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CONVERGENCE TO EQUILIBRIA FOR A THREE-DIMENSIONAL CONSERVED PHASE-FIELD SYSTEM WITH MEMORY

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Abstract. We consider a conserved phase-field system with thermal memory on a tridimensional bounded domain. Assuming that the nonlinearity is real analytic, we use a Lojasiewicz-Simon type inequality to study the convergence to steady states of single trajectories. We also give an estimate of the convergence rate.

1. Introduction

We consider a phase-field system of conserved type with thermal memory on a bounded tridimensional set Ω with smooth boundary $\partial\Omega$. Denoting by ϑ is the *relative temperature variation field*, by χ the *order parameter* (or *phase-field*) and setting some physical constants equal to one, the boundary-initial integro-differential problem we want to study reads as follows

$$\begin{gathered}
\partial_t(\vartheta+\chi)-\int_0^\infty k(s)\Delta\vartheta(t-s)ds=0, \quad \text{in } \Omega\times(0,\infty),\\
\partial_t\chi-\Delta(-\Delta\chi+\alpha\partial_t\chi+\phi(\chi)-\vartheta)=0 \quad \text{in } \Omega\times(0,\infty),\\
\partial_{\boldsymbol{n}}\vartheta=\partial_{\boldsymbol{n}}\chi=\partial_{\boldsymbol{n}}(-\Delta\chi+\alpha\partial_t\chi+\phi(\chi)-\vartheta)=0 \quad \text{on } \partial\Omega\times(0,\infty),\\
\vartheta(0)=\vartheta_0, \quad \chi(0)=\chi_0, \quad \vartheta(-s)=\vartheta_1(s) \quad (s>0) \quad \text{in } \Omega.
\end{gathered} \tag{1.1}$$

We recall that the first equation of (1.1), according to the Gurtin-Pipkin heat conduction law [20], accounts for the memory effects due to the heat propagation. Here $k:(0,\infty)\to(0,\infty)$ is the (smooth, decreasing and summable) heat conduction *relaxation kernel*. Note that all the thermal diffusion is carried out by the memory term solely and ϑ propagates at finite speed. The second equation governs the evolution of χ and it is characterized by the presence of the nonlinearity ϕ and by a viscosity term $-\alpha\Delta\partial_t\chi$, where $\alpha\geq 0$ is the *viscosity parameter*. Finally, the initial conditions $\vartheta_0,\chi_0:\Omega\to\mathbb{R}$ and $\vartheta_1:\Omega\times(0,\infty)\to\mathbb{R}$ are given functions,

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whose properties will be discussed later on. We point out that the knowledge of the unknown variable ϑ for negative times is necessary to ensure the well-posedness.

Note that the homogeneous Neumann boundary conditions we require imply that the system is thermally isolated. Moreover, thanks to such conditions, a formal application of the Green formula yields immediately the following identities

$$\int_\Omega (\vartheta(t) + \chi(t)) d\Omega = \int_\Omega (\vartheta_0 + \chi_0) d\Omega \quad \text{and} \quad \int_\Omega \chi(t) d\Omega = \int_\Omega \chi_0 d\Omega,$$

for any $t \in (0, \infty)$. The conservation of the quantities above is a structural feature of our system, which explains the reason why it is called *conserved*.

For a detailed phenomenological description of the mathematical model with the usual Fourier heat conduction law, as well as the related literature, we address the reader to [4, 5] (see also references therein). The case of heat conduction law with memory effects was studied in a number of papers (see [7, 8, 9, 13, 14, 17, 31]). In [25, 26], problem (1.1) has been considered in the framework of infinite-dimensional dynamical systems. In particular, results of well-posedness and large-time behavior for its solutions (e.g., the existence of global and exponential attractors) have been established.

The aim of this contribution is to analyze the convergence to equilibrium of single trajectories. Such a task is nontrivial, since the set of steady states of systems like (1.1) can be a continuum when the spatial dimension is greater than one (see, e.g., [21, Remark 2.3.13]). However, when the nonlinearity ϕ is real analytic, it is possible to take advantage of a Lojasiewicz-Simon type inequality originated from the theory of functions of several complex variables [23, 24, 29]. This tool allows us to prove that each trajectory converges to a single stationary point. We recall that this technique has been recently exploited in many cases (see, for instance, [1, 2, 3, 11, 12, 18, 19] and references therein). It is worth observing that the fact that ϕ is real analytic is essential. Indeed, even though the nonlinearities are C^∞, it can be shown that, for some semilinear equations, there are trajectories whose ω-limit sets are continua (see [27, 28]).

Convergence results for phase-transition systems featuring the heat conduction laws of Fourier and Coleman-Gurtin have already been achieved in [2] and [11], respectively. In particular, concerning the latter model, we point out that the contributions to the convolution integrals due to the past history of the temperature up to $t = 0$ is considered as given data, and therefore regarded as external sources. Notice that such a formulation forces the system to become non autonomous, even if the original system is autonomous. Thus, regarding the past history as a source, is not convenient to study the problem in the framework of dynamical systems. Here we will follow the dynamical system approach to take advantage of our previous results in [25]. This will be particularly helpful to overcome the lack of smoothing effects due to the Gurtin-Pipkin law.

1.1. **The past history formulation.** To prove that our problem generates a dynamical system, we follow an approach based on an idea contained in [10], and then developed by several of authors in the context of dynamical systems (see, e.g., the review papers [15, 16]). This idea consists in introducing an additional variable, usually called the *summed past history*, which in our case is

$$\eta^t(s) = -\int_0^s \Delta(e(t-y) - \chi(t-y)) dy \quad \text{in } \Omega,\ (t,s) \in [0,\infty) \times (0,\infty),$$

where we $e = \vartheta + \chi$ is the *enthalpy density*. It is immediate to check that η^t formally satisfies the first order hyperbolic equation

$$\partial_t \eta = -\partial_s \eta - \Delta(e - \chi) \quad \text{in } \Omega,\ (t,s) \in (0,\infty) \times (0,\infty).$$

Concerning the boundary and initial conditions to associate with the equation above, on account of (1.1), we deduce

$$\eta^t(0) = 0 \quad \text{and} \quad \eta^0(s) = \eta_0(s) = -\int_0^s \Delta\vartheta_1(y)dy$$

in Ω, for all $t, s \in (0,\infty)$.

Considering then the convolution term in the first equation, and making physically reasonable assumptions on the past history and the memory kernel, we observe that a formal integration by parts yields

$$-\int_0^\infty k(s)\Delta(e(t-s) - \chi(t-s))ds = \int_0^\infty \mu(s)\eta^t(s)ds \quad \text{in } \Omega,\ s \in (0,\infty),$$

where we have set $\mu = -k'$. Thus we can reformulate the original boundary and initial value problem as the following integro-partial differential system in terms of the variables (e, χ, η).

Problem P. Find a solution (e, χ, η) to the system

$$\begin{gathered}
\partial_t e + \int_0^\infty \mu(s)\eta(s)ds = 0, \\
\partial_t \chi - \Delta(-\Delta\chi + \alpha\partial_t\chi + \phi(\chi) - e + \chi) = 0, \\
\partial_t \eta = -\partial_s \eta - \Delta(e - \chi),
\end{gathered}$$

in $\Omega \times (0,\infty)$, subjected to the boundary and initial conditions

$$\begin{gathered}
\partial_{\boldsymbol{n}} e = \partial_{\boldsymbol{n}} \chi = 0, \quad \text{on } \partial\Omega \times (0,\infty), \\
\partial_{\boldsymbol{n}}(-\Delta\chi + \alpha\partial_t\chi + \phi(\chi) - e + \chi) = 0, \quad \text{on } \partial\Omega \times (0,\infty), \\
e(0) = e_0 = \vartheta_0 + \chi_0, \quad \text{in } \Omega, \\
\chi(0) = \chi_0 \quad \text{in } \Omega, \\
\eta^0 = \eta_0, \quad \text{in } \Omega \times (0,\infty).
\end{gathered}$$

The global dynamic of problem **P** has been widely analyzed in [25] and [26], where results concerning well-posedness and asymptotic behavior for large times have been provided. In particular, in [25], the existence of the global attractor has been showed, as well as its regularity and the finiteness of its fractal dimension in the viscous case ($\alpha > 0$). On the other hand, in [26] the existence of a family of exponential attractors (stable with respect to perturbation of the relaxation time) has been established.

2. Preliminary tools

This section is devoted to describe the functional setting which will be used to formulate problem **P** rigorously and to recall many results that will be useful in the sequel. Since most of the tools that we need are known, we shall omit the proofs, providing appropriate references when necessary.

2.1. **Function spaces and operators.** Let H be the real Hilbert space $L^2(\Omega)$ of the measurable functions which are square summable on Ω, endowed with the usual scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$.

Given any $\omega \in H$, we define the *spatial mean value* of ω on Ω

$$m_\omega = |\Omega|^{-1}\langle\omega, 1\rangle.$$

We then introduce

$$H_0 = \{\omega \in H : m_\omega = 0\}.$$

Denoting, as usual, by Δ the spatial Laplacian, we now define the (unbounded) operators

$$B : \mathcal{D}(B) \to H_0 \quad \text{and} \quad B_0 : \mathcal{D}(B_0) \to H_0$$

by setting

$$B = -\Delta, \quad \mathcal{D}(B) = \{\omega \in H^2(\Omega) : \partial_{\boldsymbol{n}}\omega = 0 \quad \text{a.e. on } \partial\Omega\},$$
$$B_0 = -\Delta, \quad \mathcal{D}(B_0) = \mathcal{D}(B) \cap H_0.$$

Here the symbol $\partial_{\boldsymbol{n}}$ denotes the outward normal derivative. Since B_0 is a strictly positive operator, we can set

$$V_0^r = \mathcal{D}(B_0^{r/2}), \quad \forall r \in \mathbb{R},$$

as well as the shorthand $V_0 = V_0^1$ and $W_0 = V_0^2$. For further use, we also introduce the Hilbert spaces

$$V = H^1(\Omega) \quad \text{and} \quad W = \mathcal{D}(B),$$

endowed with the norms

$$\|\omega\|_V^2 = \|\omega\|^2 + \|P\omega\|_{V_0}^2 \quad \text{and} \quad \|\omega\|_W^2 = \|\omega\|_V^2 + \|P\omega\|_{W_0}^2,$$

being $P\omega = \omega - m_\omega$ the natural projection from H to H_0. It is easy to realize that the norms defined above are equivalent, respectively, to the usual norms in $H^1(\Omega)$ and $H^2(\Omega)$.

Making the identification $H \equiv H^*$ (here and by X^* denotes the topological dual of a Banach space X), we have the compact and dense embeddings

$$W \hookrightarrow V \hookrightarrow H \hookrightarrow V^* \hookrightarrow W^*, \tag{2.1}$$

$$W_0 \hookrightarrow V_0 \hookrightarrow H_0 \hookrightarrow V_0^* \hookrightarrow W_0^*. \tag{2.2}$$

Note that, according to the notation introduced above, we have

$$V_0^* = V_0^{-1} \quad \text{and} \quad W_0^* = V_0^{-2}.$$

Moreover, there holds

$$V \hookrightarrow L^p(\Omega),\ \forall p \in [2,6], \quad W \hookrightarrow C(\overline{\Omega}), \quad V_0 \hookrightarrow V, \quad W_0 \hookrightarrow W. \tag{2.3}$$

2.2. **Assumptions on ϕ and μ.** To state our results, we need to make some structural assumptions on the nonlinearity as well as on the memory kernel. Concerning the former one, the assumptions that we consider include (and generalize) the case of the derivative of a double-well potential. Concerning the latter, the key property to ensure the dissipativity of our system (cf. Section 3) is the exponential decay of the kernel μ.

Conditions on ϕ. Let $\phi \in C^2(\mathbb{R})$ and assume that there exist $c_0 > 0$ and $c_1, c_2 \ge 0$ such that

- (H1) $r\phi(r) \ge c_0 r^4 - c_1$, for all $r \in \mathbb{R}$
- (H2) $|\phi''(r)| \le c_2(1 + |r|)$, for all $r \in \mathbb{R}$
- (H3) $\phi'(r) \ge -\ell$, for all $r \in \mathbb{R}$
- (H4) ϕ is real analytic.

Conditions on μ. Let $\mu : (0, \infty) \to (0, \infty)$ be a summable function such that

- (K1) $\mu \in C^1((0, \infty)) \cap L^1(0, \infty)$,
- (K2) $\mu(s) \ge 0$, $\mu'(s) \le 0$, for all $s \in (0, \infty)$,
- (K3) $k_0 = \int_0^\infty \mu(s)ds > 0$,
- (K4) $\mu'(s) + \lambda\mu(s) \le 0$, for all $s \in (0, \infty)$, for some $\lambda > 0$.

Remark 2.1. Note that μ is decreasing and Gronwall Lemma entails the exponential decay

$$\mu(s) \le \mu(s_0)e^{-\lambda(s-s_0)}, \quad \forall s \ge s_0 > 0. \tag{2.4}$$

Note also that μ is allowed to be unbounded in a right neighborhood of 0.

2.3. **The past history function space.** The presence of memory effects in our phase-field system requires the introduction of suitable past history spaces [15, 16].

Let $r \in \mathbb{R}$. On account of assumptions (K1)-(K2), we consider the family of weighted Hilbert spaces

$$\mathcal{M}^r = L^2_\mu(0, \infty; V_0^{r-1}),$$

endowed with the inner product

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}^r} = \int_0^\infty \mu(s)\langle \eta_1(s), \eta_2(s)\rangle_{V_0^{r-1}} ds, \quad \forall\ \eta_1, \eta_2 \in \mathcal{M}^r.$$

For the sake of clarity, from now on we will use the shorthand $\mathcal{M}$ in place of $\mathcal{M}^0$, and $\mathcal{N}$ in place of $\mathcal{M}^1$. In these cases, the norms become, respectively,

$$\|\eta\|^2_{\mathcal{M}} = \int_0^\infty \mu(s)\|\eta(s)\|^2_{V_0^*} ds \quad \text{and} \quad \|\eta\|^2_{\mathcal{N}} = \int_0^\infty \mu(s)\|\eta(s)\|^2 ds.$$

We also define the linear operator T on $\mathcal{M}$ with domain $\mathcal{D}(T) = \{\eta \in \mathcal{M} : \partial_s\eta \in \mathcal{M},\ \eta(0) = 0\}$, as

$$T\eta = -\partial_s\eta,$$

where $\partial_s\eta$ is the distributional derivative of η with respect to the internal variable s.

2.4. **The phase-space.** We are now in a position to define the phase-space for our dynamical system. We set

$$\mathcal{H} = H \times V \times \mathcal{M} \quad \text{and} \quad \mathcal{V} = V \times W \times \mathcal{N}.$$

Proposition 2.2. *There holds*

(i) *$\mathcal{H}$ is a Hilbert space, if endowed with the inner product*

$$\langle (e_1, \chi_1, \eta_1), (e_2, \chi_2, \eta_2)\rangle_{\mathcal{H}} = \langle e_1, e_2\rangle + \langle \chi_1, \chi_2\rangle_V + \langle \eta_1, \eta_2\rangle_{\mathcal{M}},$$

for all $(e_1, \chi_1, \eta_1), (e_2, \chi_2, \eta_2) \in \mathcal{H}$.

(ii) *$\mathcal{V}$ is a Hilbert space, if endowed with the inner product*

$$\langle (e_1, \chi_1, \eta_1), (e_2, \chi_2, \eta_2)\rangle_{\mathcal{V}} = \langle e_1, e_2\rangle_V + \langle \chi_1, \chi_2\rangle_W + \langle \eta_1, \eta_2\rangle_{\mathcal{N}},$$

for all $(e_1, \chi_1, \eta_1), (e_2, \chi_2, \eta_2) \in \mathcal{V}$.

(iii) *The embedding* $\mathcal{V} \hookrightarrow \mathcal{H}$ *is continuous.*

On account of the fact that the spatial means e and χ are constant in time, we also consider the function spaces

$$\mathcal{H}_{\beta,\gamma} = \{(e,\chi,\eta) \in \mathcal{H} : |m_e| \leq \beta \text{ and } |m_\chi| \leq \gamma\} \quad \text{and} \quad \mathcal{V}_{\beta,\gamma} = \mathcal{V} \cap \mathcal{H}_{\beta,\gamma}$$

for some fixed $\beta,\gamma \geq 0$. Notice that, if $\beta,\gamma > 0$, $\mathcal{H}_{\beta,\gamma}$ and $\mathcal{V}_{\beta,\gamma}$ are not linear spaces. Nevertheless, they have a metric structure, as stated in the next result.

Proposition 2.3. *Let* $\beta,\gamma \geq 0$. *Then*

(i) $\mathcal{H}_{\beta,\gamma}$ *is a complete metric space with respect to the topology induced by the norm of* $\mathcal{H}$*,*

(ii) $\mathcal{V}_{\beta,\gamma}$ *is a is a complete metric space with respect to the topology induced by the norm of* $\mathcal{Z}$*.*

(iii) *The embedding* $\mathcal{V}_{\beta,\gamma} \hookrightarrow \mathcal{H}_{\beta,\gamma}$ *is continuous.*

2.5. **The Lojasiewicz-Simon inequality.** We now recall the main tool in order to reach our goal, namely, the well-known Lojasiewicz-Simon inequality, in a convenient form for our investigation, i.e., in the space of zero-mean functions. In order to work in such a space, we set, for any fixed $\chi \in V$,

$$\overline{\phi}(P\chi) = \phi(P\chi + m_\chi) = \phi(\chi),$$

and, consequently,

$$\overline{\mathcal{F}}(x) = \int_0^x \overline{\phi}(y)dy, \quad \forall x \in \mathbb{R}.$$

It is immediate to check that $\overline{\phi}$ fulfills assumptions (H1)-(H4) as well.

If we consider the standard definition of analyticity (see [32, Vol. I, Definition 8.8] for details), then we state [1, Theorem 4.2].

Lemma 2.4. *Under assumption* (H4)*, the functional* $\overline{E} : V_0 \to \mathbb{R}$ *defined by*

$$\overline{E}(\chi) = \frac{1}{2}\|\chi\|^2_{V_0} + \langle \overline{\mathcal{F}}(\chi), 1\rangle, \quad \forall\ \chi \in V_0,$$

is real analytic. Moreover, if we denote by $\overline{E}'$ *its Fréchet derivative, the following equality holds*

$$\overline{E}'(\chi)v = \langle B_0^{1/2}\chi, B_0^{1/2}v\rangle + \langle \overline{\phi}(\chi), v\rangle, \quad \forall\ v \in V_0.$$

We are now in a position to recall the Lojasiewicz-Simon inequality we need (see [19, Lemma 4.1]).

Lemma 2.5. *Let assumptions* (H4) *hold and let* $\varphi \in W$ *be such that*

$$B_0(B_0 P\varphi + P\phi(\varphi)) = 0 \quad \text{in } W_0^*.$$

Then there exist constants $\rho \in (0,1/2)$*,* $r > 0$ *and* $\lambda > 0$*, depending on* φ*, such that*

$$\left|\overline{E}(P\chi) - \overline{E}(P\varphi)\right|^{1-\rho} \leq \lambda \|B_0 P\chi + P\phi(\chi)\|_{V_0^*} \tag{2.5}$$

for all $\chi \in V$ *such that* $\|\chi - \varphi\|_V \leq r$.

3. Well-posedness and dissipativity

On account of the previous section, we can now introduce the operator formulation of our problem; namely,

Problem P. Given $(e_0,\chi_0,\eta_0)\in\mathcal{H}$, find $z=(e,\chi,\eta)\in C([0,T];\mathcal{H})$ satisfying the equations

$$\partial_t e+\int_0^\infty \mu(s)\eta(s)ds=0, \tag{3.1}$$

$$\partial_t\chi+B_0(B_0P\chi+\alpha\partial_t\chi+P\phi(\chi)-P(e-\chi))=0, \tag{3.2}$$

$$\partial_t\eta=T\eta+B_0P(e-\chi), \tag{3.3}$$

$$(e(0),\chi(0),\eta(0))=(e_0,\chi_0,\eta_0), \tag{3.4}$$

where equation (3.3) has to be interpreted in a distributional sense.

3.1. **Semigroup generation.** By constructing a suitable approximating Faedo-Galerkin scheme, it is possible to prove the well-posedness theorem stated below. Details go exactly like in [31] (see also [17]).

Theorem 3.1. *Let assumptions* (H1)–(H2) *and* (K1)–(K3) *hold. Then problem* P *generates a strongly continuous (nonlinear) semigroup $S(t)$, both on the phase-space $\mathcal{H}$ and on the phase-space $\mathcal{H}_{\beta,\gamma}$, for any fixed $\beta,\gamma\ge 0$. Moreover, the further regularity properties hold*

$$\begin{gathered}\partial_t e\in C([0,T];V_0^*),\\ \chi\in L^2(0,T;W)\cap H^1(0,T;V^*),\\ \alpha\,\partial_t\chi\in L^2(0,T;H_0).\end{gathered}$$

3.2. **Dissipativity.** As showed in [25, Theorem 4.1], $S(t)$ is dissipative on the bounded average phase-space $\mathcal{H}_{\beta,\gamma}$. We recall that the crucial assumption to prove such a statement is (K4). More precisely, we have

Theorem 3.2. *Let assumptions* (H1)–(H2) *and* (K1)-(K4) *hold. Then there exists a bounded set $\mathcal{B}_0=\mathcal{B}_0(\beta,\gamma)$ of $\mathcal{H}_{\beta,\gamma}$ such that*

$$S(t)\mathcal{B}\subset\mathcal{B}_0,\quad \forall t\ge t_{\mathcal{B}}.$$

for all bounded set $\mathcal{B}\subset\mathcal{H}_{\beta,\gamma}$, being $t_{\mathcal{B}}$ the positive entering time (depending on $\mathcal{B}$).

Such a set $\mathcal{B}_0$ is a *bounded absorbing set* for the semigroup $S(t)$.

Remark 3.3. Besides the uniform attracting property stated in Theorem 3.2, it is possible to prove that the following energy inequality holds (see [25, Section 4])

$$\begin{aligned}&\frac{d}{dt}\big[\|P(e-\chi)\|^2+\|\eta\|^2_{\mathcal{M}}+2\overline{E}(P\chi)+2\nu L(t)\big]\\ &+c\big[\|P(e-\chi)\|^2+\|\eta\|^2_{\mathcal{M}}+\|\partial_t\chi\|^2_{V_0^*}+\alpha\|\partial_t\chi\|^2\big]\le 0,\end{aligned} \tag{3.5}$$

for some positive constant c and for some $\nu\in(0,1)$ to be chosen small enough. Here $L(t)$ is defined by

$$L(t)=-\int_0^\infty \psi(s)\langle B_0^{-1/2}\eta(s),B_0^{-1/2}P(e-\chi)\rangle ds,$$

where for any fixed $s_0\in[0,\infty)$, the function $\psi=\psi_{s_0}:[0,\infty)\to[0,\infty)$ we set

$$\psi(s)=\mu(s_0)\mathcal{I}_{(0,s_0]}(s)+\mu(s)\mathcal{I}_{[s_0,\infty)}(s),$$

being $\mathcal{I}_{\mathrm{I}}$ the indicator function of an interval $\mathrm{I}\subset[0,\infty)$. Notice that it is immediate to derive the inequality

$$|L(t)|\le c[\|P(e-\chi)\|^2+\|\eta\|^2_{\mathcal{M}}]. \tag{3.6}$$

3.3. **Global attractor.** We briefly remind that it is also possible to prove the existence of the global attractor $\mathcal{A}$ for $S(t)$ on $\mathcal{H}_{\beta,\gamma}$. More precisely, the next statement subsumes [25, Theorems 5.1 and 7.1], which provide existence of $\mathcal{A}$ as well as its regularity in the viscous case.

Theorem 3.4. *Let the assumptions of Theorem 3.2 hold. Then the strongly continuous semigroup $S(t)$ possesses a global attractor $\mathcal{A} = \mathcal{A}(\beta,\gamma)$. Moreover, assume also assumption* (H3) *to hold. Then, for any fixed $\alpha > 0$, $\mathcal{A}$ is a bounded subset of the higher order phase space $\mathcal{V}_{\beta,\gamma}$.*

We point out that Theorem 3.4, as outlined in [25, Sections 5 and 7], can be proven by means of the asymptotic compactness condition (cf., for instance, [22]). As a consequence, we immediately infer the following property, which will be crucial in the sequel

Corollary 3.5. *For all $z_0 \in \mathcal{H}_{\beta,\gamma}$, setting $z(t) = (e(t), \chi(t), \eta^t) = S(t)z_0$, we have that $\cup_{t\in[0,\infty)} z(t)$ is precompact in $\mathcal{H}_{\beta,\gamma}$.*

4. Convergence to equilibria

Here we can now state the main results of this paper. First, we need to review some preliminary result concerning the structure of equilibrium points and ω-limit sets.

4.1. **Lyapunov function and equilibrium points.** We now introduce a further invariant set, which will play a fundamental role in our investigation, namely the set of *equilibrium points* (or *steady states*)

$$\mathcal{S} = \big\{ z \in \mathcal{H}_{\beta,\gamma} : S(t)z = z \quad \forall t \in [0,\infty) \big\}.$$

It is immediate to deduce that $\mathcal{S} \subset \mathcal{A}$. Moreover, we have

$$\begin{aligned}\mathcal{S} = \Big\{ &(e,\chi,0) : e \in V \text{ and } \chi \in W, \text{ such that } B_0P(e-\chi) = 0 \text{ in } V_0^* \\ &\text{and } B_0(B_0P\chi + P\phi(\chi)) = 0 \text{ in } W_0^* \Big\}.\end{aligned}$$

We remind that a convergence result to a single equilibrium is nontrivial, since

$$\mathcal{S}_\chi = \big\{ \chi \in W : B_0(B_0P\chi + P\phi(\chi)) = 0 \text{ in } W_0^* \big\},$$

might be a continuum (see [21]). Nevertheless, we can easily realize that $\mathcal{S}_\chi$ is bounded in W.

We recall that a function $\mathbf{L} \in C(\mathcal{H}_{\beta,\gamma};\mathbb{R})$ is called a (strict) *Lyapunov function* for $S(t)$ if

(i) $\mathbf{L}(S(t)z) \le \mathbf{L}(z)$ for all $z \in \mathcal{H}_{\beta,\gamma}$ and $t \in [0,\infty)$;
(ii) $\mathbf{L}(S(t)z) = \mathbf{L}(z)$ for all $t \in (0,\infty)$ implies that $z \in \mathcal{S}$.

In our case, on account of Remark 3.3, it is natural to construct a Lyapunov function for $S(t)$ on $\mathcal{H}_{\beta,\gamma}$. Indeed, for all $z = (e,\chi,\eta) \in \mathcal{H}_{\beta,\gamma}$, we define

$$\mathbf{L}(z) = \|P(e-\chi)\|^2 + \|\eta\|^2_{\mathcal{M}} + 2\overline{E}(P\chi) + 2\nu L.$$

Proposition 4.1. *The function $\mathbf{L} \in C(\mathcal{H}_{\beta,\gamma};\mathbb{R})$ is a Lyapunov function for $S(t)$ on $\mathcal{H}_{\beta,\gamma}$.*

Proof. The continuity of $\mathbf{L}$ follows Theorem 3.1. Both assumptions **(i)** and **(ii)** follow from (3.5). $\square$

As a consequence, our dynamical system $(\mathcal{H}_{\beta,\gamma}, S(t))$ is a *gradient system*, so that $\mathcal{A}$ coincides with the unstable manifold of $\mathcal{S}$ (see, e.g., [30]).

4.2. **Preliminary results on the ω-limit sets.** Before proving the main result, it is necessary to point out some features of the $\omega-$limit sets in $\mathcal{H}_{\beta,\gamma}$.

Remark 4.2. If $(e_0, \chi_0, \eta_0) \in \mathcal{H}_{\beta,\gamma}$ and $(e_\infty, \chi_\infty, \eta_\infty) \in \omega(e_0, \chi_0, \eta_0)$, then it is immediate to check that

$$m_{e_\infty} = m_{e_0} \quad \text{and} \quad m_{\chi_\infty} = m_{\chi_0}.$$

Thanks to the existence of the Lyapunov function stated in Theorem 4.1, we can provide a further description of the $\omega-$limit sets, which is a consequence of abstract results [6, Theorems 9.2.3 and 9.2.7].

Lemma 4.3. *For any $(e_0, \chi_0, \eta_0) \in \mathcal{H}_{\beta,\gamma}$, the set $\omega(e_0, \chi_0, \eta_0)$ is nonempty, compact, invariant and connected in $\mathcal{H}_{\beta,\gamma}$ and the following inclusion holds*

$$\begin{aligned} \omega(e_0, \chi_0, \eta_0) \subset \Big\{ &(m_{e_0-\chi_0} + \chi_\infty, \chi_\infty, 0) : \chi_\infty \in W, \text{ such that} \\ &B_0(B_0 P\chi_\infty + P\phi(\chi_\infty)) = 0 \text{ in } W_0^* \Big\}. \end{aligned} \tag{4.1}$$

In addition, we have

$$\operatorname{dist}_{\mathcal{H}}\big(S(t)(e_0, \chi_0, \eta_0), \omega(e_0, \chi_0, \eta_0)\big) \to 0$$

as $t \to \infty$, where $\operatorname{dist}_{\mathcal{H}}$ denotes the usual Hausdorff semidistance. Moreover, $\mathbf{L}$ *is constant on $\omega(e_0, \chi_0, \eta_0)$.*

4.3. **Main results.** The first theorem concerns the convergence to a single equilibrium.

Theorem 4.4. *Let assumptions* (H1)–(H2), (H4), (K1)–(K4) *hold. Then, for any fixed $(e_0, \chi_0, \eta_0) \in \mathcal{H}_{\beta,\gamma}$ there exists a solution χ_∞ to the equation*

$$B_0(B_0 P\chi_\infty + P\phi(\chi_\infty)) = 0 \quad \text{in } W_0^*, \tag{4.2}$$

such that

$$e(t) \to m_{e_0-\chi_0} + \chi_\infty \quad \text{in } H\,, \tag{4.3}$$

$$\chi(t) \to \chi_\infty \quad \text{in } V\,, \tag{4.4}$$

$$\eta^t \to 0 \quad \text{in } \mathcal{M}, \tag{4.5}$$

as $t \to \infty$. Moreover, there exist $t_1 > 0$ and a positive constant $\bar{c}$ such that

$$\|\chi(t) - \chi_\infty\|_{V^*} \le \bar{c}(1+t)^{-\frac{\rho}{2(1-2\rho)}}, \quad \forall\, t \ge t_1, \tag{4.6}$$

$\rho \in (0, 1/2)$ being the same constant as in the Lojasiewicz-Simon inequality (see Lemma 2.5).

In the viscous case, supposing further (H3), a stronger convergence result holds:

Theorem 4.5. *Let assumptions* (H1)–(H4), (K1)–(K4) *hold and let $\alpha > 0$. Then there exist $t_2 \ge t_1$ and a positive constant $\bar{c}_\alpha$ (which may singularly depend on α) such that*

$$\|z(t) - z_\infty\|_{\mathcal{H}} \le \bar{c}_\alpha(1+t)^{-\frac{\rho}{4(1-2\rho)}}, \quad \forall\, t \ge t_2, \tag{4.7}$$

having set

$$z_\infty = (m_{e_0-\chi_0} + \chi_\infty, \chi_\infty, 0),$$

being t_1, χ_∞ and ρ as in Theorem 4.4.

We shall provide a complete proof of Theorems 4.4 and 4.5 in Sections 5 and 6, respectively.

Remark 4.6. Since $\vartheta = e - \chi$, from (4.3) we deduce

$$\vartheta(t) \to m_{e_0 - \chi_0} = m_{\vartheta_0} \quad \text{in } H,$$

where m_{ϑ_0} denotes the temperature mean value. This is to be expected, since the material occupying the domain Ω is assumed to be thermally isolated.

Remark 4.7. We point out that inequality (4.7) provided by Theorem 4.5 displays a convergence rate for χ to χ_∞ in V which is actually faster than the one obtained by means of interpolation inequalities.

5. Proof of Theorem 4.4

5.1. **Proof of** (4.3) **and** (4.5)**.** Integrating both members of (3.5) on the interval $(0,t)$, thanks to Theorem 3.2 and bound (3.6), we immediately infer the dissipation integral

$$\int_0^\infty [\|P(e(t)-\chi(t))\|^2 + \|\eta^t\|^2_{\mathcal{M}} + \|\partial_t\chi\|^2_{V_0^*} + \alpha\|\partial_t\chi\|^2]dt \le c. \tag{5.1}$$

Since $\|P(e(\cdot)-\chi(\cdot))\|$ and $\|\eta^{\cdot}\|_{\mathcal{M}}$ are continuous functions with bounded derivatives (cf. (3.5)), then (5.1) yields (4.3) and (4.5).

5.2. **Proof of** (4.4)**.** In the course of the proof, the following result (see [12, Lemma 7.1]) will play a fundamental role.

Lemma 5.1. *Let $\Phi \in L^2(0,\infty)$, with $\|\Phi\|_{L^2(0,\infty)} \le b$, and suppose that there exist $a \in (1,2)$, $c > 0$ and an open set $\mathcal{P} \subset (0,\infty)$ such that*

$$\Big(\int_t^\infty \Phi^2(\tau)d\tau\Big)^a \le c\Phi^2(t) \quad \textit{for a.e. } t \in \mathcal{P}.$$

Then $\Phi \in L^1(\mathcal{P})$ and there exists a constant $C = C(a,b,c)$, independent of $\mathcal{P}$, such that

$$\int_{\mathcal{P}} \Phi(\tau)d\tau \le C.$$

We define the positive functional

$$\Phi(t) = \big[\|P(e(t)-\chi(t))\|^2 + \|\eta^t\|^2_{\mathcal{M}} + \|\partial_t\chi(t)\|^2_{V_0^*} + \alpha\|\partial_t\chi(t)\|^2\big]^{1/2}, \quad \forall t \in [0,\infty).$$

Integrating inequality (3.5) from t to ∞, Lemma 4.3 and inequality (3.6) yield immediately

$$\int_t^\infty \Phi^2(\tau)d\tau \le c[\|P(e(t)-\chi(t))\|^2 + \|\eta^t\|^2_{\mathcal{M}} + \big|\overline{E}(P\chi(t)) - \overline{E}(P\chi_\infty)\big|], \tag{5.2}$$

for some χ_∞, solution to equation (4.2). Setting now

$$\mathcal{P} = \{t \in (0,\infty) : \|\chi(t) - \chi_\infty\|_V < r\},$$

we can apply Lemma 2.5 by choosing $\varphi = \chi_\infty$, to get

$$\big|\overline{E}(\chi(t)) - \overline{E}(\chi_\infty)\big|^{1-\rho} \le \lambda\big\|B_0P\chi(t) + P\overline{\phi}(P\chi(t))\big\|_{V_0^*}, \tag{5.3}$$

for all $t \in \mathcal{P}$, where $\rho \in (0,1/2)$, $r > 0$ and $\lambda > 0$ are the same as in Lemma 2.5. By means of the identity

$$(B_0 + \alpha I)^{-1}\partial_t\chi - P(e-\chi) = -B_0P\chi - P\phi(\chi) \quad \text{in } W_0^*,$$

which holds for all $\alpha \geq 0$, inequality (5.3) turns into

$$\begin{aligned}
&\big|\overline{E}(P\chi(t)) - \overline{E}(P\chi_\infty)\big|^{1-\rho} \\
&\leq \lambda\big\|(B_0+\alpha I)^{-1}\partial_t\chi(t) - P(e(t)-\chi(t))\big\|_{V_0^*} \\
&\leq \lambda[\|\partial_t\chi(t)\|^2_{W_0^*} + \alpha\|\partial_t\chi(t)\|^2 + \|P(e(t)-\chi(t))\|^2],
\end{aligned} \tag{5.4}$$

for all $t \in \mathcal{P}$. Using (5.4) and the Poincaré inequality, inequality (5.2) yields

$$\begin{aligned}
\int_t^\infty \Phi^2(\tau)d\tau &\leq c[\|P(e(t)-\chi(t))\|^2 + \|\eta^t\|^2_{\mathcal{M}}] \\
&\quad + c[\|\partial_t\chi(t)\|^2_{W_0^*} + \alpha\|\partial_t\chi(t)\|^2 + \|P(e(t)-\chi(t))\|^2]^{1/(2-2\rho)} \\
&\leq c[\|P(e(t)-\chi(t))\|^2 + \|\eta^t\|^2_{\mathcal{M}}]^{1/(2-2\rho)} \\
&\quad + c[\|\partial_t\chi(t)\|^2_{V_0^*} + \alpha\|\partial_t\chi(t)\|^2 + \|P(e(t)-\chi(t))\|^2]^{1/(2-2\rho)} \\
&\leq c[\Phi^2(t)]^{1/(2-2\rho)},
\end{aligned}$$

for all $t \in \mathcal{P}$, provided that r is small enough. Notice that in the second inequality we have used (4.3) and (4.5). Since $2-2\rho \in (1,2)$, we can apply Lemma 5.1 to conclude that

$$\int_{\mathcal{P}} \|\partial_t\chi(t)\|_{V_0^*}dt < \infty,$$

so that, for any $t_1, t_2 \in \mathcal{P}$, with $t_1 < t_2$, we have

$$\|\chi(t_2) - \chi(t_1)\|_{V^*} \leq \int_{t_1}^{t_2} \|\partial_t\chi(t)\|_{V_0^*}dt < r/4, \tag{5.5}$$

provided that t_1 is large enough and the whole interval (t_1, t_2) lies in $\mathcal{P}$. Since χ_∞ is a solution to equation (4.2), and since the trajectory is precompact in $\mathcal{H}_{\beta,\gamma}$, then we can choose $t_0 > 0$ such that

$$\|\chi(t_0) - \chi_\infty\|_{V^*} < r/4. \tag{5.6}$$

Now set

$$T_0 = \inf\big\{t > t_0 : \|\chi(t) - \chi_\infty\|_{V^*} \geq r\big\};$$

clearly we have $T_0 > t_0$. If we assume that $T_0 < \infty$, we also infer

$$\|\chi(T_0) - \chi_\infty\|_{V^*} = r.$$

On the other hand, as a consequence of (5.5) and (5.6),

$$\|\chi(t) - \chi_\infty\|_{V^*} \leq \|\chi(t) - \chi(t_0)\|_{V^*} + \|\chi(t_0) - \chi_\infty\|_{V^*} < r/2,$$

for all $t \in [t_0, T_0)$, which, by contradiction, implies $T_0 = \infty$ and, consequently, $[t_0, \infty) \subset \mathcal{P}$. Therefore, we have

$$\chi(t) \to \chi_\infty \text{ in } V^*,$$

as $t \to \infty$. Convergence (4.4) follows by the precompactness of trajectories provided by Corollary 3.5.

5.3. **Proof of inequality** (4.6)**.** We set, for all $t \in [0,\infty)$

$$\Lambda_0(t) = \frac{1}{2}\|P(e(t)-\chi(t))\|^2 + \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}} + \overline{E}(P\chi(t)) + \nu L(t) - \overline{E}(P\chi_\infty).$$

By inequality (3.5), we immediately deduce that Λ_0 is a positive monotone nonincreasing function and

$$\frac{d}{dt}\Lambda_0(t) + c(\mathcal{N}(e(t),\chi(t),\eta^t))^2 \le 0, \quad \forall\, t \in [0,\infty),$$

with

$$\mathcal{N}(e,\chi,\eta) = \|P(e-\chi)\| + \|\eta\|_{\mathcal{M}} + \|\partial_t\chi\|_{V_0^*} + \alpha\|\partial_t\chi\|.$$

On account of the convergence results (4.3), (4.4) and (4.5), there exists $t_1 > 0$ such that

$$\frac{d}{dt}\Lambda_0(t) + c[\Lambda_0(t)]^{1-\rho} \le 0, \quad \forall\, t \ge t_1,$$

$\rho \in (0,1/2)$ being as in Theorem 2.5. This yields

$$\Lambda_0(t) \le c(1+t)^{-\frac{1}{2(1-2\rho)}}, \quad \forall\, t \ge t_1. \tag{5.7}$$

On the other hand, we observe that

$$[\Lambda_0(t)]^{1-\rho} \le c\mathcal{N}(e(t),\chi(t),\eta^t), \quad \forall\, t \ge t_1,$$

and

$$\frac{d}{dt}[\Lambda_0(t)]^{\rho} = \rho[\Lambda_0(t)]^{-1+\rho}\, \frac{d}{dt}\Lambda_0(t) \le 0, \quad \forall\, t \ge t_1.$$

Therefore, for any $t \ge t_1$, we get

$$\mathcal{N}(e(t),\chi(t),\eta^t) \le -c\, \frac{d}{dt}[\Lambda_0(t)]^{\rho}.$$

Thus, integrating the above inequality from t to ∞, we obtain

$$\int_t^\infty \mathcal{N}(e(\tau),\chi(\tau),\eta^\tau)d\tau \le c[\Lambda_0(t)]^{\rho}, \quad \forall\, t \ge t_1.$$

Hence, on account of (5.7), we immediately infer

$$\int_t^\infty \|\partial_t\chi(\tau)\|_{V_0^*}d\tau \le c(1+t)^{-\frac{\rho}{2(1-2\rho)}}, \quad \forall\, t \ge t_1.$$

Finally, using

$$\chi(t) - \chi_\infty = -\int_t^\infty \partial_t\chi(\tau)d\tau \quad \text{in } V_0^*,$$

we deduce (4.6). The proof is thus complete.

6. Proof of Theorem 4.5

We first recall the decomposition already exploited in [25, Section 7]. That is

$$z(t) = z^d(t) + z^c(t),$$

where

$$z^d(t) = (e^d(t),\chi^d(t),\eta^{d,t}) \quad \text{and} \quad z^c(t) = (e^c(t),\chi^c(t),\eta^{c,t})$$

are the solutions at time $t \in [0,\infty)$ to the following problems, respectively,

$$\partial_t e^d + \int_0^\infty \mu(s)\eta^d(s)ds = 0, \tag{6.1}$$

$$\partial_t \chi^d + B_0\big(B_0\chi^d + \alpha\partial_t\chi^d + P(\psi(\chi) - \psi(\chi^c)) - (e^d - \chi^d)\big) = 0, \tag{6.2}$$

$$\partial_t \eta^d = T\eta^d + B_0(e^d - \chi^d), \tag{6.3}$$

$$z^d(0) = (Pe_0, P\chi_0, \eta_0), \tag{6.4}$$

and

$$\partial_t e^c + \int_0^\infty \mu(s)\eta^c(s)ds = 0, \tag{6.5}$$

$$\partial_t \chi^c + B_0\big(B_0 P\chi^c + \alpha\partial_t\chi^c + P\psi(\chi^c) - P(e^c - \chi^c)\big) = \theta B_0 P\chi, \tag{6.6}$$

$$\partial_t \eta^c = T\eta^c + B_0 P(e^c - \chi^c), \tag{6.7}$$

$$z^c(0) = (m_{e_0}, m_{\chi_0}, 0), \tag{6.8}$$

having set

$$\psi(r) = \phi(r) + \theta r, \quad \forall r \in \mathbb{R}.$$

for some $\theta \geq \ell$ (cf. (H3)). From [25, Lemmas 7.3 and 7.4], we know that

$$\|z^d(t)\|_{\mathcal{H}} \leq c_\alpha e^{-\kappa_d t} \quad \text{and} \quad \|z^c(t)\|_{\mathcal{V}} \leq c_\alpha, \quad \forall t \in [0,\infty), \tag{6.9}$$

for some positive κ_d and c_α, independent of $t \in [0,\infty)$, uniformly in $\mathcal{B}_0$.

We now introduce the function

$$\overline{z}^c(t) = (\overline{e}^c(t), \overline{\chi}^c(t), \overline{\eta}^{c,t}) = z^c(t) - z_\infty \in \mathcal{V}_{0,0},$$

being z_∞ as in the statement of Theorem 4.5. Notice that, as z_∞ is a stationary solution, then $\partial_t \overline{z}^c = \partial_t z^c$, and, by (6.9) we also deduce $\|\overline{z}^c(t)\|_{\mathcal{V}} \leq c_\alpha$ for all $t \in [0,\infty)$.

Arguing as in [25, Section 4], it is possible to prove the following two inequalities:

$$\frac{d}{dt}L(\overline{z}^c) + \frac{k_0}{4}\|\overline{e}^c - \overline{\chi}^c\|^2 \leq 2\|\overline{\eta}^c\|^2_{\mathcal{M}} + c\|\partial_t\overline{\chi}^c\|^2_{V_0^*} - c\int_0^\infty \mu'(s)\|\overline{\eta}^c(s)\|^2_{V_0^*}ds, \tag{6.10}$$

and

$$\begin{aligned} &\frac{d}{dt}[\|\overline{e}^c - \overline{\chi}^c\|^2 + \|\overline{\eta}^c\|^2_{\mathcal{M}}] + \lambda\|\overline{\eta}^c\|^2_{\mathcal{M}} \\ &- \int_0^\infty \mu'(s)\|\overline{\eta}^c(s)\|^2_{V_0^*}ds + 2\langle\partial_t\overline{\chi}^c, \overline{e}^c - \overline{\chi}^c\rangle \leq 0, \end{aligned} \tag{6.11}$$

L being the functional defined in Remark 3.3.

We now perform the following products of equation (6.6) by suitable test functions.

- By $B_0^{-1}\partial_t\overline{\chi}^c$, to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\overline{\chi}^c\|^2_V + \|\partial_t\overline{\chi}^c\|^2_{V_0^*} + \alpha\|\partial_t\overline{\chi}^c\|^2 - \langle\overline{e}^c - \overline{\chi}^c, \partial_t\overline{\chi}^c\rangle \\ &= -\langle\psi(\chi^c), \partial_t\overline{\chi}^c\rangle + \theta\langle\chi - \chi_\infty, \partial_t\overline{\chi}^c\rangle + (\theta-1)\langle\chi_\infty, \partial_t\overline{\chi}^c\rangle. \end{aligned} \tag{6.12}$$

Since

$$\begin{aligned} &-\langle\psi(\chi^c), \partial_t\overline{\chi}^c\rangle + \theta\langle\chi - \chi_\infty, \partial_t\overline{\chi}^c\rangle + (\theta-1)\langle\chi_\infty, \partial_t\overline{\chi}^c\rangle \\ &= \frac{d}{dt}\big[-\langle\psi(\chi^c), \overline{\chi}^c\rangle + (\theta-1)\langle\chi_\infty, \overline{\chi}^c\rangle\big] + \langle\psi'(\chi^c)\partial_t\overline{\chi}^c, \overline{\chi}^c\rangle + \theta\langle\chi - \chi_\infty, \partial_t\overline{\chi}^c\rangle, \end{aligned}$$

by means of control (6.9) and interpolation inequalities, it is immediate that

$$\begin{aligned}
&\langle \psi'(\chi^c)\partial_t\overline{\chi}^c, \overline{\chi}^c\rangle + \theta\langle \chi - \chi_\infty, \partial_t\overline{\chi}^c\rangle \\
&\le c(1 + \|\chi^c\|_W^2)\|\partial_t\overline{\chi}^c\|\|\overline{\chi}^c\| + c\|\chi - \chi_\infty\|\|\partial_t\overline{\chi}^c\| \\
&\le \alpha\|\partial_t\overline{\chi}^c\|^2 + c_\alpha\|\overline{\chi}^c\|^2 + c_\alpha\|\chi - \chi_\infty\|^2 \\
&\le \alpha\|\partial_t\overline{\chi}^c\|^2 + c_\alpha\|\overline{\chi}^c\|_{V^*} + c_\alpha\|\chi - \chi_\infty\|_{V^*},
\end{aligned}$$

so that, back to inequality above, we infer

$$\begin{aligned}
&\frac{d}{dt}[\|\overline{\chi}^c\|_V^2 + 2\langle \psi(\chi^c), \overline{\chi}^c\rangle - 2(\theta - 1)\langle \chi_\infty, \overline{\chi}^c\rangle] + 2\|\partial_t\overline{\chi}^c\|_{V_0^*}^2 - 2\langle \overline{e}^c - \overline{\chi}^c, \partial_t\overline{\chi}^c\rangle \\
&\le c_\alpha\|\overline{\chi}^c\|_{V^*} + c_\alpha\|\chi - \chi_\infty\|_{V^*}.
\end{aligned} \tag{6.13}$$

- By $B_0^{-1}\overline{\chi}^c$, to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\overline{\chi}^c\|_{V^*}^2 + \alpha\|\overline{\chi}^c\|^2] + \|\overline{\chi}^c\|_V^2 \\
&= -\langle \psi(\chi^c), \overline{\chi}^c\rangle + \langle \overline{e}^c - \overline{\chi}^c, \overline{\chi}^c\rangle + \theta\langle \chi, \overline{\chi}^c\rangle - \langle \chi_\infty, \overline{\chi}^c\rangle.
\end{aligned}$$

Once again, using (6.9), we have

$$\begin{aligned}
&- \langle \psi(\chi^c), \overline{\chi}^c\rangle + \langle \overline{e}^c - \overline{\chi}^c, \overline{\chi}^c\rangle + \theta\langle \chi, \overline{\chi}^c\rangle - \langle \chi_\infty, \overline{\chi}^c\rangle \\
&\le \|\psi(\chi^c)\|_V\|\overline{\chi}^c\|_{V^*} + \|\overline{e}^c - \overline{\chi}^c\|_V\|\overline{\chi}^c\|_{V^*} + \theta\|\chi\|_V\|\overline{\chi}^c\|_{V^*} + \|\chi_\infty\|_V\|\overline{\chi}^c\|_{V^*} \\
&\le c\|\chi^c\|_W^2\|\overline{\chi}^c\|_{V^*} + c_\alpha\|\overline{\chi}^c\|_{V^*} \le c_\alpha\|\overline{\chi}^c\|_{V^*}.
\end{aligned}$$

Thus, we deduce

$$\frac{d}{dt}[\|\overline{\chi}^c\|_{V^*}^2 + \alpha\|\overline{\chi}^c\|^2] + 2\|\overline{\chi}^c\|_V^2 \le c_\alpha\|\overline{\chi}^c\|_{V^*}. \tag{6.14}$$

Adding (6.11), (6.13), (6.14) and ν times (6.10), we have

$$\begin{aligned}
&\frac{d}{dt}\Theta(t) + \frac{k_0}{4}\|\overline{e}^c - \overline{\chi}^c\|^2 + 2\|\overline{\chi}^c\|_V^2 + \lambda\|\overline{\eta}^c\|_{\mathcal{M}}^2 + (2 - \nu c)\|\partial_t\overline{\chi}^c\|_{V_0^*}^2 \\
&- (1 - \nu c)\int_0^\infty \mu'(s)\|\overline{\eta}^c(s)\|_{V_0^*}^2 ds \\
&\le c_\alpha\|\overline{\chi}^c\|_{V^*} + c_\alpha\|\chi - \chi_\infty\|_{V^*},
\end{aligned} \tag{6.15}$$

where, for all $t \in [0, \infty)$, we have set

$$\begin{aligned}
\Theta(t) &= \|\overline{e}^c(t) - \overline{\chi}^c(t)\|^2 + \|\overline{\chi}^c(t)\|_{V^*}^2 + \alpha\|\overline{\chi}^c(t)\|^2 + \|\overline{\chi}^c(t)\|_V^2 + \|\overline{\eta}^{c,t}\|_{\mathcal{M}}^2 \\
&\quad + 2\langle \psi(\chi^c(t)), \overline{\chi}^c(t)\rangle - 2(\theta - 1)\langle \chi_\infty, \overline{\chi}^c\rangle + \nu L(\overline{z}^c(t)).
\end{aligned}$$

Since, as previously shown,

$$\langle \psi(\chi^c), \overline{\chi}^c\rangle - 2(\theta - 1)\langle \chi_\infty, \overline{\chi}^c\rangle \le c_\alpha\|\overline{\chi}^c\|_{V^*},$$

recalling also (3.6), then there exist constants $0 < c_1 < c_2$ such that

$$c_1\|\overline{z}^c(t)\|_{\mathcal{H}}^2 - c_\alpha\|\overline{\chi}^c(t)\|_{V^*} \le \Theta(t) \le c_2\|\overline{z}^c(t)\|_{\mathcal{H}}^2 + c_\alpha\|\overline{\chi}^c(t)\|_{V^*}, \quad \forall t \in [0, \infty). \tag{6.16}$$

Note that, as a consequence of inequalities (4.6) and (6.9), we have

$$\begin{aligned}
c_\alpha\|\overline{\chi}^c\|_{V^*} + c_\alpha\|\chi - \chi_\infty\|_{V^*} &\le c_\alpha\|\chi^d\|_{V^*} + c_\alpha\|\chi - \chi_\infty\|_{V^*} \\
&\le c_\alpha e^{-\kappa_d t} + c_\alpha(1 + t)^{-\frac{\rho}{2(1-2\rho)}} \\
&\le c_\alpha(1 + t)^{-\frac{\rho}{2(1-2\rho)}}, \quad \forall\ t \ge t_*,
\end{aligned}$$

for some $t_* \geq t_1$, being t_1 as in (4.6). Therefore, by (6.15), provided that we choose ν small enough, we get the inequality

$$\frac{d}{dt}\Theta(t) + \kappa\Theta(t) \leq c_\alpha (1+t)^{-\frac{\rho}{2(1-2\rho)}}.$$

By means of the Gronwall lemma and (6.16), we then derive, for all $t \geq 2t_*$

$$\begin{aligned}
\Theta(t) &\leq 2\Theta(t_*)e^{-\kappa(t-t_*)} + c_\alpha \int_{t_*}^{t} (1+\tau)^{-\frac{\rho}{2(1-2\rho)}} e^{-\kappa(t-\tau)} d\tau \\
&= 2\Theta(t_*)e^{-\kappa(t-t_*)} + c_\alpha \int_{t_*}^{t/2} (1+\tau)^{-\frac{\rho}{2(1-2\rho)}} e^{-\kappa(t-\tau)} d\tau \\
&\quad + c_\alpha \int_{t/2}^{t} (1+\tau)^{-\frac{\rho}{2(1-2\rho)}} e^{-\kappa(t-\tau)} d\tau \\
&\leq 2\Theta(t_*)e^{-\kappa(t-t_*)} + c_\alpha (1+t_*)^{-\frac{\rho}{2(1-2\rho)}} e^{-\kappa/2t} + c_\alpha (1+t/2)^{-\frac{\rho}{2(1-2\rho)}} \\
&\leq c_\alpha (1+t/2)^{-\frac{\rho}{2(1-2\rho)}},
\end{aligned}$$

so that, keeping (6.16) into account, from inequality above we deduce

$$\|\overline{z}^c(t)\|^2_{\mathcal{H}} \leq c_\alpha e^{-\kappa t} + c_\alpha (1+t)e^{-\kappa/2t} + c_\alpha (1+t/2)^{-\frac{\rho}{2(1-2\rho)}} \leq \overline{c}_\alpha (1+t)^{-\frac{\rho}{2(1-2\rho)}},$$

for all $t \geq t_2$, for some $t_2 \geq t_*$. Inequality (4.7) is then achieved by noticing that

$$z(t) - z_\infty = z^d(t) + \overline{z}^c(t),$$

and recalling (6.9).

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