $t\in(t_0,1]$

$$\begin{aligned}
&|H(t)-H(t_0)| \\
&= \Big|\int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds \\
&\quad + \int_t^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds - \int_{t_0}^1 \frac{t_0(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds \\
&\quad - \int_0^{t_0} \frac{(\alpha-1)t_0(1-s)^{\alpha-2}-(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds\Big| \\
&= \Big|\int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds \\
&\quad - \int_0^1 \frac{t_0(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds + \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds\Big| \\
&= \Big|\int_0^1 \frac{(t-t_0)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}s^\sigma F(s)ds \\
&\quad - \int_0^{t_0} \frac{(t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds - \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)ds\Big| \\
&\le \frac{M(t-t_0)}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}s^{-\sigma}ds + \frac{M}{\Gamma(\alpha)}\int_0^{t_0}\Big[(t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\Big]s^{-\sigma}ds \\
&\quad - \frac{M}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}s^{-\sigma}ds \\
&\le \frac{M(t-t_0)}{\Gamma(\alpha-1)}B(1-\sigma,\alpha-1) + \frac{Mt^{\alpha-\sigma}}{\Gamma(\alpha)}B(1-\sigma,\alpha) - \frac{Mt_0^{\alpha-\sigma}}{\Gamma(\alpha)}B(1-\sigma,\alpha)
\end{aligned}$$

$$= \frac{\Gamma(1-\sigma)M(t-t_0)}{\Gamma(\alpha-\sigma)} + \frac{\Gamma(1-\sigma)Mt^{\alpha-\sigma}}{\Gamma(1+\alpha-\sigma)} - \frac{\Gamma(1-\sigma)Mt_0^{\alpha-\sigma}}{\Gamma(1+\alpha-\sigma)} \to 0 \quad (\text{as } t \to t_0).$$

Case 3: $t_0 \in (0,1]$, for all $t \in [0,t_0)$. The proof is similar to that of Case 2; we omitted it. ☐

Lemma 3.2. *Let $0 < \sigma < 1$, $2 < \alpha \leq 3$, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $\lim_{t\to 0^+} f(t,\cdot) = +\infty$, $t^\sigma f(t,u(t))$ is continuous function on $[0,1] \times [0,+\infty)$, then the operator $T : P \to P$ is completely continuous.*

Proof. For each $u \in P$, let $Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds$. By Lemma3.1 and the fact that $f, G(t,s)$ are non-negative, we have $T : P \to P$.

Let $u_0 \in P$ and $\|u_0\| = C_0$, if $u \in P$ and $\|u - u_0\| < 1$, then $\|u\| < 1 + C_0 = C$. By the continuity of $t^\sigma f(t,u(t))$, we know that $t^\sigma f(t,u(t))$ is uniformly continuous on $[0,1] \times [0,C]$.

Thus for all $\epsilon > 0$, there exists $\delta > 0 (\delta < 1)$, such that $|t^\sigma f(t,u_2) - t^\sigma f(t,u_1)| < \epsilon$, for all $t \in [0,1]$, and $u_1, u_2 \in [0,C]$ with $|u_2 - u_1| < \delta$. Obviously, if $\|u - u_0\| < \delta$, then $u(t), u_0(t) \in [0,C]$ and $\|u(t) - u_0(t)\| < \delta$, for all $t \in [0,1]$. Hence,

$$|t^\sigma f(t,u(t)) - t^\sigma f(t,u_0(t))| < \epsilon, \quad \text{for all t} \in [0,1]. \tag{3.1}$$

$u \in P$, with $\|u - u_0\| < \delta$. It follows from (3.1) that

$$\begin{aligned}
\|Tu - Tu_0\| &= \max_{0\leq t\leq 1} \big|Tu(t) - Tu_0(t)\big| \\
&\leq \max_{0\leq t\leq 1} \int_0^1 G(t,s)s^{-\sigma}\big|s^\sigma f(s,u(s)) - s^\sigma f(s,u_0(s))\big|ds \\
&< \epsilon \int_0^1 G(t,s)s^{-\sigma}ds \\
&= \epsilon \int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds \\
&\leq \frac{\epsilon}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}s^{-\sigma}ds \\
&= \frac{\epsilon}{\Gamma(\alpha-1)} B(1-\sigma,\alpha-1) = \frac{\Gamma(1-\sigma)\epsilon}{\Gamma(\alpha-\sigma)}.
\end{aligned}$$

By the arbitrariness of u_0, $T : P \to P$ is continuous. Let $M \subset P$ be bounded; i.e., there exists a positive constant b such that $\|u\| \leq b$, for all $u \in p$.

Since $t^\sigma f(t,u)$ is continuous in $[0,1] \times [0,+\infty)$, let

$$L = \max_{0\leq t\leq 1, u\in M} t^\sigma f(t,u) + 1, \quad \forall u \in M.$$

Then

$$\big|Tu(t)\big| \leq \int_0^1 G(t,s)s^{-\sigma}|s^\sigma f(s,u(s))|ds \leq L\int_0^1 G(1,s)s^{-\sigma}ds = \frac{\Gamma(1-\sigma)L}{\Gamma(\alpha-\sigma)};$$

thus

$$\|Tu\| = \max_{0\leq t\leq 1} \big|Tu(t)\big| \leq \frac{\Gamma(1-\sigma)L}{\Gamma(\alpha-\sigma)}.$$

So, $T(M)$ is equicontinuous. For $\epsilon > 0$ set

$$\delta = \min\Big\{\frac{\epsilon}{\frac{\Gamma(1-\alpha)L}{\Gamma(\alpha-\sigma)} + \frac{\Gamma(1-\alpha)L}{\Gamma(1+\alpha-\sigma)}}, \frac{\epsilon\Gamma(\alpha-\sigma)}{2L\Gamma(1-\sigma)}, \frac{\epsilon}{\frac{\Gamma(1-\alpha)L}{\Gamma(\alpha-\sigma)} + \frac{\Gamma(1-\alpha)L2^\alpha}{\Gamma(1+\alpha-\sigma)}}\Big\}.$$

For $u \in M$, $t_1, t_2 \in [0,1]$, with $t_1 < t_2$, for $0 < t_2 - t_1 < \delta$, we have

$$
\begin{aligned}
&\big|Tu(t_2) - Tu(t_1)\big| \\
&= \Big| \int_0^1 G(t_2,s) f(s,u(s))ds - \int_0^1 G(t_1,s) f(s,u(s))ds \Big| \\
&= \Big| \int_0^1 \big[G(t_2,s) - G(t_1,s)\big] s^{-\sigma} s^{\sigma} f(s,u(s))ds \Big| \\
&\le L \int_0^1 \Big|G(t_2,s) - G(t_1,s)\Big| s^{-\sigma} ds \\
&\le L \Big| \int_0^{t_2} \frac{(\alpha-1)t_2(1-s)^{\alpha-2} - (t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} ds + \int_{t_2}^1 \frac{t_2(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} s^{-\sigma} ds \\
&\quad - \int_0^{t_1} \frac{(\alpha-1)t_1(1-s)^{\alpha-2} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} ds - \int_{t_1}^1 \frac{t_1(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} s^{-\sigma} ds \Big| \\
&\le L \Big[(t_2 - t_1) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} s^{-\sigma} ds + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} ds \\
&\quad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} ds \Big] \\
&= L \frac{(t_2 - t_1)}{\Gamma(\alpha-1)} \int_0^1 s^{-\sigma} (1-s)^{\alpha-2} ds + \frac{L}{\Gamma(\alpha)} \int_0^{t_2} s^{-\sigma} (t_2-s)^{\alpha-1} ds \\
&\quad - \frac{L}{\Gamma(\alpha)} \int_0^{t_1} s^{-\sigma} (t_1-s)^{\alpha-1} ds \\
&\le L \frac{(t_2 - t_1)\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} + \frac{L\Gamma(1-\sigma)}{\Gamma(1+\alpha-\sigma)} (t_2^{\alpha-\sigma} - t_1^{\alpha-\sigma}).
\end{aligned}
$$

Case 1: $t_1 = 0$, $t_2 < \delta$.

$$
\begin{aligned}
\big|Tu(t_2) - Tu(t_1)\big| &= L \frac{t_2 \Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} + \frac{L\Gamma(1-\sigma)}{\Gamma(1+\alpha-\sigma)} t_2^{\alpha-\sigma} \\
&< L \frac{\delta \Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} + \frac{L\Gamma(1-\sigma)}{\Gamma(1+\alpha-\sigma)} \delta < \epsilon\,.
\end{aligned}
$$

Case 2: $\delta \le t_1 < t_2 < 1$.

$$
\begin{aligned}
\big|Tu(t_2) - Tu(t_1)\big| &< L \frac{\delta \Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} + \frac{L\Gamma(1-\sigma)}{\Gamma(1+\alpha-\sigma)} \delta^{\alpha-\sigma} \\
&= \frac{L\delta\Gamma(1-\sigma) + L\Gamma(1-\sigma)\delta^{\alpha-\sigma}}{\Gamma(\alpha-\sigma)} \\
&< \frac{2L\delta\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} < \epsilon\,.
\end{aligned}
$$

Case 3: $0 < t_1 < \delta$, $t_2 < 2\delta$.

$$
\big|Tu(t_2) - Tu(t_1)\big| < L \frac{\delta \Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} + \frac{L\Gamma(1-\sigma)}{\Gamma(1+\alpha-\sigma)} 2^{\alpha} \delta < \epsilon\,.
$$

Therefore, $T(M)$ is equicontinuous. The Arzela-Ascoli theorem implies that $\overline{T(M)}$ is compact. Thus, the operator $T : P \to P$ is completely continuous. □

Theorem 3.3. *Let $0 < \sigma < 1$, $2 < \alpha \le 3$, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $\lim_{t\to 0^+} f(t,\cdot) = +\infty$, $t^\sigma f(t,y)$ is continuous function on $[0,1] \times [0,+\infty)$. Assume that there exist two distinct positive constant $\rho,\mu(\rho > \mu)$ such that*

- (H1) $t^\sigma f(t,\omega) \le \rho\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}$, *for* $(t,\omega) \in [0,1] \times [0,\rho]$;
- (H2) $t^\sigma f(t,\omega) \ge \mu\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}$, *for* $(t,\omega) \in [0,1] \times [0,\mu]$.

Then (1.1) has at least one positive solution.

Proof. From Lemma 3.2 we have $T : P \to P$ is completely continuous. We divide the proof into the following two steps.

Step1: Let $\Omega_1 = \{u \in P : \|u\| < \frac{\alpha-\sigma-1}{\alpha-\sigma}\mu\}$, for $u \in K \cap \partial\Omega_1$ and all $t \in [0,1]$, we have $0 \le u(t) \le \frac{\alpha-\sigma-1}{\alpha-\sigma}\mu$. It follows from (H2) that

$$\begin{aligned}
Tu(1) &= \int_0^1 G(1,s)f(s,u(s))ds = \int_0^1 G(1,s)s^{-\sigma}s^\sigma f(s,u(s))ds \\
&\ge \mu\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\int_0^1 G(1,s)s^{-\sigma}ds \\
&= \mu\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\Big[\int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-2}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds\Big] \\
&= \mu\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\Big[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}s^{-\sigma}ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds\Big] \\
&= \mu\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\big[\frac{B(1-\sigma,\alpha-1)}{\Gamma(\alpha-1)} - \frac{B(1-\sigma,\alpha-1)}{\Gamma(\alpha)}\big] \\
&\ge \frac{\alpha-\sigma-1}{\alpha-\sigma}\mu = \|u\|\,.
\end{aligned}$$

Hence,

$$\|Tu\| = \max_{0\le t\le 1}|Tu(t)| \ge \frac{\alpha-\sigma-1}{\alpha-\sigma}\mu = \|u\|,$$

for $u \in P \cap \partial\Omega_1$.

Step 2: Let $\Omega_2 = \{u \in P : \|u\| < \rho\}$, for $u \in K \cap \partial\Omega_2$ and all $t \in [0,1]$, we have $0 \le u(t) \le \rho$. By assumption (H1),

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\
&= \int_0^1 G(t,s)s^{-\sigma}s^\sigma f(s,u(s))ds \\
&\le \rho\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\Big[\int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds \\
&\quad + \int_t^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}ds\Big] \\
&\le \rho\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\Big[\int_0^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}s^{-\sigma}ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds\Big] \\
&\le \rho\frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}\Big[\frac{t}{\Gamma(\alpha-1)}\int_0^1 s^{-\sigma}(1-s)^{\alpha-2}ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \rho \frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)} \frac{B(1-\sigma,\alpha-1)}{\Gamma(\alpha-1)} \\
&= \rho \frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)} \frac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)} = \rho\,.
\end{aligned}
$$

So $\|Tu(t)\| \le \|u\|$, for $u \in P \cap \partial\Omega_2$. Therefore, by (ii) of Lemma 2.6, we complete the proof. ☐

Theorem 3.4. *Let $0 < \sigma < 1$, $2 < \alpha \le 3$, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $\lim_{t\to 0^+} f(t,\cdot) = +\infty$, $t^\sigma f(t,y)$ is continuous function on $[0,1] \times [0,+\infty)$. Suppose the following conditions are satisfied:*

- (H3) *there exists a continuous, nondecreasing function $\varphi : [0,+\infty) \to (0,\infty)$ with $t^\sigma f(t,\omega) \le \varphi(\omega)$, for $(t,\omega) \in [0,1] \times [0,+\infty)$*
- (H4) *there exists $r > 0$, with $\frac{r}{\varphi(r)} > \frac{\Gamma(\alpha-\sigma)}{\Gamma(1-\sigma)}$*

Then (1.1) has one positive solution.

Proof. Let $U = \{u \in P : \|u\| < r\}$, we have $U \subset P$. From Lemma 3.2, we know $T : \overline{U} \to P$ is completely continuous. If there exists $u \in \partial U$, $\lambda \in (0,1)$ such that

$$
u = \lambda T u, \tag{3.2}
$$

By (H3) and (3.2), for $t \in [0,1]$ we have

$$
\begin{aligned}
u(t) = \lambda Tu(t) = \lambda \int_0^1 G(t,s) f(s,u(s))ds &\le \int_0^1 G(t,s) s^{-\sigma} s^{\sigma} f(s,u(s))ds \\
&\le \int_0^1 G(t,s) s^{-\sigma} \varphi(u(s))ds \\
&\le \varphi(\|u\|) \int_0^1 G(t,s) s^{-\sigma} ds \\
&\le \varphi(\|u\|) \int_0^1 G(1,s) s^{-\sigma} ds \\
&= \varphi(\|u\|) \int_0^1 \frac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} ds \\
&\le \varphi(\|u\|) \frac{tB(1-\sigma,\alpha-1)}{\Gamma(\alpha-1)} \\
&\le \varphi(\|u\|) \frac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)}\,.
\end{aligned}
$$

Consequently, $\|u\| \le \varphi(\|u\|)\frac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)}$; namely,

$$
\frac{\|u\|}{\varphi(\|u\|)} \le \frac{\Gamma(1-\sigma)}{\Gamma(\alpha-\sigma)}.
$$

Combining (H4) and the above inequality, we have $\|u\| \ne r$, which is contradiction with $u \in \partial U$. According to Lemma 2.7, T has a fixed point $u \in \overline{U}$, therefore, (1.1) has a positive solution. ☐

As an example, consider the fractional differential equation

$$\begin{gathered} D_{0^+}^{\alpha}u(t) + \frac{(t-\frac{1}{2})^2\ln(2+u)}{t^{\sigma}} = 0, \quad 0<t<1 \\ u(0) = u'(1) = u''(0) = 0, \end{gathered} \tag{3.3}$$

where $0<\sigma<1$, $2<\alpha\le 3$. Then (3.3) has a positive solution.

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