QUASILINEAR NONLOCAL INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we study the existence of mild solutions for quasilinear integrodifferential equations with nonlocal conditions in Banach spaces. The results are established by using Hausdorff’s measure of noncompactness.

1. Introduction

In this paper, we discuss the existence of mild solution of the following nonlinear integrodifferential equation with nonlocal condition

\[
\frac{du(t)}{dt} = A(t, u)u + \int_0^t f(t, s, u(s))ds, \quad t \in [0, b],
\]

\[
u(0) = g(u) + u_0,
\]

where \( f : [0, b] \times [0, b] \times X \to X \) and \( A : [0, b] \times X \to X \) are continuous functions, \( g : C([0, b]; X) \to X \), \( u_0 \in X \) and \( X \) is a real Banach space with norm \( \| \cdot \| \).

The notion of “nonlocal condition” has been introduced to extend the study of the classical initial value problems; see, e.g. [4, 8, 10, 11, 19]. It is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial time. The study of abstract nonlocal initial value problems was initiated by Byszewski, we refer to some of the papers below. Byszewski [6, 7], Byszewski and Lasnikauthem [9] give the existence and uniqueness of mild solutions and classical solutions when \( f \) and \( g \) satisfy Lipschitz-type conditions. Subsequently, many authors are devoted to studying of nonlocal problems. See [1, 2, 12, 13, 15, 20] for the references and remarks about the advantage of the nonlocal problems over the classical initial value problems.

This article is motivated by the recent paper of Chandrasekaran [10]. We use some hypotheses in [10], and using the method of Hausdorff’s measure of noncompactness, we give the existence of mild solutions of quasilinear integrodifferential

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equations with nonlocal conditions (1.1)–(1.2). Our results improve and extend some corresponding results in [2, 7, 8, 10, 15].

2. Preliminaries

Throughout this paper $\mathbb{X}$ will represent a Banach space with norm $\| \cdot \|$. Denoted $\mathcal{C}([0, b]; \mathbb{X})$ by the space of $\mathbb{X}$-valued continuous functions on $[0, b]$ with the norm $\| u \| = \sup \{ \| u(t) \|, t \in [0, b] \}$ for $u \in \mathcal{C}([0, b]; \mathbb{X})$, and denoted $\mathcal{L}(0, b; \mathbb{X})$ by the space of $\mathbb{X}$-valued Bochner integrable functions on $[0, b]$ with the norm $\| u \|_L = \int_0^b \| u(t) \| \, dt$.

The Hausdorff’s measure of noncompactness $\beta_\mathcal{Y}$ is defined by $\beta_\mathcal{Y}(B) = \inf \{ r > 0, B \text{ can be covered by finite number of balls with radii } r \}$ for bounded set $B$ in a Banach space $\mathcal{Y}$.

**Lemma 2.1** (Darbo-Sadovskii [3]). Let $\mathcal{Y}$ be a real Banach space and $B, C \subseteq \mathcal{Y}$ be bounded, with the following properties:

1. $B$ is pre-compact if and only if $\beta_\mathcal{Y}(B) = 0$;
2. $\beta_\mathcal{Y}(B) = \beta_\mathcal{Y} (\overline{B}) = \beta_\mathcal{Y}(\text{conv } B)$, where $\overline{B}$ and conv $B$ mean the closure and convex hull of $B$ respectively;
3. $\beta_\mathcal{Y}(B) \leq \beta_\mathcal{Y}(C)$, where $B \subseteq C$;
4. $\beta_\mathcal{Y}(B + C) \leq \beta_\mathcal{Y}(B) + \beta_\mathcal{Y}(C)$, where $B + C = \{ x + y : x \in B, y \in C \}$;
5. $\beta_\mathcal{Y}(B \cup C) \leq \max \{ \beta_\mathcal{Y}(B), \beta_\mathcal{Y}(C) \}$;
6. $\beta_\mathcal{Y}(\lambda B) \leq |\lambda| \beta_\mathcal{Y}(B)$ for any $\lambda \in \mathbb{R}$;
7. If the map $Q : D(Q) \subseteq \mathcal{Y} \to \mathcal{Z}$ is Lipschitz continuous with constant $k$, then $\beta_\mathcal{Z}(QB) \leq k \beta_\mathcal{Y}(B)$ for any bounded subset $B \subseteq D(Q)$, where $\mathcal{Z}$ be a Banach space;
8. $\beta_\mathcal{Y}(B) = \inf \{ d_\mathcal{Y}(B, C) ; C \subseteq \mathcal{Y} \text{ is precompact } \} = \inf \{ d_\mathcal{Y}(B, C) ; C \subseteq \mathcal{Y} \text{ is finite valued } \}$, where $d_\mathcal{Y}(B, C)$ means the nonsymmetric (or symmetric) Hausdorff distance between $B$ and $C$ in $\mathcal{Y}$;
9. If $\{ W_n \}_{n=1}^{+\infty}$ is decreasing sequence of bounded closed nonempty subsets of $\mathcal{Y}$ and $\lim_{n \to +\infty} \beta_\mathcal{Y}(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in $\mathcal{Y}$.

The map $Q : W \subseteq \mathcal{Y} \to \mathcal{Y}$ is said to be a $\beta_\mathcal{Y}$-contraction if there exists a positive constant $k < 1$ such that $\beta_\mathcal{Y}(Q(B)) \leq k \beta_\mathcal{Y}(B)$ for any bounded closed subset $B \subseteq W$, where $\mathcal{Y}$ is a Banach space.

**Lemma 2.2** (Darbo-Sadovskii [3]). If $W \subseteq \mathcal{Y}$ is bounded closed and convex, the continuous map $Q : W \to W$ is a $\beta_\mathcal{Y}$-contraction, then the map $Q$ has at least one fixed point in $W$.

In this paper we denote by $\beta$ the Hausdorff’s measure of noncompactness of $\mathbb{X}$ and denote $\beta_\mathcal{C}$ by the Hausdorff’s measure of noncompactness of $\mathcal{C}([0, b]; \mathbb{X})$. To discuss the existence, we need the following Lemmas in this paper.

**Lemma 2.3** (Darbo-Sadovskii [3]). If $W \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded, then $\beta(W(t)) \leq \beta_\mathcal{C}(W)$ for all $t \in [0, b]$, where $W(t) = \{ u(t) ; u \in W \} \subseteq \mathbb{X}$. Furthermore if $W$ is equiuniformly integrable on $[a, b]$, then $\beta(W(t))$ is continuous on $[a, b]$ and $\beta_\mathcal{C}(W) = \sup \{ \beta(W(t)) , t \in [a, b] \}$.

**Lemma 2.4** (Darbo-Sadovskii [14]). If $\{ u_n \}_{n=1}^{\infty} \subseteq \mathcal{L}^1(a, b; \mathbb{X})$ is uniformly integrable, then the function $\beta(\{ u_n(t) \}_{n=1}^{\infty})$ is measurable and

$$
\beta \left( \left\{ \int_0^t u_n(s) ds \right\}_{n=1}^{\infty} \right) \leq 2 \int_0^t \beta \left( \left\{ u_n(s) \right\}_{n=1}^{\infty} \right) ds. 
$$

(2.1)
Lemma 2.5 ([3]). If $W \subseteq C([0, b]; X)$ is bounded and equicontinuous, then $\beta(W(s))$ is continuous and

\[ \beta(\int_0^t W(s)ds) \leq \int_0^t \beta(W(s))ds. \] (2.2)

From [10], we know that for any fixed $u \in C([0, b]; X)$ there exist a unique continuous function $U_u : [0, b] \times [0, b] \rightarrow B(X)$ defined on $[0, b] \times [0, b]$ such that

\[ U_u(t, s) = I + \int_s^t A_u(\omega)U_u(\omega, s)d\omega, \] (2.3)

where $B(X)$ denote the Banach space of bounded linear operators from $X$ to $X$ with the norm $\|Q\| = \sup\{\|Qu\| : \|u\| = 1\}$, and $I$ stands for the identity operator on $X$, $A_u(t) = A(t, u(t))$. From (2.3), we have

\[ U_u(t, t) = I, \quad U_u(t, s)U_u(s, r) = U_u(t, r), \quad (t, s, r) \in [0, b] \times [0, b] \times [0, b], \]

\[ \frac{\partial U_u(t, s)}{\partial t} = A_u(t)U_u(t, s) \quad \text{for almost all } t \in [0, b], \quad \forall s \in [0, b]. \]

Definition 2.6. A continuous function $u(t) \in C([0, b]; X)$ such that

\[ u(t) = U_u(t, 0)u_0 + U_u(t, 0)g(u) + \int_0^t U_u(t, s) \int_0^s f(s, \tau, u(\tau))d\tau ds \] (2.4)

and $u(0) = g(u) + u_0$ is called a mild solution of (1.1)–(1.2).

The evolution family $\{U_u(t, s)\}_{0 \leq s \leq t \leq b}$ is said to be equicontinuous if $(t, s) \rightarrow \{U_u(t, s)x : x \in B\}$ is equicontinuous for $t > 0$ and for all bounded subset $B$ in $X$. The following Lemma is obvious.

Lemma 2.7. If the evolution family $\{U_u(t, s)\}_{0 \leq s \leq t \leq b}$ is equicontinuous and $\eta \in L(0, b; \mathbb{R}^+)$, then the set $\{\int_0^t U_u(t - s, s)u(s)ds, \|u(s)\| \leq \eta(s) \text{ for a.e. } s \in [0, b]\}$ is equicontinuous for $t \in [0, b]$.

In section 3, we give some existence results when $g$ is compact and $f$ satisfies the conditions with respect to Hauardoff’s measure of noncompactness. In section 4, we use the different method to discuss the case when $g$ is Lipschitz continuous and $f$ satisfies the conditions with the Hauardoff’s measure of noncompactness.

In this paper, we denote $M = \sup\{\|U_u(t, s)\| : (t, s) \in [0, b] \times [0, b]\}$ for all $u \in X$. Without loss of generality, we let $u_0 = 0$.

3. The existence results for compact $g$

In this section by using the usual techniques of the Hauardoff’s measure of noncompactness and its applications in differential equations in Banach spaces (see, e.g., [3, 5, 14]), we give some existence results of the nonlocal problem (1.1)–(1.2).

Here we list the following hypotheses:

HA : The evolution family $\{U_u(t, s)\}_{0 \leq s \leq t \leq b}$ generated by $A(t, u)$ is equicontinuous, and $\|U_u(t, s)\| \leq M$ for almost all $t, s \in [0, b]$.

Hg (1) $g : C([0, b]; X) \rightarrow X$ is continuous and compact;

(2) There exist $N > 0$ such that $\|g(u)\| \leq N$ for all $u \in C([0, b]; X)$.

Hf (1) $f : [0, b] \times [0, b] \times X \rightarrow X$ satisfies the Carathéodory-type condition, i.e., $f(\cdot, \cdot, u)$ is measurable for all $u \in X$ and $f(t, s, \cdot)$ is continuous for a.e. $t, s \in [a, b]$.
(2) There exist two functions $h : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $q : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(\cdot, r) \in \mathcal{L}(0, b; \mathbb{R}^+)$ for every $r \geq 0$, $h(t, \cdot)$ is continuous and increasing, $q(s) \in \mathcal{L}(0, b; \mathbb{R}^+)$, and $\|f(t, s, u)\| \leq q(t)h(s, \|u\|)$ for a.e. $t \in [0, b]$, and all $u \in \mathcal{C}([0, b]; \mathbb{X})$, and for all positive constants $K_1, K_2$, the scalar equation

$$m(t) = K_1 + K_2 \int_0^t h(s, m(s)) ds, \quad t \in [0, b]$$

has at least one solution;

(3) There exist $\eta \in \mathcal{L}(0, b; \mathbb{R}^+)$, $\zeta \in \mathcal{L}(0, b; \mathbb{R}^+)$ such that $\beta(f(t, s, D)) \leq \eta(t)\zeta(s)\beta(D)$ for a.e. $t, s \in [0, b]$, and for any bounded subset $D \subseteq \mathcal{C}([0, b], \mathbb{X})$. Here we let $\int_0^t \eta(s) ds \leq K$

Now, we give an existence result under the above hypotheses.

**Theorem 3.1.** Assume the hypotheses (HA), (Hf), (Hg) are satisfied, then the nonlocal initial value problem (1.1)–(1.2) has at least one mild solution.

**Proof.** Let $m(t)$ be a solution of the scalar equation

$$m(t) = MN + RM \int_0^t h(s, m(s)) ds,$$

where $R = \int_0^t q(s) ds$. Defined a map $Q : \mathcal{C}([0, b]; \mathbb{X}) \to \mathcal{C}([0, b]; \mathbb{X})$ by

$$(Qu)(t) = U_u(t, 0)g(u) + \int_0^t U_u(t, s) \int_0^s f(s, \tau, u(\tau)) d\tau ds, \quad t \in [0, b]$$

for all $u \in \mathcal{C}([0, b]; \mathbb{X})$. We can show that $Q$ is continuous by the usual techniques (see, e.g. [10, 17]).

We denote by $W_0 = \{u \in \mathcal{C}([0, b]; \mathbb{X}), \|u(t)\| \leq m(t) \text{ for all } t \in [0, b]\}$. Then $W_0 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded and convex.

Define $W_1 = \overline{\text{conv}}K(W_0)$, where $\overline{\text{conv}}$ means the closure of the convex hull in $\mathcal{C}([0, b]; \mathbb{X})$. As $U_u(t, s)$ is equicontinuous, $q$ is compact and $W_0 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded, due to Lemma [2.7] and hypothesis (Hf)(2), $W_1 \subseteq \mathcal{C}([0, b]; \mathbb{X})$ is bounded closed convex nonempty and equicontinuous on $[0, b]$. For any $u \in Q(W_0)$, we know

$$\|u(t)\| \leq MN + M \int_0^t \int_0^s q(s) h(\tau, m(\tau)) d\tau ds$$

$$\leq MN + M \int_0^t h(\tau, m(\tau)) d\tau \int_0^t q(s) ds$$

$$\leq MN + MR \int_0^t h(s, m(s)) ds$$

$$= m(t)$$

for $t \in [0, b]$. It follows that $W_1 \subset W_0$.

We define $W_{n+1} = \overline{\text{conv}}Q(W_n)$, for $n = 1, 2, \ldots$. Form above we know that $\{W_n\}_{n=1}^\infty$ is a decreasing sequence of bounded, closed, convex, equicontinuous on $[0, b]$ and nonempty subsets in $\mathcal{C}([0, b], \mathbb{X})$.

Now for $n \geq 1$ and $t \in [0, b]$, $W_n(t)$ and $Q(W_n(t))$ are bounded subsets of $\mathbb{X}$, hence, for any $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^\infty \subset W_n$ such that (see, e.g. [5], pp.125)
\[
\beta(W_{n+1}(t)) = \beta(QW_n(t)) \\
\leq 2\beta(\int_0^t U_n(t, s) \int_0^s f(s, \tau, \{u_k(\tau)\}_{k=1}^{\infty}) d\tau ds) + \varepsilon \\
\leq 2M \int_0^t \beta(\int_0^s f(s, \tau, \{u_k(\tau)\}_{k=1}^{\infty}) d\tau ds) + \varepsilon \\
\leq 4M \int_0^t \int_0^s \beta(f(s, \tau, \{u_k(\tau)\}_{k=1}^{\infty})) d\tau ds + \varepsilon \\
\leq 4M \int_0^t \int_0^s \eta(s)\zeta(\tau)\beta(\{u_k(\tau)\}_{k=1}^{\infty}) d\tau ds + \varepsilon \\
\leq 4M \int_0^t \zeta(s)\beta(W_n(s)) ds + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, it follows from the above inequality that

\[
\beta(QW_{n+1}(t)) \leq 4MK \int_0^t \zeta(s)\beta(W_n(s)) ds
\]

for all \( t \in [0, b] \). Because \( W_n \) is decreasing for \( n \), we define

\[
\alpha(t) = \lim_{n \to \infty} \beta(W_n(t))
\]

for all \( t \in [0, b] \). From (3.4), we have

\[
\alpha(t) \leq 4MK \int_0^t \zeta(s)\alpha(s) ds
\]

for \( t \in [0, b] \), which implies that \( \alpha(t) = 0 \) for all \( t \in [0, b] \). By Lemma 2.3, we know that \( \lim_{n \to \infty} \beta_C(W_n) = 0 \). Using Lemma 2.1, we know that \( W = \bigcap_{n=1}^{\infty} W_n \) is convex compact and nonempty in \( C([0, b]; X) \) and \( Q(W) \subset W \). By the famous Schauder’s fixed point theorem, there exists at least one mild solution \( u \) of the initial value problem (1.1)–(1.2), where \( u \in W \) is a fixed point of the continuous map \( Q \). □

**Remark 3.2.** If the function \( f \) is compact or Lipschitz continuous (see, e.g. [6, 16, 18]), then (Hf)(3) is automatically satisfied.

In some of the early related results in references and above result, it is supposed that the map \( g \) is uniformly bounded. We indicate here that this condition can be released. In fact, if \( g \) is compact, then it must be bounded on bounded set. Here we give an existence result under another growth condition of \( f \) (see, [11, 20]), when \( g \) is not uniformly bounded. Precisely, we replace the hypothesis (Hf)(2) by (Hf)(2') There exists a function \( p \in C(0, b; \mathbb{R}^+) \) and an increasing function \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( ||f(t, s, u)|| \leq p(t)\psi(||u||) \), for a.e. \( t \in [0, b] \), and all \( u \in C([0, b]; X) \).
Theorem 3.3. Suppose that (HA), (Hf)(1), (Hf)(2'), (Hf)(3), (Hg)(1) are satisfied. Then the equation (1.1)–(1.2) has at least one mild solution if
\[ \lim_{k \to \infty} \sup_{t \in [0,b]} M \left( \varphi(t) + b \psi(t) \right) < 1, \]  
where \( \varphi(t) = \sup \{ \|g(u(t))\| : \|u\| \leq k \}. \)

Proof. The inequality (3.5) implies that there exists a constant \( k > 0 \) such that
\[ M(\varphi(k) + b\psi(k)) \int_0^b p(s)ds < k. \]
Just as in the proof of Theorem 3.1, let \( W \subset \text{conv} QW_0 \). Then for any \( u \in W \), we have
\[ \|u(t)\| \leq M \varphi(k) + M \int_0^t \int_0^s p(\tau)\psi(\tau)d\tau ds \leq M \varphi(k) + bM\psi(k) \int_0^b p(s)ds < k \]
for \( t \in [0,b] \). It means that \( W_1 \subset W_0 \). So we can complete the proof similarly to Theorem 3.1.

4. Existence results for Lipschitz \( g \)

In the previous section, we obtained the existence results when \( g \) is compact but without the compactness of \( \{U_u(t,s)\}_{0 \leq s \leq t \leq b} \) or \( f \). In this section, we discuss the equation (1.1)–(1.2) when \( g \) is Lipschitz and \( f \) is not Lipschitz. Precisely, we replace (Hg)(1) by (Hg)(1') There exist a constant \( L \in (0, \frac{1}{M}) \) such that \( \|g(u) - g(v)\| \leq L\|u - v\| \) for every \( u, v \in C([0,b];X) \).

Theorem 4.1. Let (HA), (Hg)(1')(2), (Hf) be satisfied. Then the equation (1.1)–(1.2) has at least one mild solution provided that
\[ ML + 4MK \int_0^b \zeta(s)ds < 1. \]  
Proof. We define \( Q_1, Q_2 : C([0,B];X) \to C([0,B];X) \) by
\[ (Q_1u)(t) = U_u(t,0)g(u), \]
\[ (Q_2u)(t) = \int_0^t U_u(t,s) \int_0^s f(s,\tau, u(\tau))d\tau ds \]
for \( u \in C([0,B];X) \). Note that \( Q_1 + Q_2 = Q \), as defined in the proof of Theorem 3.1. We define \( W_0 = \{ u \in C([0,B];X) : \|u(t)\| \leq m(t) \ \forall t \in [0,b] \} \), and let \( W = \text{conv}QW_0 \). Then from the proof of Theorem 3.1 we know that \( W \) is a bounded closed convex and equicontinuous subset of \( C([0,B];X) \) and \( QW \subset W \). We shall prove that \( Q \) is \( \beta_c \)-contraction on \( W \). Then Darbo-Sadovskii’s fixed point theorem can be used to get a fixed point of \( Q \) in \( W \), which is a mild solution of (1.1)–(1.2).

First, for every bounded subset \( B \subset W \), from the (Hg)(1') and Lemma 2.1, we have
\[ \beta_c(Q_1B) = \beta_c(U_B(t,0)g(B)) \leq M\beta_c(g(B)) \leq ML\beta_c(B). \]
Next, for every bounded subset \( B \subset W \), for \( t \in [0, b] \) and every \( \varepsilon > 0 \), there is a sequence \( \{u_k\}_{k=1}^\infty \subset B \), such that
\[
\beta(Q_2B(t)) \leq 2\beta(\{Q_2u_k(t)\}_{k=1}^\infty) + \varepsilon.
\]

Note that \( B \) and \( Q_2B \) are equicontinuous, we can get from Lemma 2.1, Lemma 2.4, Lemma 2.5 and (Hf)(3) that
\[
\beta(Q_2B(t)) \leq 2M\int_0^t \beta(\int_0^r f(s, \tau, \{u_k(\tau)\}_{k=1}^\infty)\,d\tau)\,ds + \varepsilon.
\]

Taking supremum in \( t \in [0, b] \), we have
\[
\beta_c(Q_2B) \leq 4MK\beta_c(B)\int_0^b \zeta(s)\,ds + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have
\[
\beta_c(Q_2B) \leq 4MK\beta_c(B)\int_0^b \zeta(s)\,ds
\]
for any bounded \( B \subset W \).

Now, for any subset \( B \subset W \), due to Lemma 2.1, 4.2 and 4.3 we have
\[
\beta_c(QB) \leq \beta_c(Q_1B) + \beta_c(Q_2B)
\]
\[
\leq (ML + 4MK\int_0^b \zeta(s)\,ds)\beta_c(B).
\]

By (4.1) we know that \( Q \) is a \( \beta_c \)-contraction on \( W \). By Lemma 2.2, there is a fixed point \( u \) of \( Q \) in \( W \), which is a solution of (1.1)–(1.2). This completes the proof.

Now we give an existence result without the uniform boundedness of \( g \).

**Theorem 4.2.** Suppose that (HA), (Hf)(1), (Hf)(2'), (Hf)(3), (Hg)(1') are satisfied. Then the equation (1.1)–(1.2) has at least one mild solution if
\[
ML + bM\int_0^b p(s)\,ds \limsup_{k \to \infty} \frac{\psi(k)}{k} < 1.
\]

**Proof.** From (4.4) and the fact that \( L < 1 \), there exists a constant \( k > 0 \) such that
\[
M(kL + bM\int_0^b p(s)d\psi(k) + \|g(0)\|) < k.
\]
We define $W_0 = \{ u \in C([0, b]); \mathbb{X} : \|u(t)\| \leq k, \forall t \in [0, b] \}$. Then for every $u \in W_0$, we have

\[
\|Qu(t)\| \leq M(\|g(u)\| + \psi(k) \int_0^t \int_0^s p(\tau) d\tau ds) \\
\leq M(\|g(u) - g(0) + g(0)\| + b\psi(k) \int_0^t p(s) ds) \\
\leq M(kL + \|g(0)\| + b\psi(k) \int_0^t p(\tau) d\tau) < k
\]

for $t \in [0, b]$. This means that $QW_0 \subset W_0$. Define $W = \text{conv}QW_0$. The above proof also implies that $QW \subset W$. So we can prove the theorem similar with Theorem 4.1.

\[\square\]

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