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REPRODUCING KERNEL METHODS FOR SOLVING LINEAR INITIAL-BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper, a reproducing kernel with polynomial form is used for finding analytical and approximate solutions of a second-order hyperbolic equation with linear initial-boundary conditions. The analytical solution is represented as a series in the reproducing kernel space, and the approximate solution is obtained as an n-term summation. Error estimates are proved to converge to zero in the sense of the space norm, and a numerical example is given to illustrate the method.

1. Introduction

A reproducing kernel Hilbert space is a useful framework for constructing approximate solutions of partial differential equations (PDE). Many numerical methods have been proposed for solving linear and nonlinear PDEs, but we did not find a method that uses reproducing kernels. In this paper, we focus on the exact and approximate solutions to PDEs with linear initial-boundary conditions. A reproducing kernel with polynomial form in the corresponding Hilbert space is given and the space completion is proved.

We consider the following second-order one-dimensional hyperbolic equation in a reproducing kernel space:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1,\ t \geq 0 \tag{1.1}$$

subject to the mixed boundary conditions

$$\frac{\partial u(0,t)}{\partial x} + w_1 u(0,t) = 0, \quad w_1 \in \mathbb{R},\ t \geq 0, \tag{1.2}$$

$$\frac{\partial u(1,t)}{\partial x} + w_2 u(1,t) = 0, \quad w_2 \in \mathbb{R},\ t \geq 0 \tag{1.3}$$

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and the initial conditions

$$u(x,0) = g_1(x) \quad 0 < x < 1, \tag{1.4}$$

$$\frac{\partial u(x,0)}{\partial t} = g_2(x) \quad 0 < x < 1 \tag{1.5}$$

Although the focus is on homogeneous mixed boundary conditions, we can study problems with non-homogenous mixed boundary conditions by the homogenization methods. Equation (1.1) with conditions (1.2)-(1.5) has been applied to many problems in physics, engineering, fluid mechanics, and so on. It is well known that the finite difference method [1], spline method [5], and collocation method [4] can be used to solve this equation. Now, employing the reproducing property of the kernel, we give an efficient method for solving (1.1). The analytical solution is represented in the form of series in the reproducing kernel space, and the approximate solution is obtained by the n-term intercept of the analytical solution, and error is proved to converge to zero in the sense of the space norm. In Section 2, the reproducing kernel function with polynomial form is obtained, and one-dimensional and two-dimensional reproducing kernel spaces needed in this paper are defined. After that, we devote Section 3 to solve (1.1) with initial-boundary value conditions (1.2)-(1.5). Finally, a numerical example is discussed to demonstrate the accuracy of the presented method in Section 4.

2. The reproducing kernel space

In this section, several reproducing kernel spaces are introduced. Throughout this paper, we discuss problems on the domain $\Omega = [a,b] \times [c,d]$.

One-dimensional reproducing kernel space. The reproducing kernel space $W_2^m[a,b]$ is defined as the set of functions such that $u^{(m-1)}(x)$is absolutely continuous on $[a,b]$, and $u^{(m)}(x) \in L^2[a,b]$, for $x \in [a,b]$, where m is a positive integer. This space is equipped with the inner product

$$\langle u(x), v(x)\rangle = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \tag{2.1}$$

for $u(x), v(x) \in W_2^m[a,b]$. The the norm is

$$\|u\| = \sqrt{\langle u(x), u(x)\rangle}, \quad u(x) \in W_2^m[a,b]\,. \tag{2.2}$$

Theorem 2.1. *The space $W_2^m[a,b]$ equipped with the norm (2.2), is a Hilbert space.*

Proof. Suppose that $\{f_n(x)\}_{i=1}^{\infty}$ is a Cauchy sequence of the space $W_2^m[a,b]$, that is, as $n \to \infty$, it follows that

$$\|f_{n+p} - f_n\|^2 = \sum_{i=0}^{m-1} (f_{n+p}^{(i)}(a) - f_n^{(i)}(a))^2 + \int_a^b (f_{n+p}^{(m)}(x) - f_n^{(m)}(x))^2 dx \to 0\,.$$

So, we have

$$f_{n+p}^{(i)}(a) - f_n^{(i)}(a) \to 0, \quad i = 0, 1, \dots, m-1,$$
$$\int_a^b (f_{n+p}^{(m)}(x) - f_n^{(m)}(x))^2 dx \to 0\,.$$

The above formulas show that, for $0 \le i \le m-1$, $f_n^{(i)}(a)$ $(n = 1, 2, \dots)$ are Cauchy sequences and $f_n^{(m)}(x)$ $(n = 1, 2, \dots)$ is a Cauchy sequence in $L^2[a, b]$. Hence, there exist the unique real number λ_i $(i = 0, 1, \dots, m-1)$ and the unique function $h(x) \in L^2[a, b]$. It holds that

$$\lim_{n\to\infty} f_n^{(i)}(a) = \lambda_i, \quad (i = 0, 1, \dots, m-1),$$
$$\lim_{n\to\infty} \int_a^b (f_n^{(m)}(x) - h(x))^2 dx = 0 .$$

Let

$$g(x) = \sum_{k=0}^{m-1} \frac{\lambda_k}{k!}(x-a)^k + \underbrace{\int_a^x \dots \int_a^x}_{m} h(x)dx^m,$$

from $h(x) \in L^2[a, b]$, we obtain that $g^{(m-1)}(x) = \lambda_{m-1} + \int_a^x h(x)dx$ is absolutely continuous on the interval $[a, b]$, and $g^{(m)}(x)$ is almost equal to $h(x)$ on the interval $[a, b]$. Hence, $g(x) \in W_2^m[a, b]$, and $g^{(i)}(a) = \lambda_i$ $(0 \le i \le m-1)$. Then

$$\begin{aligned}
\|f_n(x) - g(x)\|^2 &= \sum_{i=0}^{m-1} (f_n^{(i)}(a) - g^{(i)}(a))^2 + \int_a^b (f_n^{(m)}(x) - g^{(m)}(x))^2 dx \\
&= \sum_{i=0}^{m-1} (f_n^{(i)}(a) - \lambda_i)^2 + \int_a^b (f_n^{(m)}(x) - h(x))^2 dx \to 0 .
\end{aligned}$$

Hence,Space $W_2^m[a, b]$ equipped with the norm (2.2), is a Hilbert space. □

Theorem 2.2. *Hilbert space $W_2^m[a, b]$ is a reproducing kernel space, that is, for for all $f(y) \in W_2^m[a, b]$ and fixed $x \in [a, b]$, there exists $R_m(x, y) \in W_2^m[a, b]$ such that*

$$\langle f(x), R_m(x, y)\rangle = f(y) \tag{2.3}$$

and $R_m(x, y)$ is called the reproducing kernel function of space $W_2^m[a, b]$.

Proof. Let $R_m(x, y)$ be the reproducing kernel function. By (2.1) and (2.2), we have

$$\langle f(x), R_m(x, y)\rangle = \sum_{i=0}^{m-1} f^{(i)}(a) \frac{\partial^i R_m(a, y)}{\partial x^i} + \int_a^b f^{(m)}(x) \frac{\partial^m R_m(a, y)}{\partial x^m} dx . \tag{2.4}$$

Applying the integration by parts for the second scheme of the right-hand of (2.4), we obtain

$$\begin{aligned}
&\int_a^b f(x) \frac{\partial^m R_m(a, y)}{\partial x^m} dx \\
&= \sum_{i=0}^{m-1} (-1)^i f^{(m-i-1)}(x) \frac{\partial^{m+i} R_m(a, y)}{\partial x^{m+i}} |_{x=a}^b + (-1)^m \int_a^b f(x) \frac{\partial^{2m} R_m(a, y)}{\partial x^{m+i}} dx .
\end{aligned} \tag{2.5}$$

Let $j = m - i - 1$, the first term of the right-hand side of the above formula is rewritten as

$$\begin{aligned}&\sum_{i=0}^{m-1}(-1)^i f^{(m-i-1)}(x)\frac{\partial^{m+i}R_m(a,y)}{\partial x^{m+i}}|_{x=a}^{b}\\&=\sum_{j=0}^{m-1}(-1)^{m-j-1}f^{j}(x)\frac{\partial^{2m-j-1}R_m(a,y)}{\partial x^{2m-j-1}}|_{x=a}^{b}.\end{aligned}\tag{2.6}$$

Let $i = j$. Then substituting the two expressions (2.5) and (2.6) into (2.4) yields

$$\begin{aligned}&\langle f(x), R_m(x,y)\rangle\\&=\sum_{i=0}^{m-1} f^{(i)}(a)\Big(\frac{\partial^i R_m(a,y)}{\partial x^i}-(-1)^{m-i-1}\frac{\partial^{2m-i-1}R_m(a,y)}{\partial x^{2m-i-1}}\Big)\\&\quad+\sum_{i=0}^{m-1}(-1)^{m-i-1}f^{i}(b)\frac{\partial^{2m-i-1}R_m(a,y)}{\partial x^{2m-i-1}}+(-1)^m\int_a^b f(x)\frac{\partial^{2m}R_m(x,y)}{\partial x^{2m}}dx\,.\end{aligned}$$

Suppose that $R_m(x,y)$ satisfies the following generalized differential equations

$$\begin{gathered}(-1)^m\frac{\partial^{2m}R_m(x,y)}{\partial x^{2m}}=\delta(x-y)\\\frac{\partial^i R_m(a,y)}{\partial x^i}-(-1)^{m-i-1}\frac{\partial^{2m-i-1}R_m(a,y)}{\partial x^{2m-i-1}}=0,\quad i=0,1,\dots,m-1\\\frac{\partial^{2m-i-1}R_m(b,y)}{\partial x^{2m-i-1}}=0,\quad i=0,1,\dots,m-1\,.\end{gathered}\tag{2.7}$$

Then $\langle f(x), R_m(x,y)\rangle = \int_a^b f(x)\delta(x-y)dx = f(y)$. Hence, $R_m(x,y)$ is the reproducing kernel of space $W_2^m[a,b]$. □

Next, we give the expression of the reproducing kernel $R_m(x,y)$. The characteristic equation of (2.7) is $\lambda^{2m} = 0$, and the characteristic roots are $\lambda_i = 0$, $i = 1,\dots,2m$. So we write the reproducing kernel as

$$R_m(x,y)=\begin{cases}lR_m(x,y)=\sum_{i=1}^{2m}c_i(y)x^{i-1}, & x<y\\ rR_m(x,y)=\sum_{i=1}^{2m}d_i(y)x^{i-1}, & x>y\,.\end{cases}\tag{2.8}$$

By the definition of $W_2^m[a,b]$ and (2.7), the coefficients c_i, d_i, $i = 1,\dots,2m$ satisfy

$$\begin{gathered}\frac{\partial^i lR_m(y,y)}{\partial x^i}=\frac{\partial^i rR_m(y,y)}{\partial x^i},\quad i=0,1,\dots,2m-2\\(-1)^m\Big(\frac{\partial^{2m-1}rR_m(y+,y)}{\partial x^{2m-1}}-\frac{\partial^{2m-1}lR_m(y-,y)}{\partial x^{2m-1}}\Big)=1\\\frac{\partial^i R_m(a,y)}{\partial x^i}-(-1)^{m-i-1}\frac{\partial^{2m-i-1}R_m(a,y)}{\partial x^{2m-i-1}}=0,\quad i=0,1,\dots,m-1\\\frac{\partial^{2m-i-1}R_m(b,y)}{\partial x^{2m-i-1}}=0,\quad i=0,1,\dots,m-1\,.\end{gathered}\tag{2.9}$$

Then the solution of (2.9) yields the expression of the reproducing kernel $R_m(x,y)$.

In this paper we consider the case $m = 3$ and $[a,b] = [0,1]$, the corresponding kernel space is defined as $W_2^3[0,1]$ are function f such that $f^{(2)}(x)$ is absolutely continuous on $[0,1]$ and $f^{(3)}(x) \in L^2[0,1], x \in [0,1]$, and the reproducing kernel as

$$R_3(x,y) = \begin{cases} \frac{1}{120}(120 + x^5 + 120xy - 5x^4y + 30x^2y^2 + 10x^3y^2), & x < y \\ 1 + \frac{y^5}{120} + \frac{1}{12}x^2y^2(3+y) + x(y - \frac{y^4}{24}), & x > y) \end{cases}$$

Other reproducing kernel spaces needed in this paper are described similarly.

The space $W^3_{2,1}[0,1]$ is a subspace of $W^3_2[0,1]$ with $f(0) = f'(0) = 0$, and the reproducing kernel is

$$R_{31}(x,y) = \begin{cases} \frac{1}{120}x^2(x^3 - 5x^2y + 30y^2 + 10xy^2), & x < y, \\ \frac{1}{120}y^2(y^3 - 5xy^2 + 10x^2(3+y)), & x > y. \end{cases}$$

The space $W^3_{2,2}[0,1]$ is a subspace of $W^3_2[0,1]$ with $f(0) + w_1 f'(0) = 0$, $f(1) + w_2 f'(1) = 0$, and the reproducing kernel is

$$\begin{aligned} &R_{31}(x,y) \\ &= \begin{cases} \frac{1}{8400}\Big(-350x^4y + x^5(46 - 48y + 30y^2 + 10y^3 - y^5) + 24(46 + 92y \\ +30y^2 + 10y^3 - y^5) + 48x(46 + 92y + 30y^2 + 10y^3 - y^5) + 30x^2(24 + 48y \\ +40y^2 - 10y^3 + y^5) + 10x^3(24 + 48y + 40y^2 - 10y^3 + y^5)\Big), \\ \text{ if } x < y, \\ \frac{1}{8400}\Big(1104 + 2208y + 720y^2 + 240y^3 + x(2208 + 4416y + 1440y^2 + 480y^3 \\ -350y^4 - 48y^5) + 10x^3(24 + 48y - 30y^2 - 10y^3 + -y^5) - x^5(24 + 48y \\ -30y^2 - 10y^3 + y^5) + 10x^2(72 + 144y + 120y^2 + 40y^3 + 3y^5)\Big), \\ \text{ if } x > y. \end{cases} \end{aligned}$$

Two-dimensional reproducing kernel space. We construct the two-dimensional reproducing kernel spaces as in [2, 3]. Let

$$P(\Omega) = \overline{W^3_{2,1}[0,1] \otimes W^3_{2,2}[0,1]} \quad = \{\sum_{i,j=1}^{\infty} c_{i,j} g_i^{(1)}(x) g_j^{(2)}(t) : \sum_{i,j=1}^{\infty} |c_{i,j}|^2 < \infty\}$$

where $\{g_i^{(k)}\}$ is a complete orthonormal sequence in the space $W^3_{2,k}$, $k = 1, 2$, and endowed with the inner product

$$\begin{aligned} (u(x,t), v(x,t))_P &= (\sum_{k,l=1}^{\infty} c^{(1)}_{k,l} g^{(1)}_k(x) g^{(2)}_l(t), \sum_{p,q=1}^{\infty} c^{(2)}_{p,q} g^{(1)}_p(x) g^{(2)}_q(t)) \\ &= \sum_{k,l=1}^{\infty} c^{(1)}_{k,l} \sum\lrcorner_{p,q=1}^{\infty} c^{(2)}_{p,q} (g^{(1)}_k(x) g^{(2)}_l(t), g^{(1)}_p(x) g^{(2)}_q(t)) \\ &= \sum_{k,l=1}^{\infty} c^{(1)}_{k,l} c^{(2)}_{k,l} \end{aligned}$$

and the norm

$$\|u\|_P = \sqrt{(u,u)_P} = (\sum_{k,l=1}^{\infty} c^2_{k,l})^{1/2}$$

According to [4], the space $P(\Omega)$ is a Hilbert space with the norm $\|\cdot\|_P$, and possesses the reproducing kernel

$$\bar{R}((\xi,\eta),(x,t)) = R_{31}(\xi,x)\cdot R_{32}(\eta,t).$$

It is easy to prove that the following properties hold.

Property 2.3. *For $u(x)\in W_{3,1}[0,1]$, $v(y)\in W_{32}[0,1]$, it follows that $u(x)\cdot v(y)\in P(\Omega)$.*

Property 2.4. $(u_1(x)\cdot v_1(y),\, u_2(x)\cdot v_2(y))_P = \langle u_1(x),\, u_2(x)\rangle_{W_{31}}\cdot\langle v_1(y),\, v_2(y)\rangle_{W_{32}}$ *holds for any $u_1,\, u_2\in W_{3,1}[0,1]$, $v_1, v_2\in W_{32}[0,1]$.*

Similarly, the other two-dimensional reproducing kernel space can be defined as

$$\begin{aligned} P_1(\Omega) &= \overline{W_2^1[0,1]\otimes W_2^1[0,1]} \\ &= \{\sum_{i,j=1}^{\infty} c_{i,j} g_i(x) g_j(t) : \sum_{i,j=1}^{\infty} |c_{i,j}|^2 < \infty\}, \end{aligned}$$

where $\{g_i\}$ is a complete orthonormal sequence in the space W_2^1. Its reproducing kernel function $\tilde{R}((\xi,\eta),(x,t))$ can be obtain from the reproducing kernel function $R_1(x,y)$ of the space $W_2^1[0,1]$, that is, $\tilde{R}((\xi,\eta),(x,t)) = R_1(\xi,x)\cdot R_1(\eta,t)$.

3. Solution of Equation (1.1)

In this section, we consider the second-order one-dimensional hyperbolic equation (1.1) with initial-value conditions (1.4)–(1.5) and mixed boundary-value conditions (1.2)–(1.3). Without the loss of generality, we discuss equation (1.1) with homogeneous conditions, that is,

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0<x<1,\ t\ge 0 \\ \frac{\partial u(0,t)}{\partial x} + w_1 u(0,t) = 0, \quad w_1\in R,\ t\ge 0 \\ \frac{\partial u(1,t)}{\partial x} + w_2 u(1,t) = 0, \quad w_2\in R,\ t\ge 0 \\ u(x,0) = 0, \quad 0<x<1 \\ \frac{\partial u(x,0)}{\partial t} = 0, \quad 0<x<1 \end{gathered} \tag{3.1}$$

Through the homogenization, we can complete the equivalence transformation. Hence, we can solve (3.1) in the same way as (1.1).

Define the linear operator L from the reproducing kernel space $P(\Omega)$ to the reproducing kernel space $P_1(\Omega)$:

$$(Lu)(x,t) = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\,. \tag{3.2}$$

Theorem 3.1. *The operator $L : P(\Omega)\to P_1(\Omega)$ is a bounded operator.*

Proof. Note that

$$\begin{gathered}
\|(Lu)(x,t)\|^2 = \|u_{tt} - u_{xx}\|^2 \le \|u_{tt}\|^2 + \|u_{xx}\|^2, \\
u(x,t) = (u(\xi,\eta),\, R_{31}(\xi,x)R_{32}(\eta,t)) \\
u_{tt}(x,t) = (u(\xi,\eta), R_{31}(\xi,x)\frac{\partial^2}{\partial t^2}R_{32}(\eta,t)) \\
u_{xx}(x,t) = (u(\xi,\eta), \frac{\partial^2}{\partial x^2}R_{31}(\xi,x)R_{32}(\eta,t)) \\
|u_{tt}(x,t)| \le \|u\|\,\|R_{31}(\xi,x)\|\,\|\frac{\partial^2}{\partial t^2}R_{32}(\eta,t)\| \\
|u_{xx}(x,t)| \le \|u\|\,\|\frac{\partial^2}{\partial x^2}R_{31}(\xi,x)\|\,\|R_{32}(\eta,t)\|\,.
\end{gathered}$$

Also note that

$$\|R_{31}(\xi,x)\| = \sqrt{\langle R_{31}(\xi,x),\, R_{31}(\xi,x)\rangle} = \sqrt{R_{31}(x,x)}, \quad \|R_{32}(\eta,t))\| = \sqrt{R_{32}(t,t)}$$

are continuous functions on the interval $[0,1]$; that is, it holds that $\|R_{31}(\xi,x)\| \le M_1$, $\|R_{32}(\eta,t))\| \le M_2$. Meanwhile, setting $\|\frac{\partial^2}{\partial x^2}R_{31}(\xi,x)\| = M_3$, $\|\frac{\partial^2}{\partial t^2}R_{32}(\eta,t)\| = M_4$, we have

$$|u_{tt}(x,t)| \le \|u\|M_1M_4, \quad |u_{xx}(x,t)| \le \|u\|M_2M_3$$

Hence,

$$\|(Lu)(x,t)\|^2 \le \|u\|^2(M_1^2M_4^2 + M_2^2M_3^2)$$

The proof is complete. □

For a fix countable dense subset $\{M_i = (x_i, y_i)\}_{i=1}^{\infty}$ of the domain Ω, we put

$$\varphi_i(x,y) = \tilde{R}((x_i,t_i),\,(x,t)) = R_1(x_i,x)\cdot R_1(t_i,t), \tag{3.3}$$

where $R_1(x_i,x)$ is the reproducing kernel of $W_2^1[0,1]$. Let $\psi_i(x,t) = (L^*\varphi_i)(x,t)$, where L^* denotes the adjoint operator of L. By the definitions of adjoint operator and the reproducing property, the following Lemmas hold.

Lemma 3.2. $\psi_i(x,t) = (LR_{31}(\bullet,x)\cdot R_{32}(\star,t))(x_i,t_i)$

Proof. Let $\bullet$, $\star$ denote the variables corresponding to functions respectively. Then

$$\begin{aligned}
\psi_i(x,t) &= (L^*\varphi_i)(x,t) \\
&= ((L^*\varphi_i)(\bullet,\star), R_{31}(\bullet,x)\cdot R_{32}(\star,t))_{P(\Omega)} \\
&= (\varphi_i(\circ), (LR_{31}(\bullet,x)\cdot R_{32}(\star,t))(\circ))_{P(\Omega)} \\
&= (LR_{31}(\bullet,x)\cdot R_{32}(\star,t))(x_i,t_i)\,.
\end{aligned} \tag{3.4}$$

From the definition of L, we have

$$\psi_i(x,t) = R_{31}(x_i,x)\cdot\frac{\partial^2}{\partial t_i^2}R_{32}(t_i,t) - \frac{\partial^2}{\partial x_i^2}R_{31}(x_i,x)\cdot R_{32}(t_i,t) \tag{3.5}$$

□

Lemma 3.3. *If $\{M_i\}_{i=1}^{\infty}$ is dense on $P(\Omega)$, then $\{\psi_i(x,t)\}_{i=1}^{\infty}$ is a complete system of $P(\omega)$.*

Proof. For each fixed $u(x,t) \in P(\Omega)$, let $(u(x,t), \psi_i(x,t)) = 0$ $(i = 1, 2, \dots)$, which implies

$$(u(x,t), (L^*\varphi_i)(x,t)) = (Lu(x,t), \varphi_i(x,t)) = (Lu)(x_i, t_i) = 0\,. \tag{3.6}$$

Since $\{M_i\}_{i=1}^{\infty}$ is dense on $P(\Omega)$, we have $(Lu)(x,t) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1} . □

Consequently, we employ Gram-Schmidt process to orthonormalize the sequence $\{\psi_i\}_{i=1}^{\infty}$ in the reproducing kernel space $P(\Omega)$. Denote by $\{\overline{\psi}_i\}_{i=1}^{\infty}$ the orthonormalized sequence; that is,

$$\overline{\psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x,t), \quad i = 1, 2, \dots \tag{3.7}$$

where β_{ik} are orthoganal coefficients.

Theorem 3.4. *If $\{M_i\}_{i=1}^{\infty}$ is dense on $P(\Omega)$ and the solution of* (3.1) *is unique, then the solution of* (3.1) *has the form*

$$u(x,t) = \sum_{i=1}^{\infty}\Big(\sum_{k=1}^{i} \beta_{ik} f(x_k, t_k)\Big)\overline{\psi}_i(x,t) \tag{3.8}$$

Proof. Note that the Lemma 3.2 and the orthonormal system $\{\overline{\psi}_i(x,t)\}_{i=1}^{\infty}$ of $P(\Omega)$, we have

$$\begin{aligned}
u(x,t) &= \sum_{i=1}^{\infty}(u(x,t),\, \overline{\psi}_i(x,t))\overline{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}(u(x,t),\, \sum_{k=1}^{i} \beta_{ik}\psi_k(x,t))\overline{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(u(x,t),\, (L^*\varphi_k)(x,t))\overline{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(Lu(x,t),\, \varphi_k(x,t))\overline{\psi}_i(x,t) \\
&= \sum_{i=1}^{\infty}\Big(\sum_{k=1}^{i} \beta_{ik} f(x_k, t_k)\Big)\overline{\psi}_i(x,t)
\end{aligned}$$

□

We denote the approximate solution of $u(x,t)$ by

$$u_n(x,t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k, t_k)\overline{\psi}_i(x,t)\,. \tag{3.9}$$

Theorem 3.5. *For each $u(x,t) \in P(\Omega)$, let $\varepsilon_n^2 = \|u(x,t) - u_n(x,t)\|^2$, then sequence $\{\varepsilon_n\}$ is monotone decreasing and $\varepsilon_n \to 0$ $(n \to \infty)$.*

Proof. Because

$$\begin{aligned}\varepsilon_n^2 &= \|u(x,t)-u_n(x,t)\|^2\\ &= \|\sum_{i=n+1}^{\infty}(u(x,t),\overline{\psi}_i(x,t))\overline{\psi}_i(x,t)\|^2\\ &= \sum_{i=n+1}^{\infty}((u(x,t),\overline{\psi}_i(x,t)))^2,\end{aligned}$$

we have

$$\begin{aligned}\varepsilon_{n-1}^2 &= \|u(x,t)-u_{n-1}(x,t)\|^2\\ &= \|\sum_{i=n}^{\infty}(u(x,t),\overline{\psi}_i(x,t))\overline{\psi}_i(x,t)\|^2\\ &= \sum_{i=n}^{\infty}((u(x,t),\overline{\psi}_i(x,t)))^2.\end{aligned}$$

Clearly $\varepsilon_{n-1} \geq \varepsilon_n$. By Theorem 3.4, $\{\varepsilon_n\}$ is monotone decreasing and $\varepsilon_n \to 0$ $(n \to \infty)$. □

4. Numerical Example

In this Section, we employ the method introduced in Section 3 to solve (3.1) through symbolic and numerical computations are performed by using Mathematica 5.0. The results obtained by the method are compared with the analytical solution and are found to be in good agreement with each other.

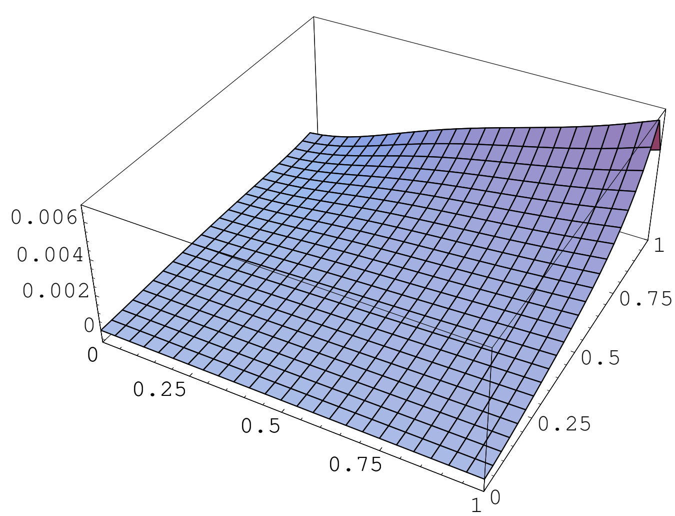

Figure 1. Error $u(x,t)-u_n(x,t)$

TABLE 1. The error in approximating $u(x,t)$

(x,t)	$u(x,t)$	$u_n(x,t)$	relative error	(x,t)	$u(x,t)$	$u_n(x,t)$	relative error
$(\frac{1}{21},\frac{1}{21})$	1.0454	1.04539	0.0000109747	$(\frac{1}{3},\frac{1}{21})$	1.64854	1.64852	0.0000125776
$(\frac{2}{3},\frac{2}{21})$	2.3769	2.3768	0.0000425437	$(\frac{20}{21},\frac{2}{21})$	3.22228	3.22214	0.0000443688
$(\frac{1}{21},\frac{1}{7})$	0.950436	0.950381	0.0000583054	$(\frac{1}{3},\frac{1}{7})$	1.49878	1.49868	0.0000678766
$(\frac{2}{3},\frac{1}{7})$	2.26637	2.26619	0.0000782034	$(\frac{20}{21},\frac{1}{7})$	3.07244	3.07219	0.0000815501
$(\frac{1}{21},\frac{5}{21})$	0.864095	0.86399	0.000122309	$(\frac{1}{3},\frac{5}{21})$	1.36263	1.36242	0.000150798
$(\frac{1}{20},\frac{2}{7})$	0.823912	0.823783	0.000156987	$(\frac{1}{3},\frac{2}{7})$	1.29926	1.299	0.000203236
$(\frac{2}{3},\frac{2}{7})$	1.96466	1.96421	0.000229523	$(\frac{20}{21},\frac{2}{7})$	2.66343	2.66279	0.000239147
$(\frac{1}{21},\frac{8}{21})$	0.749065	0.748897	0.000223804	$(\frac{1}{3},\frac{8}{21})$	1.18123	1.18084	0.000333805
$(\frac{1}{21},\frac{3}{7})$	0.714231	0.714051	0.000253051	$(\frac{1}{3},\frac{3}{7})$	1.1263	1.12584	0.000414213
$(\frac{2}{3},\frac{3}{7})$	1.70312	1.70234	0.000460057	$(\frac{20}{21},\frac{3}{7})$	2.30887	2.30776	0.000477361
$(\frac{2}{3},\frac{10}{21})$	1.62392	1.62301	0.000562991	$(\frac{20}{21},\frac{10}{21})$	2.42147	2.42054	0.000386727
$(\frac{1}{21},\frac{4}{7})$	0.619151	0.618958	0.000313067	$(\frac{1}{3},\frac{4}{7})$	0.976367	0.975648	0.000736244
$(\frac{2}{3},\frac{4}{7})$	1.4764	1.47517	0.000830423	$(\frac{20}{21},\frac{4}{7})$	2.0015	1.99979	0.000854846
$(\frac{1}{21},\frac{13}{21})$	0.590359	0.590168	0.000324107	$(\frac{1}{3},\frac{13}{21})$	0.930963	0.930144	0.000879517

As an example, we solve the equation

$$
\begin{gathered}
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1,\ t \geq 0 \\
\frac{\partial u(0,t)}{\partial x} + w_1 u(0,t) = 0, \quad w_1 \in R,\ t \geq 0 \\
\frac{\partial u(1,t)}{\partial x} + w_2 u(1,t) = 0, \quad w_2 \in R,\ t \geq 0 \\
u(x,0) = g_1(x), \quad 0 < x < 1 \\
\frac{\partial u(x,0)}{\partial t} = g_2(x), \quad 0 < x < 1
\end{gathered}
$$

where $f(x,t) = e^{-t}(-te^{x+t} - (1+4t^2)e^{x+t+t^2} + x - 2x(1+2t^2)e^{t+t^2})$, $w_1 = -2$, $w_2 = -1$, $g_1(x) = x + e^x$, $g_2(x) = -(x+e^x)$. The true solution is $u(x,t) = (x+e^x)e^{-t}$. The numerical results and error are given by table and figure, respectively.

Conclusion. In this paper, we employed a reproducing kernel and its conjugate operator to construct the complete orthonormal basis in the reproducing kernel space. By adding the initial and boundary conditions to the reproducing kernel space, we obtain the analytic solution of equation (1.1). Numerical example illustrates the accuracy and validity of the algorithm. Meanwhile, constructing the new form of the reproducing kernel function, we can obtain the analytic solution for the multi-dimensional equations because it reduces the computational complexity. In a future article will study the nonlinear problem with mixed boundary conditions by using this method.

REFERENCES

[1] K. George, E.H. Twizell; *Stable second-order finite-difference methods for linear initial-boundary-value problems*, Applied Mathematics Letters, 19(2006) 146-154.

[2] Lihong Yang, Minggen Cui; *New algorithm for a class of nonlinear integro-differential equations in the reproducing kernel space*, Applied Mathematics and Computation, 174(2006) 942-960.

[3] N. Aronszajn; *Theory of reproducing kernels*, Trans. AMS, 68(1950): 337-404.

[4] R. D. Russell, L. F. Shampine; *A collocation method for boundary value problems*, Numer. Math., 19(1972) 1-28.

[5] Samuel Jatoe, Zachariah Sinkala; *A High order B-spline collocation method for linear boundary value problems*, Applied Mathematics and Computations, DOI: 10.1016/j.amc.2007.02.027.

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