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EXISTENCE OF POSITIVE SOLUTION FOR SEMIPOSITONE SECOND-ORDER THREE-POINT BOUNDARY-VALUE PROBLEM

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Abstract. In this paper, we establish the existence of positive solution for the semipositone second-order three-point boundary value problem $u''(t) + \lambda f(t, u(t)) = 0$, $0 < t < 1$, $u(0) = \alpha u(\eta)$, $u(1) = \beta u(\eta)$. Our arguments are based on the well-known Guo-Krasnosel'skii fixed-point theorem in cones.

1. Introduction

Multi-point boundary value problems (BVPs for short), due to their applications to almost all areas of science, engineering and technology, have attracted considerable attention. For example, in 1987, Il'in and Moiseev [4] studied some multi-point BVPs first for linear second-order ordinary differential equations, and then, many authors discussed nonlinear multi-point BVPs, see [2, 3, 6, 7, 8, 9, 10] and the references therein. In particular, Ma [6] showed the existence of at least one positive solution for the three-point BVP

$$\begin{gathered} u''(t) + h(t)f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{gathered}$$

under the condition that $0 < \alpha\eta < 1$ and f was nonnegative.

Recently, when the nonlinear term is not necessarily nonnegative, Yao [9] proved the existence of at least one positive solution for the three-point BVP

$$\begin{gathered} u''(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{gathered}$$

where $0 < \alpha\eta < 1$ and $\lambda > 0$ was a parameter.

Motivated by the excellent results in [6, 9], we are concerned with the existence of positive solution for the second-order three-point BVP

$$u''(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = \alpha u(\eta), \quad u(1) = \beta u(\eta), \tag{1.2}$$

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where $0 < \eta < 1$, $0 < \beta \le \alpha < 1$, $\lambda > 0$ is a parameter. Throughout, we assume that there exists a constant $M > 0$ such that $f : [0,1] \times [0,+\infty) \to (-M,+\infty)$ is continuous. This implies that the BVP (1.1) and (1.2) is semipositone. For convenience, we denote

$$\xi = 1-\alpha+(\alpha-\beta)\eta,$$
$$\gamma = \min\big\{\frac{\alpha\eta}{1-\alpha+\alpha\eta}, \frac{(1-\eta)\alpha}{1-\beta\eta}\big\},$$
$$B = \max\{f(t,u)+M : (t,u) \in [0,1]\times[0,1]\}.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that* $\lim_{u\to+\infty}\min_{0\le t\le\eta}\frac{f(t,u)}{u} = +\infty$. *Then the BVP* (1.1) *and* (1.2) *has at least one positive solution for*

$$0 < \lambda < \min\big\{\frac{2\xi}{B(1-\alpha+\alpha\eta)},\ \frac{2\gamma\beta\xi}{\alpha M(1-\alpha+\alpha\eta-\beta\eta^2)}\big\}.$$

Our main tool is the well-known Guo-Krasnosel'skii fixed-point theorem, which we state here for convenience of the reader.

Theorem 1.2 ([1, 5])**.** *Let E be a Banach space, K a cone in E and $\Omega_c = \{u \in K : \| u \| < c\}$. Suppose that $T : K \to K$ is a completely continuous operator and $0 < a < b < +\infty$ such that either*

(1) *$Tu \nleq u$ for $u \in \partial\Omega_a$ and $u \nleq Tu$ for $u \in \partial\Omega_b$, or*
(2) *$u \nleq Tu$ for $u \in \partial\Omega_a$ and $Tu \nleq u$ for $u \in \partial\Omega_b$.*

Then T has a fixed point in $\overline{\Omega}_b \setminus \Omega_a$.

2. Preliminaries

In the remainder of this paper, we assume that $0 < \beta \le \alpha < 1$. Also let the Banach space $E = C[0,1]$ be equipped with the usual norm $\|u\| = \max_{t\in[0,1]}|u(t)|$.

Lemma 2.1. *For any fixed $y \in E$, the BVP*

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$
$$u(0) = \alpha u(\eta), \quad u(1) = \beta u(\eta) \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) = &-\int_0^t (t-s)y(s)ds + \frac{1}{\xi}[(1-\alpha)t+\alpha\eta]\int_0^1 (1-s)y(s)ds \\ &+ \frac{1}{\xi}[(\alpha-\beta)t-\alpha]\int_0^\eta (\eta-s)y(s)ds. \end{aligned}$$

Since the proof of the above lemma is easy, we omit it.

Lemma 2.2. *If $y \in E$ and $y \ge 0$, then the unique solution u of the BVP* (2.1)–(2.2) *satisfies $u(t) \ge 0$ for $t \in [0,1]$.*

Proof. Since $u''(t) = -y(t) \le 0$, $0 < t < 1$ it follows that the graph of $u(t)$ is concave dawn, we only need to prove $u(0) \ge 0$ and $u(1) \ge 0$. In view of $0 < \beta \le \alpha < 1$ and (2.2), we know that $u(0)$, $u(\eta)$ and $u(1)$ have same signs. Suppose on the contrary that $u(0) < 0$, $u(\eta) < 0$ and $u(1) < 0$. Then we have

$$u(\eta) = \frac{u(0)}{\alpha} < u(0), \quad u(\eta) = \frac{u(1)}{\beta} < u(1)\,.$$

Then

$$u(\eta) < \min\{u(0),\ u(1)\},$$

which contradicts the concavity of u. Thus, we get that

$$u(0) \geq 0 \quad \text{and} \quad u(1) \geq 0$$

as required. $\square$

Lemma 2.3. *If $y \in E$ and $y \geq 0$, then the unique solution u of the BVP* (2.1)–(2.2) *satisfies*

$$\min_{0 \leq t \leq \eta} u(t) \geq \gamma \|u\|. \tag{2.3}$$

Proof. Since $u(0) = \alpha u(\eta)$, $0 < \alpha < 1$ and Lemma 2.2 imply that $u(0) \leq u(\eta)$, we know that

$$\min_{0 \leq t \leq \eta} u(t) = u(0). \tag{2.4}$$

Set $u(\bar{t}) = \|u\|$. We consider the following two cases:

Case 1. $\eta \leq \bar{t}$. It follows from the concavity of u that

$$\frac{u(\eta) - u(0)}{\eta - 0} \geq \frac{u(\bar{t}) - u(0)}{\bar{t} - 0}.$$

Combining the boundary condition $u(0) = \alpha u(\eta)$, we conclude that

$$u(0) \geq \frac{\alpha\eta}{1 - \alpha + \alpha\eta} u(\bar{t}) = \frac{\alpha\eta}{1 - \alpha + \alpha\eta} \|u\|,$$

which together with (2.4) implies

$$\min_{0 \leq t \leq \eta} u(t) \geq \frac{\alpha\eta}{1 - \alpha + \alpha\eta} \|u\|. \tag{2.5}$$

Case 2. $\bar{t} < \eta$. It follows from the concavity of u that

$$u(\bar{t}) \leq \frac{u(1) - u(\eta)}{1 - \eta}(0 - \eta) + u(\eta),$$

which together with (2.4) and the boundary conditions $u(0) = \alpha u(\eta)$ and $u(1) = \beta u(\eta)$ implies

$$\min_{0 \leq t \leq \eta} u(t) \geq \frac{(1 - \eta)\alpha}{1 - \beta\eta} \|u\|. \tag{2.6}$$

By (2.5) and (2.6), we know that (2.3) is fulfilled. $\square$

Lemma 2.4. *The BVP*

$$\widetilde{u}''(t) + 1 = 0, \quad 0 < t < 1, \tag{2.7}$$

$$\widetilde{u}(0) = \alpha \widetilde{u}(\eta), \quad \widetilde{u}(1) = \beta \widetilde{u}(\eta) \tag{2.8}$$

has a unique solution

$$\widetilde{u}(t) = -\frac{t^2}{2} + \frac{(1 - \alpha)t + \alpha\eta + [(\alpha - \beta)t - \alpha]\eta^2}{2\xi}, \quad t \in [0, 1].$$

Remark 2.5. The unique solution $\widetilde{u}$ of the BVP (2.7)–(2.8) satisfies

$$\widetilde{u}(t) \leq \frac{1 - \alpha + \alpha\eta - \beta\eta^2}{2\xi}, \quad t \in [0, 1].$$

3. Proof of Theorem 1.1

Let

$$\begin{aligned} g(t,u) &= f(t,u)+M, \quad (t,u)\in[0,1]\times[0,+\infty),\\ \overline{g}(t,u) &= g(t,\max\{u,0\}), \quad (t,u)\in[0,1]\times(-\infty,+\infty). \end{aligned}$$

Obviously, $\overline{g}:[0,1]\times(-\infty,+\infty)\to(0,+\infty)$ is continuous. We consider the BVP

$$u''(t)+\lambda\overline{g}(t,u(t)-w(t))=0, \quad 0<t<1, \tag{3.1}$$
$$u(0)=\alpha u(\eta), \quad u(1)=\beta u(\eta), \tag{3.2}$$

where $w(t)=\lambda M\widetilde{u}(t)$ and $\widetilde{u}(t)$ is the solution of the BVP (2.7)–(2.8). It is not difficult to prove that u^* is a positive solution of the BVP (1.1)–(1.2) if and only if $\overline{u}=u^*+w$ is a solution of the BVP (3.1)–(3.2) and $\overline{u}(t)>w(t)$, $0<t<1$.

We define an operator $T_\lambda:E\to E$:

$$\begin{aligned} (T_\lambda u)(t) &= -\lambda\int_0^t(t-s)\overline{g}(s,u(s)-w(s))ds\\ &\quad+\frac{\lambda}{\xi}[(1-\alpha)t+\alpha\eta]\int_0^1(1-s)\overline{g}(s,u(s)-w(s))ds\\ &\quad+\frac{\lambda}{\xi}[(\alpha-\beta)t-\alpha]\int_0^\eta(\eta-s)\overline{g}(s,u(s)-w(s))ds, \quad t\in[0,1]. \end{aligned}$$

It is easy to check that $\overline{u}\in E$ is a solution of the BVP (3.1)–(3.2) if and only if $\overline{u}$ is a fixed point of the operator T_λ in E. Therefore, we only need to prove that the operator T_λ has a fixed point $\overline{u}\in E$ and $\overline{u}(t)>w(t)$, $0<t<1$. Denote

$$K=\{u\in E:\min_{0\le t\le 1}u(t)\ge 0,\ \min_{0\le t\le\eta}u(t)\ge\gamma\|u\|\}.$$

Obviously, K is a cone in E. It follows from Lemma 2.3 that $T_\lambda K\subset K$. Furthermore, we can prove that $T_\lambda:K\to K$ is completely continuous. Now, we introduce a partial order in E. Let x_1, $x_2\in E$. We say $x_1\le x_2$ if and only if $x_2-x_1\in K$.

If we let $\Omega_1=\{u\in K:\|u\|<1\}$, then we may assert that

$$u\nleq T_\lambda u \quad \text{for any } u\in\partial\Omega_1. \tag{3.3}$$

Suppose on the contrary that there exists a $u_0 \in \partial\Omega_1$ such that $u_0 \le T_\lambda u_0$. Since $u_0(t) - w(t) \le 1$ and $(\alpha-\beta)t - \alpha < 0$, $0 \le t \le 1$, we have

$$\begin{aligned}
u_0(t) &\le (T_\lambda u_0)(t)\\
&= -\lambda \int_0^t (t-s)\overline{g}(s, u_0(s)-w(s))ds\\
&\quad + \frac{\lambda}{\xi}[(1-\alpha)t + \alpha\eta]\int_0^1 (1-s)\overline{g}(s, u_0(s)-w(s))ds\\
&\quad + \frac{\lambda}{\xi}[(\alpha-\beta)t - \alpha]\int_0^\eta (\eta-s)\overline{g}(s, u_0(s)-w(s))ds\\
&\le \frac{\lambda}{\xi}[(1-\alpha)t + \alpha\eta]\int_0^1 (1-s)\overline{g}(s, u_0(s)-w(s))ds\\
&\le \frac{\lambda}{\xi}[(1-\alpha)t + \alpha\eta]\int_0^1 (1-s)\max_{0\le s\le 1}\overline{g}(s, u_0(s)-w(s))ds\\
&= \frac{\lambda}{\xi}[(1-\alpha)t + \alpha\eta]\int_0^1 (1-s)\max_{0\le s\le 1}\left[f(s, \max\{u_0(s)-w(s), 0\}) + M\right]ds\\
&\le \frac{B\lambda}{2\xi}[(1-\alpha)t + \alpha\eta],\ t \in [0,1],
\end{aligned}$$

which leads to a contradiction:

$$1 = \|u_0\| \le \frac{B\lambda}{2\xi}(1 - \alpha + \alpha\eta) < 1.$$

So, (3.3) is satisfied.

On the other hand, we claim that there exists a constant $\sigma > 1$ such that

$$T_\lambda u \nleq u \quad \text{for any } u \in \partial\Omega_\sigma. \tag{3.4}$$

In fact, if we let $V_\lambda = \{u \in K : T_\lambda u \le u\}$ and $m_\lambda = \sup\{\|u\| : u \in V_\lambda\}$, then we only need to prove $m_\lambda < +\infty$. Suppose on the contrary that there exists a sequence $\{u_n\}_{n=1}^\infty \subset K$ such that $T_\lambda u_n \le u_n$ and $\|u_n\| \to +\infty$ $(n \to +\infty)$. Then for any $t \in [0, \eta]$, we have

$$u_n(t) - w(t) \ge \gamma\|u_n\| - \|w\| \to +\infty \quad (n \to +\infty). \tag{3.5}$$

In view of (3.5) and $\lim_{u\to+\infty} \min_{0\le t\le\eta} \frac{f(t,u)}{u} = +\infty$, we know that

$$\lim_{n\to+\infty} \min_{0\le t\le\eta} \frac{\overline{g}(t, u_n(t)-w(t))}{u_n(t)-w(t)} = +\infty. \tag{3.6}$$

So, there exists a positive integer N such that for any $n \ge N$,

$$\min_{0\le t\le\eta}[u_n(t) - w(t)] \ge \frac{\gamma}{2}\|u_n\| \tag{3.7}$$

and

$$\min_{0\le t\le\eta} \frac{\overline{g}(t, u_n(t)-w(t))}{u_n(t)-w(t)} \ge \frac{4\xi}{\lambda\gamma}\Big[(1-\eta)\int_0^\eta t\,dt\Big]^{-1}. \tag{3.8}$$

For the rest of this article, we let $n \ge N$. Noticing $T_\lambda u_n \in K$, we have $0 \le (T_\lambda u_n)(t) \le u_n(t)$, $t \in [0,1]$. And so,

$$\|u_n\| = \max_{0\le t\le 1} u_n(t) \ge \max_{0\le t\le 1}(T_\lambda u_n)(t) \ge (T_\lambda u_n)(\eta). \tag{3.9}$$

At the same time, by (3.7) and (3.8), we also obtain

$$
\begin{aligned}
&(T_\lambda u_n)(\eta)\\
&= -\lambda \int_0^\eta (\eta - s)\overline{g}(s, u_n(s) - w(s))ds + \frac{\lambda}{\xi}\eta \int_0^1 (1-s)\overline{g}(s, u_n(s) - w(s))ds\\
&\quad + \frac{\lambda}{\xi}[(\alpha - \beta)\eta - \alpha] \int_0^\eta (\eta - s)\overline{g}(s, u_n(s) - w(s))ds\\
&= \frac{\lambda}{\xi}(1-\eta) \int_0^\eta s\overline{g}(s, u_n(s) - w(s))ds + \frac{\lambda}{\xi}\eta \int_\eta^1 (1-s)\overline{g}(s, u_n(s) - w(s))ds\\
&\geq \frac{\lambda}{\xi}(1-\eta) \int_0^\eta s\overline{g}(s, u_n(s) - w(s))ds\\
&\geq \frac{\lambda}{\xi}(1-\eta) \int_0^\eta s \min_{0\leq s\leq\eta} \big[\frac{\overline{g}(s, u_n(s) - w(s))}{u_n(s) - w(s)}\big] \min_{0\leq s\leq\eta}[u_n(s) - w(s)]ds\\
&\geq \frac{\lambda}{\xi}(1-\eta)\frac{4\xi}{\lambda\gamma}\Big[(1-\eta)\int_0^\eta tdt\Big]^{-1}\frac{\gamma}{2}\|u_n\| \cdot \int_0^\eta sds\\
&= 2\|u_n\|,
\end{aligned}
$$

which together with (3.9) implies

$$\|u_n\| \geq (T_\lambda u_n)(\eta) \geq 2\|u_n\|.$$

This is impossible. So, $m_\lambda < +\infty$. And so, (3.4) is fulfilled.

It follows from (3.3), (3.4) and Theorem 1.2 that T_λ has a fixed point $\overline{u} \in \overline{\Omega}_\sigma \setminus \Omega_1$. With the similar arguments as in Lemma 2.3, we know that

$$\min_{0\leq t\leq 1} \overline{u}(t) = \overline{u}(1) = \frac{\beta}{\alpha}\overline{u}(0) \geq \frac{\beta\gamma}{\alpha}\|\overline{u}\|,$$

which together with Remark 2.5 implies

$$\overline{u}(t) \geq \frac{\beta\gamma}{\alpha}\|\overline{u}\| \geq \frac{\beta\gamma}{\alpha} > \lambda M \cdot \frac{1-\alpha+\alpha\eta-\beta\eta^2}{2\xi} \geq \lambda M \cdot \widetilde{u}(t) = w(t),$$

for $t \in (0,1)$. Therefore, $u^* = \overline{u} - w$ is a positive solution of the BVP (1.1)–(1.2).

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