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NEW APPROACH TO STREAMING SEMIGROUPS WITH MULTIPLYING BOUNDARY CONDITIONS

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Abstract. This paper concerns the generation of a C_0-semigroup by the streaming operator with general multiplying boundary conditions. A first approach, presented in [2], is based on the Hille-Yosida's Theorem. Here, we present a second approach based on the construction of the generated semigroup, without using the Hille-Yosida's Theorem.

1. Introduction

Let us consider a particle population (neutrons, photons, molecules of gas,...) in some domain of $\mathbb{R}^n$. Each particle is distinguished by its position $x \in X \subset \mathbb{R}^n$ and its directional velocity $v \in V \subset \mathbb{R}^n$. If we denote by $f(t, x, v)$ the density of particles having, at the time t, the position x with the directional velocity v, then particle population is governed by the following evolution equation

$$\frac{\partial f}{\partial t}(t) = -v \cdot \nabla_x f(t) =: T_K f(t), \tag{1.1}$$

where $(x, y) \in \Omega = X \times V$ and $t \geq 0$. The operator T_K is called the streaming operator describing the transport of particles and it is equipped with following general boundary conditions

$$f(t)\big|_{\Gamma_-} = K\big(f(t)\big|_{\Gamma_+}\big) \tag{1.2}$$

where $f(t)|_{\Gamma_-}$ (resp. $f(t)|_{\Gamma_+}$) is the incoming (resp. outgoing) particle flux which is the restriction of the density $f(t)$ on the subset Γ_- (resp. Γ_+) of $\partial X \times V$. The boundary operator K is linear and bounded on suitable function spaces. All of known boundary conditions (vacuum, specular reflections, periodic,...) are special examples of our general context. (see the next section for more explanations).

When $\|K\| \leq 1$, the existence of a strongly continuous semigroup has been investigated by several authors and important results have been cleared in [1, 7, 8].

However, the case $\|K\| \geq 1$ has been rarely studied and the first approach, based on Hille-Yosida's Theorem, is given in [2] according to some geometrical restrictions on X and V that we have expressed in the definition 2.1. Namely, the difficulty regarding the case $\|K\| > 1$ is linked to the increasing number of incoming particles.

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In this case, the time sojourn of particles in X may be arbitrary small and intuitively the boundary operator K does not take too much into account such as particles.

The motivation, of this present work, is to give a second approach when $\|K\| \geq 1$ without using the Hille-Yosida's Theorem. This approach is concerned by two steps. The first one is devoted to the construction of a C_0-semigroup. In the second one, we show that T_K is the infinitesimal generator of this semigroup.

To obtain our objective, we use our technics successfully applied in [3, 4]. We point out that this work is new and gives the explicit expression of the generated semigroup.

2. Estatement of the problem

We consider Banach space $L^p(\Omega)$ $(1 \leq p < \infty)$ with its natural norm

$$\|\varphi\|_p = \Big[\int_\Omega |\varphi(x,v)|^p dx d\mu\Big]^{1/p}, \tag{2.1}$$

where $\Omega = X \times V$ with $X \subset \mathbb{R}^n$ be a smoothly bounded open subset and $d\mu$ be a Radon measure on $\mathbb{R}^n$ with support V. We also consider the partial Sobolev space

$$W^p(\Omega) = \{\varphi \in L^p(\Omega),\ v \cdot \nabla_x \varphi \in L^p(\Omega)\},$$

with the norm $\|\varphi\|_{W^p(\Omega)} = [\|\varphi\|_p^p + \|v \cdot \nabla_x \varphi\|_p^p]^{1/p}$. We set $n(x)$ the outer unit normal at $x \in \partial X$, where ∂X is the boundary of X equipped with the measure of surface $d\gamma$. We denote

$$\Gamma = \partial X \times V, \quad \Gamma_0 = \{(x,v) \in \Gamma,\ v \cdot n(x) = 0\},$$
$$\Gamma_+ = \{(x,v) \in \Gamma,\ v \cdot n(x) > 0\}, \quad \Gamma_- = \{(x,v) \in \Gamma,\ v \cdot n(x) < 0\},$$

and suppose that $d\gamma d\mu(\Gamma_0) = 0$. For $(x,v) \in \Omega$, the time which a particle starting at x with velocity $-v$ needs until it reaches the boundary ∂X of X is denoted by

$$t(x,v) = \inf\{t > 0,\ x - tv \notin X\}.$$

Similarly, if $(x,v) \in \Gamma_+$ we set

$$\tau(x,v) = \inf\{t > 0,\ x - tv \notin X\}.$$

Now, we use the context of [2] as follows

Definition 2.1. The pair (X,V) is regular if

$$\tau_0 := \inf_{(x,v)\in\Gamma_+} \tau(x,v) > 0.$$

We also consider the trace spaces $L^p(\Gamma_\pm)$ equipped with the norm

$$\|\varphi\|_{L^p(\Gamma_\pm)} = \Big[\int_{\Gamma_\pm} |\varphi(x,v)|^p d\xi\Big]^{1/p},$$

where $d\xi = |v \cdot n(x)| d\gamma d\mu$. The first consequence of the regularity of the pair (X,V) is as follows.

Lemma 2.2 ([2])**.** *If the pair (X,V) is regular, then the trace applications*

$$\gamma_+ : W^p(\Omega) \longrightarrow L^p(\Gamma_+), \quad \gamma_- : W^p(\Omega) \longrightarrow L^p(\Gamma_-),$$

are linear and continuous.

Finally, if we consider the boundary operator

$$K \in \mathcal{L}\left(L^p(\Gamma_+), L^p(\Gamma_-)\right), \tag{2.2}$$

then the previous Lemma gives a sense to the operator

$$\begin{gathered} T_K\varphi = -v\cdot\nabla_x\varphi \quad \text{defined on the domain} \\ D(T_K) = \{\varphi \in W^p(X\times V),\ \gamma_-\varphi = K\gamma_+\varphi\}. \end{gathered}$$

We set $\|K\| := \|K\|_{\mathcal{L}(L^p(\Gamma_+),L^p(\Gamma_-))}$ for the rest of this article. If $K = 0$, the operator T_0 has properties that we summarize as follows.

Lemma 2.3. *The operator T_0, on $L^p(\Omega)$ ($p \geq 1$), generates a contraction C_0-semigroup $\{U_0(t)\}_{t\geq 0}$ given by*

$$U_0(t)\varphi(x,v) = \chi\left(t - t(x,v)\right)\varphi\left(x - tv, v\right), \tag{2.3}$$

where

$$\chi\left(t - t(x,v)\right) = \begin{cases} 1 & \text{if } t(x,v) - t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

We conclude this section with the following lemma that we will need later.

Lemma 2.4. *Suppose that the pair (X,V) is regular and let $\varphi \in W^p(\Omega)$ and $\lambda > 0$. If we set*

$$\begin{gathered} \Psi(x,v) = \epsilon_\lambda(x,v)\gamma_-\varphi(x - t(x,v)v, v), \\ \Phi = \varphi - \Psi, \end{gathered} \tag{2.5}$$

where $\epsilon_\lambda(x,v) = e^{-\lambda t(x,v)}$, then the following statements hold

1. *$\Psi \in W^p(\Omega)$ and $\Phi \in D(T_0)$;*
2. *the application $t \geq 0 \to \gamma_+[U_0(t)\varphi] \in L^p(\Gamma_+)$ is continuous.*

Proof. (1) Let $\varphi \in W^p(\Omega)$ and $\lambda > 0$. As we have $v\cdot\nabla_x\Psi + \lambda\Psi = 0$ with $\gamma_-\Psi = \gamma_-\varphi \in L^p(\Gamma_-)$, then a simple calculation gives us

$$\|v\cdot\nabla_x\Psi\|_p^p = \lambda\|\Psi\|_p^p \leq \lambda\Big[\frac{1}{p\lambda}\Big]^{1/p}\|\gamma_-\varphi\|^p_{L^p(\Gamma_-)} < \infty$$

which implies

$$\begin{gathered} \|\Psi\|_{W^p(\Omega)} = [\|\Psi\|_p^p + \|v\cdot\nabla_x\Psi\|_p^p]^{1/p} < \infty, \\ \|\Phi\|_{W^p(\Omega)} = \|\varphi - \Psi\|_{W^p(\Omega)} \leq \|\varphi\|_{W^p(\Omega)} + \|\Psi\|_{W^p(\Omega)} < \infty, \end{gathered}$$

and therefore Ψ and Φ belong to $W^p(\Omega)$. Furthermore, we trivially have $\gamma_-\Phi = \gamma_-(\varphi - \Psi) = \gamma_-\varphi - \gamma_-\varphi = 0$ and thus $\Phi \in D(T_0)$.

(2) Let $\varphi \in W^p(\Omega)$ and $\lambda > 0$. For all $h > 0$ and all $t \geq 0$ we have

$$\begin{aligned} &\|\gamma_+U_0(t+h)\varphi - \gamma_+U_0(t)\varphi\|_{L^p(\Gamma_+)} \\ &= \|\gamma_+U_0(t+h)\Psi - \gamma_+U_0(t)\Psi + \gamma_+U_0(t+h)\Phi - \gamma_+U_0(t)\Phi\|_{L^p(\Gamma_+)} \\ &\leq \|\gamma_+U_0(t+h)\Psi - \gamma_+U_0(t)\Psi\|_{L^p(\Gamma_+)} + \|\gamma_+U_0(t+h)\Phi - \gamma_+U_0(t)\Phi\|_{L^p(\Gamma_+)} \\ &=: I_1(h) + I_2(h). \end{aligned} \tag{2.6}$$

As $\Phi \in D(T_0)$, Lemmas 2.2 and 2.3, imply

$$\begin{aligned}
\lim_{h\to 0} I_2(h) &= \lim_{h\to 0} \|\gamma_+ U_0(t+h)\Phi - \gamma_+ U_0(t)\Phi\|_{L^p(\Gamma_+)} \\
&\le \|\gamma_+\|_{\mathcal{L}(D(T_0),L^p(\Gamma_+))} \lim_{h\to 0} \|U_0(t+h)\Phi - U_0(t)\Phi\|_{D(T_0)} \\
&= 0.
\end{aligned}$$

Next, a simple calculation shows that

$$\begin{aligned}
\lim_{h\to 0} I_1(h)^p &= \lim_{h\to 0} \|\gamma_+ U_0(t+h)\Psi - \gamma_+ U_0(t)\Psi\|^p_{L^p(\Gamma_+)} \\
&= \lim_{h\to 0} \int_{\Gamma_+} |\chi(t+h-t(x,v))e^{\lambda(t+h)} - \chi(t-t(x,v))e^{\lambda t}|^p |\Psi(x,v)|^p d\xi \\
&= 0
\end{aligned}$$

This completes the proof. $\square$

3. Construction of the semigroup

In this section, we construct the semigroup $\{U_K(t)\}_{t\ge 0}$ when $\|K\| \ge 1$. In order to show Theorem 3.5 which is the main result, we begin by

Lemma 3.1. *The following Cauchy's problem*

$$\begin{gathered}
\frac{du}{dt} + v\cdot\nabla_x u = 0, \quad (t,x,v) \in (0,\infty)\times\Omega; \\
\gamma_- u = f_- \in L^p\left(\mathbb{R}_+, L^p(\Gamma_-)\right); \\
u(0) = f_0 \in L^p(\Omega),
\end{gathered} \tag{3.1}$$

admits a unique solution $u = u(t,x,v) = u(t)(x,v)$. Furthermore, for all $t \ge 0$, we have

$$\|u(t)\|_p^p + \int_0^t \|\gamma_+ u(s)\|^p_{L^p(\Gamma_+)} ds = \int_0^t \|f_-(s)\|^p_{L^p(\Gamma_-)} ds + \|f_0\|_p^p. \tag{3.2}$$

Proof. Let $f_- \in L^p\left(\mathbb{R}_+, L^p(\Gamma_-)\right)$ and $f_0 \in L^p(\Omega)$. First. Using [6, pp.1124] it follows that Cauchy's problem $\mathrm{P(f_-, f_0)}$ has a unique solution given by

$$u(t,x,v) = \xi\left(t - t(x,v)\right) f_-(t-t(x,v), x - t(x,v)v, v) + U_0(t) f_0(x,v). \tag{3.3}$$

where ξ is given in Lemma 3.3. Next. Multiplying first equation of Cauchy's problem $(\mathrm{P})(f_-, f_0)$ by $\operatorname{sgn} u|u|^{p-1}$ and using

$$\operatorname{sgn} u|u|^{p-1} v\cdot\nabla_x u = \frac{1}{p} v\cdot\nabla_x |u|^p,$$

with an integrating over Ω, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d\|u(t)\|_p^p}{dt} &= \frac{1}{p}\int_{\Gamma_-} |\gamma_- u(t,x,v)|^p d\xi - \frac{1}{p}\int_{\Gamma_+} |\gamma_+ u(t,x,v)|^p d\xi \\
&= \frac{1}{p}\int_{\Gamma_-} |f_-(t,x,v)|^p d\xi - \frac{1}{p}\int_{\Gamma_+} |\gamma_+ u(t,x,v)|^p d\xi \\
&= \frac{1}{p}\|f_-(t)\|^p_{L^p(\Gamma_-)} - \frac{1}{p}\|\gamma_+ u(t)\|^p_{L^p(\Gamma_+)}
\end{aligned}$$

which implies, by integration with respect to t, that

$$\|u(t)\|_p^p - \|f_0\|_p^p = \int_0^t \|f_-(s)\|^p_{L^p(\Gamma_-)} ds - \int_0^t \|\gamma_+ u(s)\|^p_{L^p(\Gamma_+)} ds$$

and achieves the proof. □

Remark 3.2. In the sequel, we use the fact that all expression on the form of (3.3) is automatically solution of Cauchy's problem $\mathrm{P}(\mathrm{f}_-, \mathrm{f}_0)$.

The second consequence of the regularity of the pair (X, V) is as follows.

Lemma 3.3. *Suppose that the pair (X, V) is regular and let*

$$\xi\left(t - t(x, v)\right) = \begin{cases} 1 & \text{if } t(x, v) - t \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $0 \le t \le \tau_0$, then we have $\gamma_+ \xi\left(t - t(\cdot, \cdot)\right) = 0$.

Proof. By the regularity of the pair (X, V), we have

$$0 \le t \le \tau_0 = \inf_{(x,v) \in \Gamma_+} \tau(x, v) \le \tau(x, v)$$

a.e. $(x, v) \in \Gamma_+$, and therefore

$$\gamma_+ \left[\xi(t - t(\cdot, \cdot))\right](x, v) = \xi(t - \tau(x, v)) = 0,$$

a.e. $(x, v) \in \Gamma_+$. □

Lemma 3.4. *Suppose that the pair (X, V) is regular. For all $0 \le t \le \tau_0$, the operator $A_K(t)$ given by*

$$A_K(t)\varphi(x, v) = \xi\left(t - t(x, v)\right) K\left[\gamma_+ U_0\left(t - t(x, v)\right)\varphi\right](x - t(x, v)v, v)$$

is a linear and bounded from $L^p(\Omega)$ into itself. Furthermore, we have

1. *$A_K(0) = 0$;*
2. *$\lim_{t \searrow 0} \|A_K(t)\varphi\|_p = 0$ for all $\varphi \in L^p(\Omega)$;*
3. *$\gamma_+ A_K(t) = 0$ for $0 \le t \le \tau_0$;*
4. *$A_K(t)A_K(s) = 0$ for all $0 \le t, s \le \tau_0$ such that $0 \le t + s \le \tau_0$.*

Proof. Let $0 \le t \le \tau_0$ and $\varphi \in L^p(\Omega)$. As $u(t) = A_K(t)\varphi$ is the solution of Cauchy's problem $\mathrm{P}(f_- = K\left[\gamma_+ U_0(\cdot)\varphi\right], f_0 = 0)$ then (3.2) and the boundedness of K implies

$$\|A_K(t)\varphi\|_p^p \le \int_0^t \|K\left[\gamma_+ U_0(s)\varphi\right]\|_{L^p(\Gamma_-)}^p ds \le \|K\|^p \int_0^t \|\gamma_+ U_0(s)\varphi\|_{L^p(\Gamma_+)}^p.$$

However, $u(t) = U_0(t)\varphi$ is solution of Cauchy's problem (3.1) with $f_- = 0, f_0 = \varphi$, and therefore (3.2) implies

$$\int_0^t \|\gamma_+ U_0(s)\varphi\|_{L^p(\Gamma_+)}^p = \|\varphi\|_p^p - \|U_0(t)\varphi\|_p^p. \tag{3.4}$$

From the previous two relations we obtain

$$\|A_K(t)\varphi\|_p^p \le \|K\|^p \left[\|\varphi\|_p^p - \|U_0(t)\varphi\|_p^p\right] \le \|K\|^p \|\varphi\|_p^p$$

which implies that $A_K(t)\varphi \in L^p(\Omega)$ and the boundedness of the operator $A_K(t)$ follows. Points (1) and (2) follow from the fact that $\{U_0(t)\}_{t \ge 0}$ is a C_0-semigroup.

(3) This point obviously follows from previous Lemma.

(4) Let $0 \le t, s \le \tau_0$ such that $0 \le t+s \le \tau_0$ and $\varphi \in L^p(\Omega)$. A simple calculation shows that the expression of $A_K(t)A_K(s)\varphi$ contains the following function

$$\alpha(x, v, x', v') := \xi\Big(s - t\Big(x' - \big(t - t(x, v)\big)v', v'\Big)\Big)$$

for a.e $(x,v)\in\Omega$ and a.e $(x',v')\in\Gamma_+$. Using the definition of ξ in previous Lemma, we get that

$$\begin{aligned}\alpha(x,v,x',v')=0 &\Longleftrightarrow s<t\Big(x'-\big(t-t(x,v)\big)v',v'\Big)\\ &\Longleftrightarrow s<\tau(x',v')-(t-t(x,v))\\ &\Longleftrightarrow s+t<\tau(x',v')+t(x,v)\end{aligned}$$

for a.e $(x,v)\in\Omega$ and a.e $(x',v')\in\Gamma_+$. But, the regularity of the pair (X,V) in the sense of Definition 2.1 gives us

$$t+s\le\tau_0=\inf_{(x',v')\in\Gamma_+}\tau(x',v')\le\tau(x',v')<\tau(x',v')+t(x,v)$$

for a.e. $(x,v)\in\Omega$ and a.e. $(x',v')\in\Gamma_+$ which implies that $\alpha(\cdot,\cdot,\cdot,\cdot)=0$ and therefore $A_K(t)A_K(s)=0$. The fourth point is proved. □

The main result of this section is given as follows.

Theorem 3.5. *Suppose that the pair (X,V) is regular. The family of operators $\{U_K(t)\}_{t\ge0}$ defined by*

$$\begin{gathered}U_K(t)=\left[U_0(\tau_0)+A_K(\tau_0)\right]^n\left[U_0(r)+A_K(r)\right],\\ \text{if } t=n\tau_0+r \text{ with } 0\le r<\tau_0 \text{ and } n\in\mathbb{N},\end{gathered}\tag{3.5}$$

is a C_0-semigroup on $L^p(\Omega)$. Furthermore, we have

$$\begin{gathered}U_K(t)\varphi(x,v)=U_0(t)(x,v)+\\ \xi\left(t-t(x,v)\right)K\left[\gamma_+U_K\left(t-t(x,v)\right)\varphi\right](x-t(x,v)v,v)\end{gathered}\tag{3.6}$$

for all $t\ge0$, a.e. $(x,v)\in\Omega$ and all $\varphi\in L^p(\Omega)$, where ξ is given in Lemma (3.3).

Proof. Note that from previous Lemma and Lemma 2.3 the operator $U_0(t)+A_K(t)$ $(0\le t\le\tau_0)$ is a linear bounded from $L^p(\Omega)$ into itself. Thus $U_K(t)$ is also linear bounded for all $t\ge0$, $U_K(0)=U_0(0)+A_K(0)=I$. Furthermore, if $t\le\tau_0$ then we trivially have

$$\lim_{t\searrow0}\|U_K(t)\varphi-\varphi\|_p=\lim_{t\searrow0}\|U_0(t)\varphi-\varphi\|_p+\lim_{t\searrow0}\|A_K(t)\varphi t\|_p=0,$$

for all $\varphi\in L^p(X\times V)$. Now, let us show that $U_K(t)U_K(s)=U_K(t+s)$ for all $t,s\ge0$.

First, note that if $0\le t,s\le\tau_0$ such that $0\le t+s\le\tau_0$, a simple calculation shows that $U_0(t)A_K(s)+A_K(t)U_0(s)=A_K(t+s)$ and therefore

$$\begin{aligned}U_K(t)U_K(s)&=\left[U_0(t)+A_K(t)\right]\left[U_0(s)+A_K(s)\right]\\ &=U_0(t+s)+U_0(t)A_K(s)+A_K(t)U_0(s)+A_K(t)A_K(s)\\ &=U_0(t+s)+A_K(t+s)\\ &=U_K(t+s),\end{aligned}\tag{3.7}$$

where we have used the relation $A_K(t)A_K(s)=0$ in previous lemma. Thus

$$U_K(t)U_K(s)=U_K(t+s),\tag{3.8}$$

for all $0\le t,s\le\tau_0$ such that $0\le t+s\le\tau_0$.

Next, for all $t, s \geq 0$, there exists $n_t, n_s \in \mathbb{N}$ and $0 \leq r_t, r_s < \tau_0$ such that $t = n_t\tau_0 + r_t$ and $t = n_s\tau_0 + r_s$. In this case (3.8) implies

$$\begin{aligned} U_K(t)U_K(s) &= [U_K(\tau_0)]^{n_t}U_K(r_t)[U_K(\tau_0)]^{n_s}U_K(r_t) \\ &= [U_K(\tau_0/2)]^{2n_t}[U_K(r_t/2)]^2[U_K(\tau_0/2)]^{2n_s}[U_K(r_t/2)]^2 \\ &= [U_K(\tau_0/2)]^{2n_t}[U_K(\tau_0/2)]^{2n_s}[U_K(r_t/2)]^2[U_K(r_t/2)]^2 \\ &= [U_K(\tau_0)]^{n_t}[U_K(\tau_0)]^{n_s}[U_K((r_t+r_t)/2)]^2 \\ &= [U_K(\tau_0)]^{n_t+n_s}[U_K((r_t+r_t)/2)]^2 \end{aligned}$$

because

$$0 \leq (\tau_0/2), (r_t/2), (r_s/2), (r_t+\tau_0)/2, (r_s+\tau_0)/2, (r_t+r_s)/2 < \tau_0$$

Now, if $r_t + r_s < \tau_0$ then $(((r_t + r_s)/2) + ((r_t + r_s)/2)) < \tau_0$ which implies, by (3.8), that $[U_K((r_t+r_t)/2)]^2 = U_K(r_t+r_t)$ and therefore $U_K(t)U_K(s) = U_K(t+s)$ because $t+s = (n_t+n_s)\tau_0 + (r_t+r_s)$ with $0 \leq (r_t+r_s) < \tau_0$.

If $\tau_0 \leq r_t + r_s < 2\tau_0$ then $r_t + r_s = \tau_0 + r$ with $0 \leq r < \tau_0$. As we have,

$$0 \leq (r_t/2), (r_s/2), (r_t+r_s)/2, (r/2) < \tau_0$$

then (3.8) implies

$$\begin{aligned} U_K(t)U_K(s) &= [U_K(\tau_0)]^{n_t+n_s}[U_K((r_t+r_t)/2)]^2 \\ &= [U_K(\tau_0)]^{n_t+n_s}[U_K((\tau_0/2)+(r/2))]^2 \\ &= [U_K(\tau_0)]^{n_t+n_s}[U_K(\tau_0/2)]^2[U_K(r/2)]^2 \\ &= [U_K(\tau_0)]^{n_t+n_s}U_K(\tau_0)U_K(r) \\ &= [U_K(\tau_0)]^{n_t+n_s+1}U_K(r) \\ &= U_K(t+s) \end{aligned}$$

because $(t+s) = (n_t+n_s)\tau_0+(r_t+r_s) = (n_t+n_s)\tau_0+(\tau_0+r) = (n_t+n_s+1)\tau_0+r$ with $0 \leq r < \tau_0$.

Now, let us show (3.6). If we denote

$$\overline{A}_K(t)\varphi(x,v) = \xi\left(t - t(x,v)\right) K\left[\gamma_+ U_K\left(t - t(x,v)\right)\varphi\right](x - t(x,v)v, v)\,,$$

then, we have to show the following formula

$$U_K(t) = U_0(t) + \overline{A}_K(t), \quad t \geq 0. \tag{3.9}$$

Let $\varphi \in W^p(\Omega)$. If $0 \leq t < \tau_0$, the third point of Lemma 3.4 infers that $\gamma_+U_K(t)\varphi = \gamma_+U_0(t)\varphi + \gamma_+A_K(t)\varphi = \gamma_+U_0(t)\varphi$ which implies

$$\gamma_+U_K(t-t(x,v))\varphi = \gamma_+U_0(t-t(x,v))\varphi$$

for a.e. $(x,v) \in \Omega$ such that $t(x,v) \leq t$ and therefore

$$\begin{aligned} \overline{A}_K(t)\varphi(x,v) &= \xi(t-t(x,v))K\left[\gamma_+U_K(t-t(x,v))\varphi\right](x-t(x,v)v,v) \\ &= \xi(t-t(x,v))K\left[\gamma_+U_0(t-t(x,v))\varphi\right](x-t(x,v)v,v) \\ &= A_K(t)\varphi(x,v) \end{aligned}$$

for a.e. $(x,v) \in \Omega$ and all $0 \leq t < \tau_0$. Thus, we have

$$U_K(t)\varphi = U_0(t)\varphi + A_K(t)\varphi = U_0(t)\varphi + \overline{A}_K(t)\varphi$$

for all $0 \leq t < \tau_0$ and all $\varphi \in W^p(\Omega)$. Now the density of $W^p(\Omega)$ in $L^p(\Omega)$ implies that (3.9) holds for all $0 \leq t < \tau_0$.

Next. Suppose that (3.9) holds for $(n-1)\tau_0 \le t < n\tau_0$ and all $\varphi \in L^p(\Omega)$. If $n\tau_0 \le t < (n+1)\tau_0$ then $(n-1)\tau_0 \le t-\tau_0 < n\tau_0$ and therefore

$$\begin{aligned} U_K(t)\varphi &= U_K(t-\tau_0)U_K(\tau_0)\varphi \\ &= \big[U_0(t-\tau_0) + \overline{A}_K(t-\tau_0)\big]\big[U_0(\tau_0) + \overline{A}_K(\tau_0)\big]\varphi \\ &= U_0(t)\varphi + U_0(t-\tau_0)\overline{A}_K(\tau_0)\varphi + \overline{A}_K(t-\tau_0)U_K(\tau_0)\varphi. \end{aligned}$$

Using the definition of ξ given Lemma 3.3, a simple calculation implies $U_0(t-\tau_0)\overline{A}_K(\tau_0)\varphi = 0$ and $\overline{A}_K(t-\tau_0)U_K(\tau_0)\varphi = \overline{A}_K(t)\varphi$. Thus

$$U_K(t)\varphi = U_0(t)\varphi + \overline{A}_K(t)\varphi$$

which prove (3.9) for $n\tau_0 \le t < (n+1)\tau_0$ and therefore for all $t \ge 0$. □

4. Generation Theorem

To show the main result of this work, we needed the following result.

Lemma 4.1. *Suppose that the pair (X,V) is regular. If $\varphi \in W^p(\Omega)$ and $\lambda > 0$, then we have*

$$\lim_{t\searrow 0} \Big\| \frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi \Big\|_p = \|K\gamma_+\varphi - \gamma_-\varphi\|_{L^p(\Gamma_-)}$$

where Ψ is given by (2.5).

Proof. Let $0 < t \le \tau_0$ and $\varphi \in W^p(\Omega)$. Using (2.5), a simple calculation gives us

$$\begin{aligned} &\Big[\frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi\Big](x,v) \\ &= \Big(\frac{e^{\lambda t}-1}{t} - \lambda\Big)\Psi(x,v) \\ &\quad + \frac{1}{t}\Big(A_K(t)\varphi(x,v) - \xi(t-t(x,v))e^{\lambda t}\epsilon_\lambda(x,v)\gamma_-\varphi\left(x-t(x,v)v,v\right)\Big) \\ &=: I_1(t)\varphi + I_2(t)\varphi \end{aligned}$$

a.e. $(x,v) \in \Omega$, which implies

$$\lim_{t\searrow 0}\|I_2(t)\varphi\|_p - \lim_{t\searrow 0}\|I_1(t)\varphi\|_p \le \lim_{t\searrow 0}\Big\|\frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi\Big\|_p$$

and

$$\lim_{t\searrow 0}\Big\|\frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi\Big\|_p \le \lim_{t\searrow 0}\|I_2(t)\varphi\|_p + \lim_{t\searrow 0}\|I_1(t)\varphi\|_p.$$

As we obviously have $\lim_{t\to 0} I_1(t) = 0$, then we get

$$\lim_{t\searrow 0}\|\frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi\|_p = \lim_{t\searrow 0}\|I_2(t)\varphi\|_p. \tag{4.1}$$

But $u(t) = tI_2(t)\varphi$ is the solution of Cauchy's problem $\mathrm{P}(f_- = K\left[\gamma_+U_0(t)\varphi\right] - e^{\lambda t}\gamma_-\varphi, f_0 = 0)$, thus (3.2) implies

$$\|I_2(t)\varphi\|_p^p = \frac{1}{t}\int_0^t \|K\left[\gamma_+U_0(s)\varphi\right] - e^{\lambda s}\gamma_-\varphi\|_{L^p(\Gamma_-)}^p ds - \frac{1}{t}\int_0^t \|\gamma_+I_2(t)\varphi\|_{L^p(\Gamma_-)}^p ds.$$

From the regularity of the pair (X, V) we get, by Lemma 3.3 and the third point of Lemma 3.4, that $\gamma_+ I_2(t)\varphi = 0$ and therefore the previous relation becomes

$$\|I_2(t)\varphi\|_p^p = \frac{1}{t}\int_0^t \|K[\gamma_+ U_0(s)\varphi] - e^{\lambda s}\gamma_-\varphi\|_{L^p(\Gamma_-)}^p ds. \tag{4.2}$$

Using the boundedness of K and the second point of Lemma 2.4 we obtain the continuity of the application

$$0 \le t \le \tau_0 \longrightarrow K[\gamma_+ U_0(s)\varphi] - e^{\lambda s}\gamma_-\varphi \in L^p(\Gamma_-)$$

for all $\varphi \in W^p(\Omega)$ and therefore (4.2) becomes

$$\lim_{t\searrow 0} \|I_2(t)\varphi\|_p^p = \|K[\gamma_+\varphi] - \gamma_-\varphi\|_{L^p(\Gamma_-)}^p. \tag{4.3}$$

which achieves the proof by (4.1). $\square$

Now, we are able to state the main result of this work.

Theorem 4.2. *Suppose that the pair (X, V) is regular. Then the operator T_K given by*

$$\begin{aligned} &T_K\varphi(x,v) = -v\cdot\nabla_x\varphi(x,v), \quad \text{on the domain} \\ &D(T_K) = \{\varphi \in W^p(\Omega),\ \gamma_-\varphi = K\gamma_+\varphi\} \end{aligned} \tag{4.4}$$

is the infinitesimal generator of the C_0-semigroup $\{U_K(t)\}_{t\ge 0}$ satisfying

$$U_K(t)\varphi(x,v) = U_0(t)(x,v) + \xi(t - t(x,v))\,K[\gamma_+ U_K(t - t(x,v))\varphi](x - t(x,v)v, v) \tag{4.5}$$

for all $t \ge 0$ and a.e. $(x,v) \in \Omega$ and all $\varphi \in L^p(\Omega)$, where ξ is given the Lemma 3.3. Furthermore, if $\|K\| \ge 1$, then we have

$$\|U_K(t)\|_{\mathcal{L}(L^p(\Omega))} \le \|K\| \exp\big(\frac{t}{\tau_0}\ln\|K\|\big), \quad t \ge 0. \tag{4.6}$$

Proof. Note that the existence of the C_0-semigroup $\{U_K(t)\}_{t\ge 0}$ and (4.5) are already proved in Theorem 3.5. Let us shown that the operator T_K is the generator of our semigroup.

Let $0 < t < \tau_0$ and $\varphi \in W^p(\Omega)$. Using (2.5) we easily get

$$\big[\frac{U_K(t)\varphi - \varphi}{t} + v\cdot\nabla_x\varphi\big] = \big[\frac{U_0(t)\Phi - \Phi}{t} - T_0\Phi\big] + \frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi \tag{4.7}$$

which implies

$$\begin{aligned} &\big\|\frac{U_K(t)\varphi + \varphi}{t} + v\cdot\nabla_x\varphi\big\|_p \\ &\le \big\|\frac{U_0(t)\Phi - \Phi}{t} - T_0\Phi\big\|_p + \big\|\frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi\big\|_p \end{aligned}$$

and

$$\begin{aligned} &\big\|\frac{A_K(t)\varphi + U_0(t)\Psi - \Psi}{t} - \lambda\Psi\big\|_p - \big\|\frac{U_0(t)\Phi - \Phi}{t} - T_0\Phi\|_p \\ &\le \big\|\frac{U_K(t)\varphi - \varphi}{t} + v\cdot\nabla_x\varphi\big\|_p. \end{aligned}$$

Lemmas 2.3 and 4.1 imply

$$\lim_{t\searrow 0}\big\|\frac{U_K(t)\varphi - \varphi}{t} + v\cdot\nabla_x\varphi\big\|_p = \lim_{t\searrow 0}\|K\gamma_+\varphi - \gamma_-\varphi\|_{L^p(\Gamma_-)}^p \tag{4.8}$$

which implies

$$\varphi \in D(T_K) \Longleftrightarrow K\gamma_+\varphi - \gamma_-\varphi = 0 \Longleftrightarrow \lim_{t\searrow 0} \Big\| \frac{U_K(t)\varphi - \varphi}{t} + v \cdot \nabla_x \varphi \Big\|_p = 0$$

and therefore (4.8) gives us

$$\lim_{t\searrow 0} \Big\| \frac{U_K(t)\varphi - \varphi}{t} + T_K\varphi \Big\|_p = 0.$$

Thus T_K is the generator of the semigroup $\{U_K(t)\}_{t\geq 0}$. Now, let us show (4.6).

Let $0 \leq t \leq \tau_0$ and $\varphi \in D(T_K)$. As, $u(t) = U_K(t)\varphi = U_0(t)\varphi + A_K(t)\varphi$ is the solution of Cauchy's problem $\mathrm{P}(f_- = K[\gamma_+ U_0(t)\varphi], f_0 = \varphi)$, then (3.2) and the boundedness of the operator K infer that

$$\begin{aligned}
\|U_K(t)\varphi\|_p^p &= \int_0^t \|K[\gamma_+ U_0(s)\varphi]\|_{L^p(\Gamma_-)}^p ds + \|\varphi\|_p^p - \int_0^t \|\gamma_+ U_0(s)\varphi\|_{L^p(\Gamma_+)}^p ds \\
&\leq [\|K\|^p - 1] \int_0^t \|\gamma_+ U_0(s)\varphi\|_{L^p(\Gamma_+)}^p ds + \|\varphi\|_p^p
\end{aligned}$$

where we have used the third point of Lemma 3.4. Using (3.4) and the fact that $\|K\| \geq 1$, the previous relation becomes

$$\|U_K(t)\varphi\|_p^p \leq [\|K\|^p - 1] \|\varphi\|_p^p + \|\varphi\|_p^p = \|K\|^p \|\varphi\|_p^p$$

and therefore

$$\|U_K(t)\|_{\mathcal{L}(L^p(\Omega))} \leq \|K\|, \quad \text{for all } 0 \leq t \leq \tau_0,$$

because of the density of $D(T_K)$ in $L^p(\Omega)$. Now, for all $t \geq 0$, there exists $n \in \mathbb{N}$ and $0 \leq r < \tau_0$ such that $t = n\tau_0 + r$. Using previous relation and (3.5) we get

$$\begin{aligned}
\|U_K(t)\|_{\mathcal{L}(L^p(\Omega))} &= \| [U_K(\tau_0)]^n U_K(r)\|_{\mathcal{L}(L^p(\Omega))} \\
&\leq \|U_K(\tau_0)\|_{\mathcal{L}(L^p(\Omega))}^n \|U_K(r)\|_{\mathcal{L}(L^p(\Omega))} \\
&\leq \|K\|^n \|K\| \\
&\leq \|K\|^{t/\tau_0} \|K\|.
\end{aligned}$$

which prove (4.6) and completes the proof. □

Remark 4.3. Recall that the case $\|K\| < 1$ is already studied in [5] without the hypothesis : (X, V) is regular.

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