

DISPERSIVE ESTIMATES FOR A LINEAR WAVE EQUATION WITH ELECTROMAGNETIC POTENTIAL

DAVIDE CATANIA

ABSTRACT. We consider radial solutions to the Cauchy problem for a linear wave equation with a small short-range electromagnetic potential (depending on space and time) and zero initial data. We present two dispersive estimates that provide, in particular, an optimal decay rate in time t^{-1} for the solution. Also, we apply these estimates to obtain similar results for the linear massless Dirac equation perturbed by a potential.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we investigate the dispersive properties of the linear wave equation with an electromagnetic potential

$$(\square_A - B)u = F \quad (t, x) \in [0, \infty[\times \mathbb{R}^3, \quad (1.1)$$

where $x = (x_1, x_2, x_3)$ and

$$\square_A = \partial^\mu \partial_{\mu, A}.$$

Here and in the following, sum over repeated up-down indices is assumed (according to the Einstein's convention), the covariant derivatives ∂^μ and $\partial_{\mu, A}$ are defined by

$$\begin{aligned} \partial^\mu &= \partial^\mu - iA^\mu, & \partial_{\mu, A} &= \partial_\mu - iA_\mu \quad \mu = 0, 1, 2, 3, \\ \partial_0 &= \partial_t, & \partial_k &= \partial_{x_k} \quad k = 1, 2, 3, \end{aligned}$$

where “ i ” is the imaginary unit, and we raise and lower indices according to

$$X^\mu = \eta^{\mu\nu} X_\nu, \quad X_\mu = \eta_{\mu\nu} X^\nu,$$

where the 4×4 matrix

$$(\eta_{\mu\nu})_{0 \leq \mu, \nu \leq 3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

represents the standard Minkowski metric in $\mathbb{R}_t \times \mathbb{R}_x^3$ and

$$(\eta^{\mu\nu}) = (\eta_{\mu\nu})^{-1} = (\eta_{\mu\nu}).$$

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The fact that the potential $A = A(t, x)$, depending on space and time, is electromagnetic means that the components A_μ of $A = (A_0, A_1, A_2, A_3)$ assume real values. This will play a crucial role in the development of the proof, since electromagnetic potentials make the operator \square_A gauge invariant (see what follows).

We assume further that the potential decreases sufficiently fast when

$$r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

approaches infinity; more precisely, we suppose that

$$\sum_{j \in \mathbb{Z}} 2^{-j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j A\|_{L_{t,x}^\infty} \leq \delta_0 \quad (1.2)$$

(that is, A is a short-range potential), where ϵ_0 and δ_0 are positive constants independent of r (see Section 2) and the sequence $(\phi_j)_{j \in \mathbb{Z}}$ is a Paley-Littlewood partition of unity, which means that $\phi_j(r) = \phi(2^j r)$ and $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ (\mathbb{R}^+ is the set of all nonnegative real numbers) is a function such that

- (a) $\text{supp } \phi = \{r \in \mathbb{R} : 2^{-1} \leq r \leq 2\}$;
- (b) $\phi(r) > 0$ for $2^{-1} < r < 2$;
- (c) $\sum_{j \in \mathbb{Z}} \phi(2^j r) = 1$ for each $r \in \mathbb{R}^+$.

In other words, $\sum_{j \in \mathbb{Z}} \phi_j(r) = 1$ for all $r \in \mathbb{R}^+$ and

$$\text{supp } \phi_j = \{r \in \mathbb{R} : 2^{-j-1} \leq r \leq 2^{-j+1}\}.$$

Roughly speaking, condition (1.2) means that the potential A decays at least like $r^{-(1+\epsilon_0)}$ as r tends to infinity, while a singularity as r tends to 0 is allowed.

Additionally, we assume that the jacobian matrix $\nabla A = (\partial_\nu A_\mu)_{0 \leq \mu, \nu \leq 3}$ is well-defined, and that ∇A and the potential term $B = B(t, x)$ satisfy the smallness hypothesis

$$\sum_{j \in \mathbb{Z}} 2^{-2j} \langle 2^{-j} \rangle^{\epsilon_0} (\|\phi_j B\|_{L_{t,x}^\infty} + \|\phi_j \nabla A\|_{L_{t,x}^\infty}) \leq \delta_0. \quad (1.3)$$

Essentially, ∇A and B decay at least like $r^{-(2+\epsilon_0)}$ as r tends to infinity.

The possible values for δ_0 and ϵ_0 will be made precise in the statement of the main result.

Moreover, we shall restrict ourselves to radial solutions $u = u(t, r)$. Since the Cauchy problem

$$\begin{aligned} (\square_A - B)u &= F & (t, x) &\in [0, \infty[\times \mathbb{R}^3, \\ u(0, x) &= \partial_t u(0, x) = 0 & x &\in \mathbb{R}^3 \end{aligned} \quad (1.4)$$

is linear, its solution exists globally in time; in particular, this fact holds for the smaller class of radial solutions, that is to say for the problem

$$\begin{aligned} (\square_A - B)u &= F & (t, r) &\in [0, \infty[\times \mathbb{R}^+, \\ u(0, r) &= \partial_t u(0, r) = 0 & r &\in \mathbb{R}^+. \end{aligned} \quad (1.5)$$

Let us introduce the change of coordinates

$$\tau_\pm := \frac{t \pm r}{2}$$

and the standard notation $\langle s \rangle := \sqrt{1 + s^2}$; our main result can be expressed as follows.

Theorem 1.1. *Let u be a radial solution to (1.4); i.e., a solution to (1.5), where A and B satisfy (1.2) and (1.3) for some $\delta_0 > 0$ and $\epsilon_0 > 0$. Then, for every $\epsilon \in]0, \epsilon_0]$, there exist two positive constants δ and C (depending on ϵ) such that, for each $\delta_0 \in]0, \delta]$, one has*

$$\|\tau_+ u\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}.$$

Note that $\epsilon_0 > 0$ can be chosen freely. Let us introduce the differential operators

$$\nabla_\pm := \partial_t \pm \partial_r.$$

The proof of the previous a priori estimate follows easily from the following estimate.

Lemma 1.1. *Under the conditions of Theorem 1.1, for every $\epsilon \in]0, \epsilon_0]$ there exist two positive constants δ and C (depending on ϵ) such that, for each $\delta_0 \in]0, \delta]$, one has*

$$\|\tau_+ r \nabla_- u\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}. \quad (1.6)$$

An immediate consequence of Theorem 1.1 is the following dispersive estimate.

Corollary 1.1. *Under the same conditions of Theorem 1.1, for every $\epsilon \in]0, \epsilon_0]$ there exist two positive constants δ and C (depending on ϵ) such that, for each $\delta_0 \in]0, \delta]$, one has*

$$|u(t, r)| \leq \frac{C}{t} \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}$$

for every $t > 0$.

The strategy for proving the lemma is the following. First of all, the potential terms in (1.5) can be thought as part of the forcing term, that is, $(\square_A - B)u = F$ can be viewed as

$$\square u = F_1 := F + Bu + i(\partial^\mu A_\mu)u + A^\mu A_\mu u + 2iA^\mu \partial_\mu u, \quad (1.7)$$

where

$$\square = \partial_t^2 - \Delta = \partial_t^2 - (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2)$$

is the standard d'Alembert operator. Moreover, if we introduce the gradient operators $\nabla_{t,x} = (\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\nabla_{t,r} = (\partial_t, \partial_r)$, setting

$$\tilde{A} = (\tilde{A}^0, \tilde{A}^1), \quad \tilde{A}^0 = A^0, \quad \tilde{A}^1 = \frac{A^1 x_1 + A^2 x_2 + A^3 x_3}{r},$$

one has

$$A^\mu \partial_\mu = (A^0, A^1, A^2, A^3) \cdot \nabla_{t,x} = \tilde{A} \cdot \nabla_{t,r}$$

and

$$F_1 = F + Bu + i(\partial^\mu A_\mu)u + A^\mu A_\mu u + 2i\tilde{A} \cdot \nabla_{t,r} u. \quad (1.8)$$

Then we can rewrite (1.7) in terms of τ_\pm and ∇_\pm (see Section 2), obtaining

$$\nabla_+ \nabla_- v = G,$$

where

$$v(t, r) := ru(t, r) \quad \text{and} \quad G(t, r) := rF_1(t, r). \quad (1.9)$$

This last equation can be easily integrated to obtain a relatively simple explicit representation of $(\nabla_- v)(\tau_+, \tau_-)$ in terms of G .

Another fundamental step consists in taking advantage of the gauge invariance property of the operator \square_A , which means that, if we consider the potential \dot{A} of components

$$\dot{A}^\mu = A^\mu + \partial^\mu \phi, \quad \phi \in \mathbb{R},$$

we have

$$\square_{\dot{A}}(e^{i\phi}u) = e^{i\phi}\square_A u$$

(see [2, p. 34]). Because of this property, which can be easily verified, since $|e^{i\phi}| = 1$, we can modify through ϕ the potential A and get dispersive estimates for the solution to $\square_{\dot{A}}$ that extend to the solution to \square_A .

More precisely, set

$$A_{\pm} := \frac{\tilde{A}^0 \pm \tilde{A}^1}{2},$$

we can assume, without loss of generality, that $A_+ \equiv 0$. Indeed, it is sufficient to choose ϕ such that

$$\dot{A}^0 = \tilde{A}^0 + \partial^t \phi = \tilde{A}^0 + \partial_t \phi \quad \text{and} \quad \dot{A}^1 = \tilde{A}^1 + \partial^r \phi = \tilde{A}^1 - \partial_r \phi$$

are opposite; i.e.,

$$\nabla_- \phi = -(\tilde{A}^0 + \tilde{A}^1).$$

Hence we can take any ϕ of the form

$$\phi(\tau_+, \tau_-) = \phi_0 - \int_{\tau_0}^{\tau_-} (\tilde{A}^0 + \tilde{A}^1)(\tau_+, s) ds,$$

where ϕ_0 and τ_0 are real numbers. This implies

$$\tilde{A} \cdot \nabla_{t,r} u = A_- \nabla_- u + A_+ \nabla_+ u = A_- \nabla_- u,$$

and hence

$$F_1 = F + Bu + i(\partial^\mu A_\mu)u + A^\mu A_\mu u + 2iA_- \nabla_- u, \quad (1.10)$$

thus

$$G = rF_1 = rF + Bv + i(\partial^\mu A_\mu)v + A^\mu A_\mu v + 2iA_- \nabla_- v + \frac{2i}{r}A_- v. \quad (1.11)$$

Obviously, one still has

$$\sum_{j \in \mathbb{Z}} 2^{-j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j A_-\|_{L_{t,x}^\infty} \leq \delta_0. \quad (1.12)$$

These simplifications, combined with the technical Lemma 2.1 and the estimate of Lemma 2.2, allow us easily to obtain Lemma 1.1 and Theorem 1.1.

Application: The Dirac Equation. As an application, we use Theorem 1.1 to obtain a similar result for radial solutions to the massless Dirac equation with a suitable potential. Let us introduce some notations. First of all, the Dirac matrices are defined by

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The relativistic invariant form of the nonperturbed massless Dirac operator, applied to a vector function $\psi : \mathbb{R}^{1+3} \rightarrow \mathbb{C}^4$ (generally called spinor), is

$$D = \gamma^\mu \partial_\mu$$

while, for the perturbed case, we consider the operator

$$D_A = \gamma^\mu \partial_{\mu,A}$$

(with the notations introduced for the wave equation).

We consider radial solutions $u : \mathbb{R}^{1+3} \rightarrow \mathbb{C}^4$ to the Cauchy problem

$$\begin{aligned} D_A u &= F \quad (t, x) \in [0, \infty[\times \mathbb{R}^3, \\ u(0, x) &= 0 \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.13}$$

assuming that each potential matrix $A_\mu \in \mathbb{R}^{4 \times 4}$ (i.e., it is real), $\mu = 0, 1, 2, 3$, ∇A_μ is well-defined and the following smallness hypotheses are satisfied for suitable δ_0 and $\epsilon_0 > 0$:

$$\sum_{j \in \mathbb{Z}} 2^{-j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j A_\mu\|_{L_{t,x}^\infty} \leq \delta_0, \tag{1.14}$$

$$\sum_{j \in \mathbb{Z}} 2^{-2j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j \nabla A_\mu\|_{L_{t,x}^\infty} \leq \delta_0, \tag{1.15}$$

where the sequence $(\phi_j)_{j \in \mathbb{Z}}$ is the Paley–Littlewood partition of unity previously defined.

Under these hypotheses, we have the following result.

Theorem 1.2. *Let u be a radial solution to (1.13), where, for each $\mu = 0, 1, 2, 3$, A_μ is real and satisfies (1.14) and (1.15) for some $\delta_0 > 0$ and $\epsilon_0 > 0$. Then, for every $\epsilon \in]0, \epsilon_0]$, there exist two positive constants δ and C (depending on ϵ) such that, for each $\delta_0 \in]0, \delta]$, one has*

$$\|\tau_+ u\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon D_A F\|_{L_{t,x}^\infty}.$$

In particular, one has

$$|u(t, r)| \leq \frac{C}{t} \|\tau_+ r^2 \langle r \rangle^\epsilon D_A F\|_{L_{t,x}^\infty}$$

for every $t > 0$. Moreover, if $\|A\|_{L_{t,x}^\infty} < \infty$, one has

$$\|\tau_+ u\|_{L_{t,x}^\infty} \leq C (\|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty} + \|\tau_+ r^2 \langle r \rangle^\epsilon \nabla F\|_{L_{t,x}^\infty}). \tag{1.16}$$

To prove these estimates, we observe that the solution u to the Cauchy problem (1.13) is a solution to

$$\begin{aligned} D_A^2 u &= D_A F, \\ u(0, r) &= \partial_t u(0, r) = 0, \end{aligned}$$

and this problem can be recast in the form

$$\begin{aligned} (\square_A - B)u &= D_A F, \\ u(0, r) &= \partial_t u(0, r) = 0, \end{aligned}$$

for $A = (A_0, A_1, A_2, A_3)$ and B suitably chosen, where in this case A_μ and B are complex 4×4 matrices such that we can apply a slight variation of Theorem 1.1. In other words, the operator D_A can be viewed as the square root of the operator $\square_A - B$, where

$$B = -\frac{\gamma^\mu \gamma^\nu}{2i} ((\partial_\mu A_\nu) - (\partial_\nu A_\mu))$$

(for further details, see Section 3). In this sense, the massless Dirac equation (with potential) can be viewed as the square root of the wave equation (with potential); i.e., $D^2 = \square$.

Motivation. The dispersive properties of evolution equations are important for their physical meaning and, consequently, they have been deeply studied, though the problem in its generality is still open. The dispersive estimate obtained in Corollary 1.1 provides the natural decay rate, that is the same rate that one has for the nonperturbed wave equation (see [11, 13]); i.e., a $t^{-(n-1)/2}$ decay in time, where n is the space dimension (in our case, $n = 3$). The generalization to the case of a potential-like perturbation has been considered widely (see [1, 3, 4, 5, 7, 10, 14, 17, 18, 19, 20]), also for the Schrödinger equation (see [8, 9, 12, 15, 16]). Recently, D'Ancona and Fanelli have considered in [6] the case

$$\begin{aligned} \partial_t^2 u(t, x) + Hu &= 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0, x) &= 0, & \partial_t u(0, x) = g(x), \end{aligned}$$

where

$$\begin{aligned} H &:= -(\nabla + iA(x))^2 + B(x), \\ A : \mathbb{R}^3 &\rightarrow \mathbb{R}^3, & B : \mathbb{R}^3 \rightarrow \mathbb{R}. \end{aligned}$$

Under suitable conditions on A , ∇A and B , in particular

$$|A(x)| \leq \frac{C_0}{r\langle r \rangle(1 + |\lg r|)^\beta}, \quad \sum_{j=1}^3 |\partial_j A_j(x)| + |B(x)| \leq \frac{C_0}{r^2(1 + |\lg r|)^\beta}, \quad (1.17)$$

with $C_0 > 0$ sufficiently small, $\beta > 1$ and $r = |x|$, they have obtained the dispersive estimate

$$|u(t, x)| \leq \frac{C}{t} \sum_{j \geq 0} 2^{2j} \|\langle r \rangle w_\beta^{1/2} \phi_j(\sqrt{H})g\|_{L^2}, \quad (1.18)$$

where $w_\beta := r(1 + |\log r|)^\beta$ and $(\phi_j)_{j \geq 0}$ is a nonhomogeneous Paley-Littlewood partition of unity on \mathbb{R}^3 .

In this paper, restricting ourselves to radial solutions, we are able to obtain the result in Corollary 1.1, which is optimal from the point of view of the estimate decay rate t^{-1} and improve essentially the assumptions on the potential, assuming the weaker conditions (1.2) and (1.3) instead of (1.17) and allowing that it could depend on time.

This article is structured as follows: In Section 2 we prove the main results (concerning the wave equation), while Section 3 is devoted to the proof of Theorem 1.2 (the application to the Dirac equation).

2. PROOF OF THE MAIN RESULTS

First of all, we reformulate our problem taking advantage of the radiality of the solution u to (1.5). Indeed, since $\Delta_{S^2} u(t, r) = 0$ and $v = ru$, we have

$$\begin{aligned} \square u(t, r) &= (\partial_t^2 - \Delta_x)u = \left(\partial_t^2 - \partial_r^2 - \frac{2}{r} \partial_r - \frac{1}{r^2} \Delta_{S^2} \right) u(t, r) \\ &= \frac{1}{r} \partial_t^2 v(t, r) - \frac{1}{r} \partial_r^2 v(t, r) \\ &= \frac{1}{r} \nabla_+ \nabla_- v(t, r) = \frac{1}{r} \nabla_- \nabla_+ v(t, r). \end{aligned}$$

Recalling (1.7), (1.9) and (1.10), we get that the (1.5) is equivalent to

$$\nabla_+ \nabla_- v = G.$$

Note that the support of $u(t, r)$ is contained in the domain $\{(t, r) \in \mathbb{R}^2 : r > 0, t > r\}$, therefore we have

$$\text{supp } v(\tau_+, \tau_-) \subseteq \{(\tau_+, \tau_-) \in \mathbb{R}^2 : \tau_- > 0, \tau_+ > \tau_-\}. \quad (2.1)$$

From this fact, we get

$$\nabla_- v(\tau_+, \tau_-) = \nabla_- v(\tau_-, \tau_-) + \int_{\tau_-}^{\tau_+} G(s, \tau_-) ds = \int_{\tau_-}^{\tau_+} G(s, \tau_-) ds.$$

Note that G depends on (t, x) or, in spherical coordinates, on $(t, r, \theta_1, \theta_2)$, or even on $(\tau_+, \tau_-, \theta_1, \theta_2)$; since the angular components are not relevant to our computations, we shall write briefly $G(\tau_+, \tau_-)$ and proceed similarly for other terms. However, it is important to remember that u and v are effectively radial.

Let us observe that, for each $s \in [\tau_-, \tau_+]$, we have

$$s \leq \tau_+, \quad s - \tau_- \leq \tau_+ - \tau_- = r,$$

hence

$$\begin{aligned} \left| \int_{\tau_-}^{\tau_+} G(s, \tau_-) ds \right| &\leq \int_{\tau_-}^{\tau_+} \frac{s \langle s - \tau_- \rangle^\epsilon |G(s, \tau_-)|}{\langle s \rangle \langle s - \tau_- \rangle^\epsilon} ds \\ &\leq \|\tau_+ \langle r \rangle^\epsilon G\|_{L_{t,x}^\infty} \int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\epsilon} ds \end{aligned}$$

for every $\epsilon > 0$; applying Lemma 2.1 (see the end of this section), we conclude that

$$\tau_+ |\nabla_- v(\tau_+, \tau_-)| \leq Cr \|\tau_+ \langle r \rangle^\epsilon G\|_{L_{t,x}^\infty}.$$

Now, we recall that G satisfies (1.11) and we note that, set

$$\tilde{B} = B + i(\partial^\mu A_\mu) + A^\mu A_\mu,$$

we have

$$\sum_{j \in \mathbb{Z}} 2^{-2j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j \tilde{B}\|_{L_{t,x}^\infty} \leq \delta_0, \quad (2.2)$$

as one easily deduces from (1.2) and (1.3). Hence we obtain

$$\begin{aligned} \tau_+ |\nabla_- v(\tau_+, \tau_-)| &\leq Cr \left(\|\tau_+ \langle r \rangle^\epsilon A_- \nabla_- v\|_{L_{t,x}^\infty} + \|\tau_+ \langle r \rangle^\epsilon r^{-1} A_- v\|_{L_{t,x}^\infty} \right. \\ &\quad \left. + \|\tau_+ \langle r \rangle^\epsilon \tilde{B} v\|_{L_{t,x}^\infty} + \|\tau_+ \langle r \rangle^\epsilon r F\|_{L_{t,x}^\infty} \right). \end{aligned} \quad (2.3)$$

Now, if we take $\epsilon \leq \epsilon_0$, we have

$$r \langle r \rangle^\epsilon \phi_j(r) |A_-(t, r)| \leq C 2^{-j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j A_-\|_{L_{t,x}^\infty} \quad (2.4)$$

(here and in the following, we assume that $C = C(\epsilon) > 0$ could change time by time), thus

$$\begin{aligned} r \|\tau_+ \langle r \rangle^\epsilon A_- \nabla_- v\|_{L_{t,x}^\infty} &\leq C \|\tau_+ \nabla_- v\|_{L_{t,x}^\infty} \sum_{j \in \mathbb{Z}} 2^{-j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j A_-\|_{L_{t,x}^\infty} \\ &\leq C \delta_0 \|\tau_+ \nabla_- v\|_{L_{t,x}^\infty}, \end{aligned} \quad (2.5)$$

where we have used the fact that $(\phi_j)_{j \in \mathbb{Z}}$ is a Paley-Littlewood partition of unity and property (1.12).

Moreover, $v(\tau_+, \tau_+) = 0$ because of (2.1), whence

$$v(\tau_+, \tau_-) = - \int_{\tau_-}^{\tau_+} \nabla_- v(\tau_+, s) ds$$

and, consequently,

$$|v(\tau_+, \tau_-)| \leq \int_{\tau_-}^{\tau_+} |\nabla_- v(\tau_+, s)| ds \leq r \|\nabla_- v\|_{L_{t,x}^\infty}. \quad (2.6)$$

Thus we have

$$\langle r \rangle^\epsilon \phi_j(r) |A_-(t, r) v(\tau_+, \tau_-)| \leq C 2^{-j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j A_-\|_{L_{t,x}^\infty} \|\nabla_- v\|_{L_{t,x}^\infty},$$

which implies

$$r \|\tau_+ \langle r \rangle^\epsilon r^{-1} A_- v\|_{L_{t,x}^\infty} \leq C \delta_0 \|\tau_+ \nabla_- v\|_{L_{t,x}^\infty}. \quad (2.7)$$

Similarly, from (2.6) and (2.2), we get

$$\begin{aligned} r \|\tau_+ \langle r \rangle^\epsilon \tilde{B} v\|_{L_{t,x}^\infty} &\leq C \|\tau_+ \nabla_- v\|_{L_{t,x}^\infty} \sum_{j \in \mathbb{Z}} 2^{-2j} \langle 2^{-j} \rangle^{\epsilon_0} \|\phi_j \tilde{B}\|_{L_{t,x}^\infty} \\ &\leq C \delta_0 \|\tau_+ \nabla_- v\|_{L_{t,x}^\infty}. \end{aligned}$$

Combining this estimate with (2.5) and (2.7) in (2.3), we deduce

$$\|\tau_+ \nabla_- v\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}, \quad (2.8)$$

provided δ_0 is sufficiently small. For instance, one can take δ_0 such that $4C^2 \delta_0 \leq 1$ (it is sufficient that $3C^2 \delta_0 < 1$).

From the definition of v , we have

$$r \nabla_- u = \nabla_- v + u \quad (2.9)$$

and, hence,

$$|\tau_+ r \nabla_- u| \leq |\tau_+ \nabla_- v| + |\tau_+ u|. \quad (2.10)$$

Now, thanks to the inequality in Lemma 2.2, we have

$$\begin{aligned} \|\tau_+ u\|_{L_{t,x}^\infty} &\leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F_1\|_{L_{t,x}^\infty} \\ &\leq C (\|\tau_+ r^2 \langle r \rangle^\epsilon A_- \nabla_- u\|_{L_{t,x}^\infty} + \|\tau_+ r^2 \langle r \rangle^\epsilon \tilde{B} u\|_{L_{t,x}^\infty}) \\ &\quad + C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty} \\ &\leq C \left\| \sum_{j \in \mathbb{Z}} r \langle r \rangle^{\epsilon_0} \phi_j A_- \right\|_{L_{t,x}^\infty} \|\tau_+ r \nabla_- u\|_{L_{t,x}^\infty} \\ &\quad + C \left\| \sum_{j \in \mathbb{Z}} r^2 \langle r \rangle^{\epsilon_0} \phi_j \tilde{B} \right\|_{L_{t,x}^\infty} \|\tau_+ u\|_{L_{t,x}^\infty} + C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty} \\ &\leq C \delta_0 (\|\tau_+ r \nabla_- u\|_{L_{t,x}^\infty} + \|\tau_+ u\|_{L_{t,x}^\infty}) + C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}, \end{aligned}$$

and thus

$$\|\tau_+ u\|_{L_{t,x}^\infty} \leq C (\delta_0 \|\tau_+ r \nabla_- u\|_{L_{t,x}^\infty} + \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}).$$

Combining this result with (2.8) in (2.10), we conclude

$$\|\tau_+ r \nabla_- u\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}, \quad (2.11)$$

provided $\delta_0 > 0$ small enough, that is, Lemma 1.1.

Now we use the fact that, because of (2.9), we have

$$|\tau_+ u| \leq |\tau_+ r \nabla_- u| + |\tau_+ \nabla_- v|; \quad (2.12)$$

combining this estimate with (2.11) and (2.8), we finally conclude

$$\|\tau_+ u\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}, \quad (2.13)$$

and also Theorem 1.1 is proven.

Now we prove the two lemmas that we have used previously in this section.

Lemma 2.1. *For each $\epsilon > 0$, there exists a positive constant $C = C(\epsilon)$ such that*

$$\int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\epsilon} ds \leq \frac{Cr}{\tau_+} \quad \forall \tau_- > 0.$$

Proof. We distinguish two cases.

case 1: $\tau_+ \geq 2\tau_-$. Note that, since $r = \tau_+ - \tau_- \geq \tau_+/2$, it is sufficient to prove that

$$\int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\epsilon} ds \leq C_0(\epsilon).$$

We observe that $s - \tau_- \geq s/2$ provided $s \geq 2\tau_-$, so

$$\begin{aligned} \int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\epsilon} ds &\leq \int_{\tau_-}^{2\tau_-+1} \langle s \rangle^{-1} ds + 2^\epsilon \int_{2\tau_-+1}^{\tau_++1} s^{-(1+\epsilon)} ds \\ &\leq \frac{\tau_- + 1}{\langle \tau_- \rangle} + 2^\epsilon \int_1^\infty s^{-(1+\epsilon)} ds \\ &\leq C_1(\epsilon). \end{aligned}$$

case 2: $\tau_+ < 2\tau_-$. We use the estimates $\langle s \rangle^{-1} < 2/\tau_+$ and $\langle s - \tau_- \rangle^{-\epsilon} \leq 1$ to get

$$\int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\epsilon} ds \leq \frac{2}{\tau_+} (\tau_+ - \tau_-) = \frac{2r}{\tau_+}. \quad (2.14)$$

This completes the proof. \square

Let us note that, because of the property (2.1) about the support of the solution v , we are interested only in the case $\tau_- > 0$.

In the case $A \equiv B \equiv 0$ (nonperturbed equation), we have the following version of the estimate in Theorem 1.1. It consists in a slight modification of estimate (1.8) shown in [7], p. 2269.

Lemma 2.2. *Let u be the solution to*

$$\begin{aligned} \square u &= F \quad (t, r) \in [0, \infty[\times \mathbb{R}^+, \\ u(0, r) &= \partial_t u(0, r) = 0 \quad r \in \mathbb{R}^+. \end{aligned}$$

Then, for every $\epsilon > 0$, there exists $C > 0$ such that

$$\|\tau_+ u\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}.$$

Proof. Note that u is the solution to (1.5) with $A \equiv B \equiv 0$. Then, from (2.3), we have

$$\tau_+ |\nabla_- v(\tau_+, \tau_-)| \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty}, \quad (2.15)$$

where $v = ru$. Using (2.6), we deduce

$$\tau_+ |u| = \tau_+ |v| r^{-1} \leq \|\tau_+ \nabla_- v\|_{L_{t,x}^\infty}$$

and hence the claim follows. \square

3. PROOF FOR THE DIRAC EQUATION

First of all, let us observe that a radial solution u to the Cauchy problem (1.13) is also a radial solution to the Cauchy problem

$$\begin{aligned} D_A^2 u &= D_A F, \\ u(0, r) &= \partial_t u(0, r) = 0, \end{aligned} \quad (3.1)$$

and that

$$D_A^2 = \gamma^\mu \gamma^\nu \partial_{\mu,A} \partial_{\nu,A} = \frac{\gamma^\mu \gamma^\nu}{2} (\{\partial_{\mu,A}, \partial_{\nu,A}\} + [\partial_{\mu,A}, \partial_{\nu,A}]), \quad (3.2)$$

where

$$\{X, Y\} = XY + YX, \quad [X, Y] = XY - YX$$

represent respectively the symmetric and the antisymmetric part of $2XY$. On one hand, we have

$$\begin{aligned} \frac{\gamma^\mu \gamma^\nu}{2} \{\partial_{\mu,A}, \partial_{\nu,A}\} &= \frac{\{\gamma^\mu, \gamma^\nu\}}{4} \{\partial_{\mu,A}, \partial_{\nu,A}\} \\ &= \frac{\eta^{\mu\nu}}{2} \{\partial_{\mu,A}, \partial_{\nu,A}\} \\ &= \frac{\partial_A^\nu \partial_{\nu,A} + \partial_A^\mu \partial_{\mu,A}}{2} = \square_A. \end{aligned}$$

On the other hand, since

$$[\partial_\mu, \partial_\nu] = [A_\mu, A_\nu] = 0,$$

we have also

$$\begin{aligned} \frac{\gamma^\mu \gamma^\nu}{2} [\partial_{\mu,A}, \partial_{\nu,A}] &= \frac{\gamma^\mu \gamma^\nu}{2i} ([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu]) \\ &= \frac{\gamma^\mu \gamma^\nu}{2i} ((\partial_\mu A_\nu) - (\partial_\nu A_\mu)). \end{aligned}$$

Consequently, setting

$$B = -\frac{\gamma^\mu \gamma^\nu}{2i} ((\partial_\mu A_\nu) - (\partial_\nu A_\mu)),$$

the Cauchy problem (3.1) can be recast in the form

$$\begin{aligned} (\square_A - B)u &= D_A F, \\ u(0, r) &= \partial_t u(0, r) = 0, \end{aligned}$$

that is the form of (1.5), the one for which Theorem 1.1 holds.

To conclude, we need two remarks. First, Theorem 1.1 can be easily generalized (with essentially the same proof) to the case of a system of wave equations where $u, F \in \mathbb{C}^N$, while $A^\mu \in \mathbb{R}^{N \times N}$ and $B \in \mathbb{C}^{N \times N}$ are matrices that satisfy the hypotheses of the theorem. In particular, this holds for $N = 4$.

Second, thanks to the decay assumptions on A and ∇A , that is the conditions (1.14) and (1.15), the smallness conditions on A^μ and B are satisfied, that is estimates similar to (1.2) and (1.3) hold.

Hence we can apply the generalization of Theorem 1.1 and get, for every $\epsilon \in]0, \epsilon_0]$, the existence of two positive constants δ and C (depending on ϵ) such that, for each $\delta_0 \in]0, \delta]$, one has

$$\|\tau_+ u\|_{L_{t,x}^\infty} \leq C \|\tau_+ r^2 \langle r \rangle^\epsilon D_A F\|_{L_{t,x}^\infty}, \quad (3.3)$$

where u is a radial solution to the Cauchy problem (1.13). Moreover, if A is essentially bounded; i.e.,

$$\|A(t, x)\|_{L_{t,x}^\infty} < \infty,$$

we have immediately

$$\|\tau_+ r^2 \langle r \rangle^\epsilon D_A F\|_{L_{t,x}^\infty} \leq C(\|\tau_+ r^2 \langle r \rangle^\epsilon F\|_{L_{t,x}^\infty} + \|\tau_+ r^2 \langle r \rangle^\epsilon \nabla F\|_{L_{t,x}^\infty}).$$

This concludes the proof of Theorem 1.2.

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DAVIDE CATANIA
DIPARTIMENTO DI MATEMATICA, FACOLTÀ DI INGEGNERIA, UNIVERSITÀ DI BRESCIA, VIA VALOTTI
9, 25133 BRESCIA, ITALY
E-mail address: `catania@ing.unibs.it`