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NONLINEAR VOLTERRA INTEGRAL EQUATIONS IN HENSTOCK INTEGRABILITY SETTING

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Abstract. We prove the existence of global solutions for nonlinear Volterra-type integral equations involving the Henstock integral in Banach spaces. The functions governing the equation are supposed to be continuous only with respect to some variables and to satisfy some integrability or boundedness conditions. Our result improves the similar one given in [16] (where uniform continuity was required), as well as those referred therein. Then a related existence result is deduced in the set-valued case.

1. Introduction

In the previous two decades, many authors showed interest in studying differential and integral problems under hypothesis of integrability in a weaker sense than the classical ones; by classical meaning Lebesgue integral on the real line, respectively Bochner and Pettis integral in the vector case. To be more precise, integrals of quite oscillating functions were considered. Thus, on the real line, we recall the results obtained using the Henstock-Kurzweil integral (e.g. [3, 6, 7, 24]) and in the general case of Banach spaces, the solutions given for similar problems under Henstock-Lebesgue integrability assumptions (see [26, 27, 29]), in Henstock setting (e.g. [21]) or imposing some Henstock-Kurzweil-Pettis integrability conditions (see [4, 23, 28, 29]).

In the present paper, applying a Darbo-type fixed point theorem established in [16], we obtain the existence of global continuous solutions for the nonlinear Volterra integral equation

$$u(t) = (H)\int_0^t G(t,s) f\Big(s, u(s), \int_0^s k(s,\tau)u(\tau)d\tau, \int_0^1 h(s,\tau)u(\tau)d\tau\Big)ds$$

as well as for a related integral inclusion, considering the Henstock integral (introduced in [2]). The setting is that of a separable Banach space and the assumptions made on the operators are much weaker than those previously imposed for similar results (see [16] and the papers cited there). Mainly, we require some partial continuity to f and G, along with some integrability and boundedness conditions.

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Let us recall that Volterra-type integral equations appearing in various nonlinear problems in science were studied by many authors (see e.g. [12, 13, 15, 16, 17, 18, 19]) via different fixed point theorems, such as Darbo-type theorems (in [12, 16]) or Mönch-type fixed point theorems (in [15, 18, 19]).

2. Notation and preliminary facts

Through this paper, X is a real separable Banach space with norm $\|\cdot\|$ and R-closed ball T_R and X^* is its topological dual with unit ball B^*. $\mathcal{P}_0(X)$ stands for the family of nonempty subsets of X. D denotes the Pompeiu-Hausdorff distance and, for any $A \in \mathcal{P}_0(X)$, $|A| = D(A, \{0\})$.

The space $C([0,1], X)$ of continuous functions is endowed with the usual (Banach space) norm $\|f\|_C = \sup_{t\in[0,1]}\|f(t)\|$. Also, $L^\infty([0,1],\mathbb{R})$ is the space of essentially bounded real functions with essential supremum $\|\cdot\|_{L^\infty}$ and $BV([0,1],\mathbb{R})$ stands for the space of real bounded variation function with its classical norm $\|\cdot\|_{BV}$.

Let us now introduce the Henstock-type integrals in Banach spaces, concepts that extend the real Henstock-Kurzweil integrability (for which the reader is referred to [10]). A gauge δ on $[0,1]$ is a positive function. A partition of $[0,1]$ is a finite family $(I_i, t_i)_{i=1}^n$ of non-overlapping intervals covering $[0,1]$ with the tags $t_i \in I_i$; a partition is said to be δ-fine if for each $i = 1, \dots, n$, $I_i \subset]t_i - \delta(t_i), t_i + \delta(t_i)[$.

Definition 2.1. (see [2])

(i) A function $f : [0,1] \to X$ is called Henstock-integrable on $[0,1]$ if there exists an element (H) $\int_0^1 f(s)ds \in X$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε with

$$\Big\|\sum_{1=1}^{n} f(t_i)\mu(I_i) - (\mathrm{H})\int_0^1 f(s)ds\Big\| < \varepsilon$$

for every δ_ε-fine partition;

(ii) f is called Henstock-Lebesgue-integrable (shortly, HL-integrable) on $[0,1]$ if there exists $\widetilde{f} : [0,1] \to X$ such that, for every $\varepsilon > 0$, there is a gauge $\delta_\varepsilon > 0$ satisfying, for each δ_ε-fine partition $((x_{i-1}, x_i), t_i)_{i=1}^n$,

$$\sum_{i=1}^{n} \big\|f(t_i)(x_i - x_{i-1}) - \big[\widetilde{f}(x_i) - \tilde{f}(x_{i-1})\big]\big\| < \varepsilon.$$

In this case, we denote $\widetilde{f}(t) = (\mathrm{HL})\int_0^t f(s)ds$ and call it the HL-integral of f on $[0,t]$.

Remark 2.2. The Henstock and the Henstock-Lebesgue integrability are preserved on sub-intervals of $[0,1]$, but not on any measurable subset. As about the relationship between this integral and the classical ones, it is well known that:

(i) Every Bochner integrable function is HL-integrable, and the converse is not valid;
(ii) the HL-integrability implies the Henstock integrability;
(iii) following a result given in [8], any Pettis integrable function is Henstock integrable, but the implication in the other sense is not true;
(iv) there exist Henstock-Lebesgue-integrable functions that are not Pettis integrable (see the real case) and vice-versa (as Example 42 in [9] shows).

In finite dimensional spaces, the two notions (of Henstock and Henstock-Lebesgue integral) are equivalent. In the particular real case, the preceding definition gives the Henstock-Kurzweil (shortly, HK-) integral.

The space of Henstock integrable X-valued functions is denoted by $\mathcal{H}([0,1],X)$, and is equipped with the Alexiewicz norm:

$$\|f\|_A = \sup_{[a,b]\subset[0,1]} \Big\|(\mathrm{H})\int_a^b f(s)ds\Big\|.$$

It was proved that:

Lemma 2.3 ([20])**.** *T is a linear continuous functional on the space of real HK-integrable functions if and only if there exists a real function g of bounded variation such that, for every HK-integrable function f, $T(f) = (\mathrm{HK})\int_0^1 f(s)g(s)ds$.*

In order to extend this property to the vector Henstock integral, let us recall several notions and results.

Definition 2.4. A family $\mathcal{F}$ of HK-integrable functions is said to be uniformly HK-integrable if, for each $\varepsilon > 0$ there exists a gauge δ_ε such that for every δ_ε-fine partition and every $f \in \mathcal{F}$, $\big|\sum_{i=1}^n f(t_i)\mu(I_i) - (\mathrm{HK})\int_0^1 f(t)dt\big| < \varepsilon$.

Remark 2.5. The uniform HK-integrability is sometimes called HK-equi-integrability (e.g. [25, Definition 3.5.1]).

The following auxiliary result can be found in [23].

Lemma 2.6. *Let $(f_n)_n$ be an uniformly HK-integrable, pointwisely bounded sequence of real functions and g be a function of bounded variation. Then the sequence $(gf_n)_n$ is uniformly HK-integrable.*

A consequence of Proposition 1 in [5] is the characterization result given in the sequel.

Proposition 2.7. *If $f : [0,1] \to X$ is scalarly HK-integrable (i.e. for any $x^* \in X^*$, $\langle x^*, f(\cdot)\rangle$ is HK-integrable), then the following conditions are equivalent:*

- (i) *f is Henstock-integrable;*
- (ii) *the collection $\{\langle x^*, f(\cdot)\rangle; x^* \in B^*\}$ is uniformly HK-integrable;*
- (iii) *each countable subset of $\{\langle x^*, f(\cdot)\rangle; x^* \in B^*\}$ is uniformly HK-integrable.*

From here, one deduces the following result.

Proposition 2.8. *If $f : [0,1] \to X$ is Henstock-integrable and $g : [0,1] \to \mathbb{R}$ is of bounded variation, then fg is Henstock-integrable.*

In the set-valued case, an Aumann-type integral will be considered, via Henstock-integrable selections.

Definition 2.9. A set-valued function $F : [0,1] \to \mathcal{P}_0(X)$ is Aumann-Henstock integrable if the collection of its Henstock-integrable selections S_F^H is non-empty. In this case, the Aumann-Henstock integral of F is defined by

$$(AH)\int_0^1 F(s)ds = \big\{(H)\int_0^1 f(s)ds, f \in S_F^H\big\}$$

The Hausdorff measure of non-compactness α will play an essential role in establishing the main results. For this, the reader is referred to [14].

Theorem 2.10 ([1]). *Let $\mathcal{K} \subset C([0,1],X)$ be bounded and equi-continuous. Then $\alpha(\mathcal{K}) = \sup_{t\in[0,1]}\alpha(\mathcal{K}(t))$.*

The following result was proved in [26] for the Henstock-Lebesgue integral, but the proof works for the Henstock integral to. It is a generalization of some similar inequality available for Bochner integrable functions, which can be found in [14].

Theorem 2.11. *If $M \subset \mathcal{H}([0,1],X)$ is a countable family satisfying that, for some $h \in L^1([0,1],\mathbb{R})$, $\alpha(M(t)) \le h(t)$ for a.e. $t \in [0,1]$, then $\alpha(M(\cdot)) \in L^1([0,1],\mathbb{R})$ and*

$$\alpha\Big((H)\int_0^t M(s)ds\Big) \le \int_0^t \alpha(M(s))ds, \quad \forall t \in [0,1].$$

The following generalization of Darbo's fixed point theorem was given in [16].

Lemma 2.12. *Let F be a closed convex subset of a Banach space and the operator $A : F \to F$ be continuous with $A(F)$ bounded. Suppose that for the sequence defined for any bounded $B \subset F$ by*

$$\tilde{A}^1(B) = A(B) \quad \text{and} \quad \tilde{A}^n(B) = A\left(\overline{co}\left(\tilde{A}^{n-1}(B)\right)\right), \quad \forall n \ge 2$$

there exist a positive constant $0 \le k < 1$ and a natural number n_0 such that for every bounded $B \subset F$, $\alpha(\tilde{A}^{n_0}(B)) \le k\alpha(B)$. Then A has a fixed point.

3. Main results

Theorem 3.1. *Let $f : [0,1]\times X^3 \to X$ and $h, k, G : [0,1]^2 \to \mathbb{R}$ satisfy the following conditions:*

(i) *for each $t \in [0,1]$, $h(t,\cdot)$, $k(t,\cdot)$ are in $L^\infty([0,1],\mathbb{R})$, $G(t,\cdot) \in BV([0,1],\mathbb{R})$, the application $t \mapsto G(t,\cdot)$ is $\|\cdot\|_{BV}$-continuous and $t \mapsto h(t,\cdot)$, $t \mapsto k(t,\cdot)$ are L^∞-bounded;*

(ii) *for any $x, y, z \in C([0,1],X)$, $f(\cdot, x(\cdot), y(\cdot), z(\cdot))$ is Henstock-integrable, and*

(ii)(1) *for each $R > 0$ and $\varepsilon > 0$, one can find $\delta_{\varepsilon,R} > 0$ such that*

$$\big\|(H)\int_{t_1}^{t_2} f(s,x,y,z)ds\big\| \le \varepsilon, \quad \forall\, |t_1 - t_2| \le \delta_{\varepsilon,R}, \;\; \forall x,y,z \in T_R$$

and

$$\limsup_{R\to\infty} \frac{1}{R\delta_{1,R}} < \frac{1}{\sup_{t\in[0,1]}\|G(t,\cdot)\|_{BV} \max\big\{1, \sup_{t\in[0,1]}\|k(t,\cdot)\|_{L^\infty}, \sup_{t\in[0,1]}\|h(t,\cdot)\|_{L^\infty}\big\}};$$

(ii)(2) *the application $(x,y,z) \mapsto f(\cdot,x,y,z)$ from X^3 towards $\mathcal{H}([0,1],X)$ is $\|\cdot\|_A$-uniformly continuous;*

(iii) *there exist three positive integrable functions such that for any bounded $D_i \subset X$ and any $t \in [0,1]$,*

$$\alpha\left(f(t,D_1,D_2,D_3)\right) \le \sum_{i=1}^3 L_i(t)\alpha(D_i).$$

Then the Volterra-type integral equation

$$u(t) = (H)\int_0^t G(t,s) f\Big(s, u(s), \int_0^s k(s,\tau)u(\tau)d\tau, \int_0^1 h(s,\tau)u(\tau)d\tau\Big)ds$$

has a continuous solution on $[0,1]$.

Proof. We follow the ideas of proof in [16, Theorem 3.1]. To simplify the calculus, denote by

$$(Tu)(t) = \int_0^t k(t,s)u(s)ds, \quad (Su)(t) = \int_0^1 h(t,s)u(s)ds.$$

By hypothesis (i), let $a = \sup_{t\in[0,1]}\|k(t,\cdot)\|_{L^\infty}$, $b = \sup_{t\in[0,1]}\|G(t,\cdot)\|_{BV}$ and $c = \sup_{t\in[0,1]}\|h(t,\cdot)\|_{L^\infty}$. One can find $0 < r < \frac{1}{b\max\{1,a,c\}}$ and $R_0 > 0$ such that for any $R \geq R_0 \max\{1,a,c\}$,

$$\frac{1}{\delta_{1,R}} < rR.$$

Consider $A : C([0,1],X) \to C([0,1],X)$ defined by

$$Au(t) = (H)\int_0^t G(t,s)f(s,u(s),(Tu)(s),(Su)(s))ds.$$

We claim that A is a continuous operator applying the closed ball B_{R_0} of $C([0,1],X)$ into itself. Indeed, from (ii)(1), for any $u \in C([0,1],X)$ with $\|u\|_C \leq R_0$,

$$\begin{aligned}\|Au\|_C &\leq \sup_{t\in[0,1]}\|G(t,\cdot)\|_{BV}\Big\|(H)\int_0^t f(s,u(s),(Tu)(s),(Su)(s))ds\Big\| \\ &\leq b\frac{1}{\delta_{1,R_0\max\{1,a,c\}}},\end{aligned}$$

since $\|Tu\|_C \leq aR_0$ and $\|Su\|_C \leq cR_0$. It follows that

$$\|Au\|_C < brR_0\max\{1,a,c\} < R_0.$$

About the continuity, from hypothesis (ii)(2), one obtains that for every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$\sup_{t\in[0,1]}\Big\|(H)\int_0^t f(s,x_1,y_1,z_1) - f(s,x_2,y_2,z_2)ds\Big\| < \frac{\varepsilon}{b}$$

for any x_i, y_i satisfying $\max\{\|x_1-x_2\|, \|y_1-y_2\|, \|z_1-z_2\|\} < \max\{1,a,c\}\delta_\varepsilon$. Then

$$\begin{aligned}\|Au_1 - Au_2\|_C &= \sup_{t\in[0,1]}\|Au_1(t) - Au_2(t)\| \\ &= \sup_{t\in[0,1]}\Big\|(H)\int_0^t G(t,s)\Big(f(s,u_1(s),(Tu_1)(s),(Su_1)(s)) \\ &\quad - f(s,u_2(s),(Tu_2)(s),(Su_2)(s))\Big)ds\Big\| \\ &\leq b\sup_{t\in[0,1]}\Big\|(H)\int_0^t f(s,u_1(s),(Tu_1)(s),(Su_1)(s)) \\ &\quad - f(s,u_2(s),(Tu_2)(s),(Su_2)(s))ds\Big\|.\end{aligned}$$

If $\|u_1 - u_2\|_C < \delta_\varepsilon$, then $\|u_1(s) - u_2(s)\| < \delta_\varepsilon$, $\|(Tu_1)(s) - (Tu_2)(s)\| < a\delta_\varepsilon$ and $\|(Su_1)(s) - (Su_2)(s)\| < c\delta_\varepsilon$, thus

$$\|Au_1 - Au_2\|_C < \varepsilon.$$

Next, we prove that $F = \overline{co}A(B_{R_0})$ is equi-continuous. Using [16, Lemma2.1], it is sufficient to show that $A(B_{R_0})$ is equi-continuous. For all $u \in B_{R_0}$ and all $0 \le t_1 < t_2 \le 1$, we have

$$\begin{aligned}
\|Au(t_1) - Au(t_2)\| &= \|(H)\int_0^{t_1} G(t_1,s)f(s,u(s),(Tu)(s),(Su)(s))ds - \\
&\quad - (H)\int_0^{t_2} G(t_2,s)f(s,u(s),(Tu)(s),(Su)(s))ds\| \\
&\le \big\|(H)\int_0^{t_1} (G(t_1,s) - G(t_2,s))f(s,u(s),(Tu)(s),(Su)(s))ds\big\| \\
&\quad + \big\|(H)\int_{t_1}^{t_2} G(t_2,s)f(s,u(s),(Tu)(s),(Su)(s))ds\big\| \\
&\le \|G(t_1,\cdot) - G(t_2,\cdot)\|_{BV}\frac{1}{\delta_{1,R_0\max\{1,a,c\}}} \\
&\quad + b\sup_{t_1 \le t_1' < t_2' \le t_2}\big\|(H)\int_{t_1'}^{t_2'} f(s,u(s),(Tu)(s),(Su)(s))ds\big\|
\end{aligned}$$

and, by hypothesis (i) and (ii)(1), this can be made less than some fixed ε for t_1, t_2 with an appropriately small distance between them. So, the equi-continuity follows.

Obviously, $A : F \to F$ is bounded and continuous. Let us prove, by mathematical induction, that for every $B \subset F$ and any $n \in \mathbb{N}$, $\tilde{A}^n(B) \subset A(B_{R_0})$, so $\tilde{A}^n(B)$ is bounded and equi-continuous. For $n = 1$, this is valid, since $A(B) \subset A(F) \subset A(B_{R_0})$. Suppose now that this is true for $n-1$ and prove it for n:

$$\tilde{A}^n(B) = A(\overline{co}(\tilde{A}^{n-1}(B))) \subset A(\overline{co}(A(B_{R_0}))) \subset A(\overline{co}(B_{R_0})) = A(B_{R_0}).$$

Then, by Theorem 2.10,

$$\alpha\left(\tilde{A}^n(B)\right) = \sup_{t\in[0,1]}\alpha\left(\tilde{A}^n(B)(t)\right), \quad \forall n \in \mathbb{N}.$$

Similarly to the second part of the proof of Theorem 3.1 in [16], one can show that there exist a constant $0 \le k < 1$ and a positive integer n_0 such that for any $B \subset F$, $\alpha(\tilde{A}^{n_0}(B)) \le k\alpha(B)$.

Let $(v_n)_n$ be an arbitrary countable subset of $\tilde{A}^1(B) = A(B)$. There exists a sequence $(u_n)_n \subset B$ such that $v_n = Au_n$. Hypothesis (ii)(1) allows us to use Theorem 2.11 in order to obtain that

$$\begin{aligned}
\alpha\left(\{v_n(t), n \in \mathbb{N}\}\right) &= \alpha\big(\{Au_n(t), n \in \mathbb{N}\}\big) \\
&= \alpha\Big(\int_0^t G(t,s)f(s,\{u_n(s),n\},\{(Tu_n)(s),n\},\{(Su_n)(s),n\})ds\Big) \\
&\le b\int_0^t \alpha\left(f(s,\{u_n(s),n\},\{(Tu_n)(s),n\},\{(Su_n)(s),n\})\right)ds.
\end{aligned}$$

By hypothesis (iii) it follows that

$$\begin{aligned}&\alpha(\{v_n(t), n \in \mathbb{N}\})\\ &\le b\int_0^t L_1(s)\alpha(\{u_n(s)\}) + L_2(s)\alpha(\{(Tu_n)(s)\}) + L_3(s)\alpha(\{(Su_n)(s)\})ds.\end{aligned}$$

and, applying again Theorem 2.11,

$$\begin{aligned}\alpha(\{v_n(t), n \in \mathbb{N}\}) &\le b\int_0^t (L_1(s) + aL_2(s) + cL_3(s))\,\alpha(\{u_n(s)\})ds\\ &\le b\int_0^t (L_1(s) + aL_2(s) + cL_3(s)))\,ds\ \alpha(B).\end{aligned}$$

Since the Banach space is separable and the Hausdorff measure of non-compactness is preserved when the set under discussion is replaced by its adherence, this implies that

$$\alpha\left(\tilde{A}^1(B)(t)\right) \le b\int_0^t (L_1(s) + aL_2(s) + cL_3(s)))\,ds\ \alpha(B).$$

As $L_1(s) + aL_2(s) + cL_3(s) \in L^1([0,1],\mathbb{R})$ and continuous functions are dense in $L^1([0,1],\mathbb{R})$ with respect to the usual norm, for any $\varepsilon > 0$ one can make an evaluation of the form $\alpha(\tilde{A}^1(B)(t)) \le (\varepsilon + Mt)\alpha(B)$. It can be shown, by mathematical induction, that for every $m \in \mathbb{N}$,

$$\alpha\left(\tilde{A}^m(B)(t)\right) \le \Big(\varepsilon^m + C_m^1\varepsilon^{m-1}Mt + \dots + \frac{(Mt)^m}{m!}\Big)\alpha(B), \quad \forall t \in [0,1].$$

Suppose the inequality is valid for m and prove it for $m+1$. For any countable subset $(v_n)_n$ of $\tilde{A}^{m+1}(B) = A\left(\overline{co}\left(\tilde{A}^m(B)\right)\right)$, there exist $(u_n)_n \subset \overline{co}\left(\tilde{A}^m(B)\right)$ such that $v_n = Au_n$. Then, as before,

$$\alpha(\{v_n(t), n \in \mathbb{N}\}) \le b\int_0^t (L_1(s) + aL_2(s) + cL_3(s))\,ds\ \alpha\left(\tilde{A}^m(B)\right),$$

whence

$$\alpha\left(\tilde{A}^{m+1}(B)(t)\right) \le b\int_0^t (L_1(s) + aL_2(s) + cL_3(s))\,ds\ \alpha\left(\tilde{A}^m(B)\right)$$

and so the assertion follows. The rest of the calculus goes as in [16]: for some integer n_0 the evaluation term $\varepsilon^{n_0} + C_{n_0}^1\varepsilon^{n_0-1}Mt + \dots + \frac{(Mt)^{n_0}}{n_0!}$ can be made less than 1 and so, by Lemma 2.12, A has a fixed point, which is a global solution to our equation. □

An existence result can be deduced in the set-valued case.

Theorem 3.2. *Let $F : [0,1] \times X^3 \to \mathcal{P}_0(X)$ and $h, k, G : [0,1]^2 \to \mathbb{R}$ satisfy the following conditions:*

(i) *for each $t \in [0,1]$, $h(t,\cdot)$, $k(t,\cdot)$ are in $L^\infty([0,1],\mathbb{R})$, $G(t,\cdot) \in BV([0,1],\mathbb{R})$, the application $t \mapsto G(t,\cdot)$ is $\|\cdot\|_{BV}$-continuous and $t \mapsto h(t,\cdot)$, $t \mapsto k(t,\cdot)$ are L^∞-bounded;*

(ii) *for any $x, y, z \in C([0,1],X)$, $F(\cdot, x(\cdot), y(\cdot), z(\cdot))$ is Aumann-Henstock integrable, and* (ii)(1) *for each $R>0$ and $\varepsilon>0$, one can find $\delta_{\varepsilon,R}>0$ such that*

$$\Big|(AH)\int_{t_1}^{t_2}F(s,x,y,z)ds\Big|\le\varepsilon,\quad\forall\,|t_1-t_2|\le\delta_{\varepsilon,R},\ \ \forall x,y,z\in T_R$$

and

$$\limsup_{R\to\infty}\frac{1}{R\delta_{1,R}}$$

$$<\frac{1}{\sup_{t\in[0,1]}\|G(t,\cdot)\|_{BV}\max\big\{1,\sup_{t\in[0,1]}\|k(t,\cdot)\|_{L^\infty},\sup_{t\in[0,1]}\|h(t,\cdot)\|_{L^\infty}\big\}};$$

(ii)(2) *the application $(x,y,z)\mapsto S^H_{F(\cdot,x,y,z)}$ from X^3 towards $\mathcal{P}_0(\mathcal{H}([0,1],X))$ possess $\|\cdot\|_A$-uniformly continuous selections;*

(iii) *there exist three positive integrable functions such that for any bounded $D_i\subset X$ and any $t\in[0,1]$,*

$$\alpha\left(F(t,D_1,D_2,D_3)\right)\le\sum_{i=1}^{3}L_i(t)\alpha(D_i).$$

Then the integral inclusion

$$u(t)\in(AH)\int_0^t G(t,s)F\Big(s,u(s),\int_0^s k(s,\tau)u(\tau)d\tau,\int_0^1 h(s,\tau)u(\tau)d\tau\Big)ds$$

has global continuous solutions.

Proof. Every $\|\cdot\|_A$-uniformly continuous selection f of $(x,y,z)\mapsto S^H_{F(\cdot,x,y,z)}$ satisfies the hypothesis of Theorem 3.1, and so, the integral equation

$$u(t)=(H)\int_0^t G(t,s)f\Big(s,u(s),\int_0^s k(s,\tau)u(\tau)d\tau,\int_0^1 h(s,\tau)u(\tau)d\tau\Big)ds$$

has a continuous solution on $[0,1]$. It follows that our inclusion possess continuous solutions. □

Remark 3.3. Theorems 3.1 and 3.2 improve the related results given in [12, 13, 15, 16, 17], where the involved functions are supposed to be uniformly continuous with respect to all arguments.

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