Electronic Journal of Differential Equations, Vol. 2008(2008), No. 45, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR SECOND-ORDER M-POINT BOUNDARY-VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITIES

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Abstract. In this paper, we study the nonlinear second-order m-point boundary value problem

$$u''(t) + f(t,u) = 0, \quad 0 \le t \le 1,$$
$$\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$

where the nonlinear term f is allowed to change sign. We impose growth conditions on f which yield the existence of at least two positive solutions by using a fixed-point theorem in double cones. Moreover, the associated Green's function for the above problem is given.

1. Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moviseev [6, 7]. Motivated by the study of [6, 7], Gupta [3] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [4, 8, 9] for some references along this line. Multi-point boundary value problems describe many phenomena in the applied mathematical sciences. For example, the vibrations of a guy wire composed of N parts with a uniform cross-section throughout but different densities in different parts can be set up as a multi-point boundary value problems (see [11]). Many problems in the theory of elastic stability can be handle by the method of multi-point boundary value problems (see [5]).

In 1997, Henderson and Wang [5] studied the existence of positive solutions for nonlinear eigenvalue problem

$$u''(t) + \lambda h(t) f(u) = 0, \quad 0 \le t \le 1,$$
$$u(0) = 0, \quad u(1) = 0,$$

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. m-point; boundary-value problem; Green's function; fixed point theorem in double cones.

Submitted December 27, 2007. Published March 29, 2008.
Supported by grant 10471075 from the the National Natural Science Foundation of China.

where $f \in C([0,+\infty),[0,+\infty))$ and $h \in C([0,1],[0,+\infty))$. The authors establish the existence of positive solutions under the condition that f is either superlinear or sublinear.

Ma [9] investigated the second-order three-point boundary value problem(BVP)

$$u''(t) + a(t)f(u) = 0, \quad 0 \le t \le 1,$$
$$u(0) = 0, \quad u(1) = \alpha u(\eta),$$

where $0 < \eta < 1$, $0 < \alpha\eta < 1$, $f \in C([0,+\infty),[0,+\infty))$, $a \in C([0,1],[0,+\infty))$. The existence of at least one positive solution is obtained under the condition that f is either superlinear or sublinear by applying Guo-Krasnoselskii's fixed point theorem.

Recently, Ma [10] studied the second-order m-point boundary-value problem

$$u''(t) + a(t)f(u) = 0, \quad 0 \le t \le 1,$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$

where $\alpha_i \ge 0$, $i = 1, 2, \dots, m-3$, $\alpha_{m-2} > 0$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \Sigma_{i=1}^{m-2}\alpha_i\xi_i < 1$, $f \in C([0,+\infty),[0,+\infty))$, $a \in C([0,1],[0,+\infty))$. The author obtained the existence of at least one positive solution if f is either superlinear or sublinear by applying a fixed-point theorem in cones.

All the above works were done under the assumption that the nonlinear term is nonnegative, applying the concavity of solutions in the proofs. In this paper we study the nonlinear second-order m-point boundary value problem

$$u''(t) + f(t,u) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \tag{1.2}$$

where the nonlinear term f is allowed to change sign. Firstly we give the associated Green's function for the above problem which makes later discussions more precise. Then certain growth conditions are imposed on f which yield the existence of at least two positive solutions by using a new fixed-point theorem in double cones. In this way we removed the usual restriction on $f \ge 0$.

For a cone K in a Banach space X with norm $\|\cdot\|$ and a constant $r > 0$, let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$. Suppose $\alpha : K \to \mathbb{R}^+$ is a continuously increasing functional; i.e., α is continuous and $\alpha(\lambda x) \le \alpha(x)$ for $\lambda \in (0,1)$. Let

$$K(b) = \{x \in K : \alpha(x) < b\}, \partial K(b) = \{x \in K : \alpha(x) = b\}.$$

and $K_a(b) = \{x \in K : a < \|x\|, \alpha(x) < b\}$. The origin in X is denoted by θ.

Our main tool of this paper is the following fixed point theorem in double cones.

Theorem 1.1 ([1]). *Let X be a real Banach space with norm $\|\cdot\|$ and $K, K' \subset X$ two solid cones with $K' \subset K$. Suppose $T : K \to K$ and $T' : K' \to K'$ are two completely continuous operators and $\alpha : K' \to R^+$ is a continuously increasing functional satisfying $\alpha(x) \le \|x\| \le M\alpha(x)$ for all $x \in K'$, where $M \ge 1$ is a constant. If there are constants $b > a > 0$ such that*

- (C1) *$\|Tx\| < a$, for $x \in \partial K_a$;*
- (C2) *$\|T'x\| < a$, for $x \in \partial K'_a$ and $\alpha(T'x) > b$ for $x \in \partial K'(b)$;*
- (C3) *$Tx = T'x$, for $x \in K'_a(b) \cap \{u : T'u = u\}$.*

Then T has at least two fixed points y_1 and y_2 in K, such that

$$0 \le \|y_1\| < a < \|y_2\|, \quad \alpha(y_2) < b.$$

2. Preliminaries

In this section, we present some lemmas that are important to prove our main results.

Lemma 2.1. *Suppose that $d = \beta(1 - \sum_{i=1}^{m-2} a_i\xi_i) + \gamma(1 - \sum_{i=1}^{m-2} a_i) \neq 0$ and $y(t) \in C[0,1]$. Then boundary-value problem*

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) = &-\int_0^t (t-s)y(s)ds + \frac{\beta t + \gamma}{d}\int_0^1 (1-s)y(s)ds \\ &- \frac{\beta t + \gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds \end{aligned} \tag{2.3}$$

Proof. Integrating both sides of (2.1) on $[0, t]$, we have

$$u'(t) = -\int_0^t y(s)ds + u'(0). \tag{2.4}$$

Again integrating (2.4) from 0 to t, we get

$$u(t) = -\int_0^t (t-s)y(s)ds + u'(0)t + u(0). \tag{2.5}$$

In particular,

$$\begin{gathered} u(1) = -\int_0^1 (1-s)y(s)ds + u'(0) + u(0), \\ u(\xi_i) = -\int_0^{\xi_i} (\xi_i - s)y(s)ds + u'(0)\xi_i + u(0). \end{gathered}$$

By (2.2) we get

$$u'(0) = \frac{\beta}{d}\Big[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\Big].$$

The proof is complete. □

Lemma 2.2. *Let $0 < \sum_{i=1}^{m-2} a_i\xi_i < 1$, $d > 0$. If $y \in C[0,1]$ and $y \ge 0$, then the unique solution u of (2.1)-(2.2) satisfies $u(t) \ge 0$.*

Proof. Since $u''(t) = -y(t) \le 0$, we know that the graph of $u(t)$ is concave down on $(0,1)$. So we only prove $u(0) \ge 0$, $u(1) \ge 0$.

Firstly, we shall prove $u(0) \ge 0$ in the following two cases

Case i: If $0 < \sum_{i=1}^{m-2} a_i \le 1$, by (2.3) we have

$$\begin{aligned}
u(0) &= \frac{\gamma}{d}\Big[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\Big] \\
&\ge \frac{\gamma}{d}\Big[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{1} (1 - s)y(s)ds\Big] \\
&= \frac{\gamma}{d}\Big(1 - \sum_{i=1}^{m-2} a_i\Big)\int_0^1 (1-s)y(s)ds \ge 0.
\end{aligned}$$

Case ii: If $\sum_{i=1}^{m-2} a_i > 1$, by (2.3) we have

$$\begin{aligned}
u(0) &= \frac{\gamma}{d}\Big[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\Big] \\
&\ge \frac{\gamma}{d}\Big[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{1} (\xi_i - s)y(s)ds\Big] \\
&= \frac{\gamma}{d}\int_0^1 \Big[(1 - \sum_{i=1}^{m-2} a_i\xi_i) + (\sum_{i=1}^{m-2} a_i - 1)s\Big]y(s)ds \ge 0.
\end{aligned}$$

On the other hand, by (2.3) we have

$$\begin{aligned}
u(1) &= -\int_0^1 (1-s)y(s)ds + \frac{\beta+\gamma}{d}\int_0^1 (1-s)y(s)ds \\
&\quad - \frac{\beta+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds \\
&\ge \frac{\beta}{d}\Big[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i(1-s) - (\xi_i - s))y(s)ds + \sum_{i=1}^{m-2} a_i\xi_i \int_{\xi_i}^1 (1-s)y(s)ds\Big] \\
&\quad + \frac{\gamma}{d}\sum_{i=1}^{m-2} a_i \Big[\int_0^1 (1-s)y(s)ds - \int_0^1 (\xi_i - s)y(s)ds\Big] \\
&= \frac{\beta}{d}\sum_{i=1}^{m-2} a_i \Big[\int_0^{\xi_i} (1-\xi_i)sy(s)ds + \xi_i\int_{\xi_i}^1 (1-s)y(s)ds\Big] \\
&\quad + \frac{\gamma}{d}\sum_{i=1}^{m-2} a_i \Big[\int_0^1 (1-\xi_i)y(s)ds\Big] \ge 0.
\end{aligned}$$

The proof is complete. □

Lemma 2.3. *Let* $\sum_{i=1}^{m-2} a_i\xi_i > 1$, $d \ne 0$. *If* $y \in C[0,1]$ *and* $y \ge 0$, *then* (2.1)-(2.2) *has no positive solution.*

Proof. On the contrary, suppose that (2.1)-(2.2) has a positive solution u, then $u(\xi_i) > 0$, $i = 1, 2, \dots, m-2$ and

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) = \sum_{i=1}^{m-2} a_i\xi_i \frac{u(\xi_i)}{\xi_i} \ge \sum_{i=1}^{m-2} a_i\xi_i \frac{u(\bar\xi)}{\bar\xi} > \frac{u(\bar\xi)}{\bar\xi},$$

where $\overline{\xi} = \min\{\xi_1, \xi_2, \dots, \xi_{m-2}\}$ satisfies

$$\frac{u(\overline{\xi})}{\overline{\xi}} = \min\big\{\frac{u(\xi_1)}{\xi_1}, \frac{u(\xi_2)}{\xi_2}, \dots, \frac{u(\xi_{m-2})}{\xi_{m-2}}\big\},$$

which contradicts to the concave of $u(t)$. The proof is complete. $\square$

Lemma 2.4. *Let $a_i \geq 0$, $i = 1, \dots, m-2$, $0 < \sum_{i=1}^{m-2} a_i\xi_i < 1$, $d > 0$. If $y \in C[0,1]$ and $y \geq 0$, then the unique positive solution $u(t)$ of (2.1)-(2.2) satisfies*

$$\inf_{t\in[\xi_{m-2},1]} u(t) \geq \sigma\|u\|,$$

where

$$\sigma = \min\big\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}},\ a_{m-2}\xi_{m-2},\ \xi_{m-2}\big\}, \quad \|u\| = \sup_{t\in[0,1]} |u(t)|.$$

Proof. Let $u(\overline{t}) = \max_{t\in[0,1]} u(t) = \|u\|$, we shall discuss it from the following two cases:

Case 1: If $0 < \sum_{i=1}^{m-2} a_i < 1$. Firstly, assume that $\overline{t} < \xi_{m-2} < 1$, so that $\min_{t\in[\xi_{m-2},1]} u(t) = u(1)$. By $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq a_{m-2}u(\xi_{m-2})$ we have

$$\begin{aligned}
u(\overline{t}) &\leq u(1) + \frac{u(1) - u(\xi_{m-2})}{1-\xi_{m-2}}(0-1)\\
&= u(1) - \frac{1}{1-\xi_{m-2}}u(1) + \frac{1}{1-\xi_{m-2}}u(\xi_{m-2})\\
&\leq u(1)\Big(1 - \frac{1}{1-\xi_{m-2}} + \frac{1}{a_{m-2}(1-\xi_{m-2})}\Big)\\
&= u(1)\frac{1-a_{m-2}\xi_{m-2}}{a_{m-2}(1-\xi_{m-2})}.
\end{aligned}$$

So that

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}}\|u\|. \tag{2.6}$$

Secondly, assume $\xi_{m-2} < \overline{t} < 1$, then $\min_{t\in[\xi_{m-2},1]} u(t) = u(1)$. Otherwise, we have $\min_{t\in[\xi_{m-2},1]} u(t) = u(\xi_{m-2})$, then $\overline{t} \in [\xi_{m-2}, 1], u(\xi_{m-2}) \geq u(\xi_{m-1}) \geq \dots \geq u(\xi_2) \geq u(\xi_1)$. By $0 < \sum_{i=1}^{m-2} a_i < 1$ we have

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \leq \sum_{i=1}^{m-2} a_i u(\xi_{m-2}) < u(\xi_{m-2}) \leq u(1)$$

which is a contradiction. Since $u(t)$ is concave,

$$\frac{u(\xi_{m-2})}{\xi_{m-2}} \geq \frac{u(\overline{t})}{\overline{t}} \geq u(\overline{t}).$$

In fact, since $u(1) \geq a_{m-2}u(\xi_{m-2})$, then $\frac{u(1)}{a_{m-2}\xi_{m-2}} \geq u(\overline{t})$, which implies

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq a_{m-2}\xi_{m-2}\|u\|. \tag{2.7}$$

Case 2: If $\sum_{i=1}^{m-2} a_i > 1$. Firstly, assume $u(\xi_{m-2}) \leq u(1)$, then $\min_{t\in[\xi_{m-2},1]} u(t) = u(\xi_{m-2})$. By concave of $u(t)$ we have $\overline{t} \in [\xi_{m-2}, 1]$, which implies

$$\frac{u(\xi_{m-2})}{\xi_{m-2}} \geq \frac{u(\overline{t})}{\overline{t}} \geq u(\overline{t}),$$

then

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \xi_{m-2}\|u\|. \tag{2.8}$$

Secondly, assume $u(\xi_{m-2}) > u(1)$, and so $\min_{t\in[\xi_{m-2},1]} u(t) = u(1)$, and $\bar{t} \in [\xi_1, 1]$. If not, $\bar{t} \in [0, \xi_1)$, then $u(\xi_1) \geq \dots \geq u(\xi_{m-2}) > u(1)$. So we have

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) > u(1) \sum_{i=1}^{m-2} a_i \geq u(1)$$

which is a contradiction. Since $\sum_{i=1}^{m-2} a_i > 1$, there exists $\bar{\xi} \in \{\xi_1, \xi_2, \dots, \xi_{m-2}\}$ such that $u(\bar{\xi}) \leq u(1)$, then $u(\xi_1) \leq u(\xi_2) \leq \dots \leq u(\xi_{m-2}) \leq u(1)$. Since $u(t)$ is concave, we have $\frac{u(1)}{\xi_1} \geq \frac{u(\xi_1)}{\xi_1} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t})$, then

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \xi_1\|u\|. \tag{2.9}$$

Therefore, by (2.6)-(2.9) we have $\inf_{t\in[\xi_{m-2},1]} u(t) \geq \sigma\|u\|$, where

$$\sigma = \min\big\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}},\ a_{m-2}\xi_{m-2},\ \xi_{m-2}\big\}.$$

The proof is complete. $\square$

Lemma 2.5. *Suppose that $d \neq 0$. Then the boundary value problem*

$$-u''(t) = 0, \quad 0 < t < 1,$$
$$\beta u(0) - \gamma u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$$

has Green's function

$$G(t,s) = \begin{cases} \frac{(\beta s+\gamma)\big[(1-t)-\sum_{j=1}^{m-2} a_j(\xi_j-t)\big]}{d}, \\ \textit{if } 0 \leq t \leq 1,\ s \leq \xi_1,\ s \leq t; \\ \frac{(\beta s+\gamma)(1-t)-\sum_{j=i}^{m-2} a_j(\xi_j-t)(\beta s+\gamma)+\sum_{j=1}^{i-1} a_j(\beta\xi_j+\gamma)(t-s)}{d}, \\ \textit{if } \xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m-1,\ \xi_{i-1} \leq s \leq \xi_i,\ 2 \leq i \leq r, s \leq t; \\ \frac{(\beta t+\gamma)\big[(1-s)-\sum_{j=i}^{m-2} a_j(\xi_j-s)\big]}{d}, \\ \textit{if } \xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m-1,\ \xi_{i-1} \leq s \leq \xi_i,\ 2 \leq i \leq r, t \leq s; \\ \frac{(\beta t+\gamma)(1-s)}{d}, \\ \textit{if } 0 \leq t \leq 1,\ \xi_{m-2} \leq s \leq 1,\ t \leq s. \end{cases} \tag{2.10}$$

Here for the sake of convenience, we write $\xi_0 = 0, \xi_{m-1} = 1$.

Proof. If $0 \leq t \leq \xi_1$, the unique solution (2.3) given by Lemma 2.1 can be rewritten as

$$\begin{aligned} u(t) &= \int_0^t \frac{(\beta s+\gamma)\big[(1-t)-\sum_{j=1}^{m-2} a_j(\xi_j-t)\big]}{d} y(s)ds \\ &\quad + \int_t^{\xi_1} \frac{(\beta t+\gamma)\big[(1-s)-\sum_{j=1}^{m-2} a_j(\xi_j-s)\big]}{d} y(s)ds \end{aligned}$$

$$+\sum_{i=2}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\frac{(\beta t+\gamma)\big[(1-s)-\sum_{j=i}^{m-2}a_j(\xi_j-s)\big]}{d}y(s)ds$$
$$+\int_{\xi_{m-2}}^{1}\frac{(\beta t+\gamma)(1-s)}{d}y(s)ds.$$

Similarly, if $\xi_{r-1}\le t\le \xi_r, 2\le r\le m-2$, the unique solution (2.3) can be expressed

$$
\begin{aligned}
u(t) &= \int_0^{\xi_1}\frac{(\beta s+\gamma)\big[(1-t)-\sum_{j=1}^{m-2}a_j(\xi_j-t)\big]}{d}y(s)ds\\
&+\sum_{i=2}^{r-1}\int_{\xi_{i-1}}^{\xi_i}\Big[(\beta s+\gamma)(1-t)-\sum_{j=i}^{m-2}a_j(\xi_j-t)(\beta s+\gamma)\\
&+\sum_{j=1}^{i-1}a_j(\beta_1\xi_j+\gamma)(t-s)\Big]\frac{y(s)}{d}ds\\
&+\int_{\xi_{r-1}}^{t}\Big[(\beta s+\gamma)(1-t)-\sum_{j=r}^{m-2}a_j(\xi_j-t)(\beta s+\gamma)\\
&+\sum_{j=1}^{i-1}a_j(\beta\xi_j+\gamma)(t-s)\Big]\frac{y(s)}{d}ds\\
&+\int_t^{\xi_r}\frac{(\beta t+\gamma)\big[(1-s)-\sum_{j=r}^{m-2}a_j(\xi_j-s)\big]}{d}y(s)ds\\
&+\sum_{i=r+1}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\frac{(\beta t+\gamma)\big[(1-s)-\sum_{j=i}^{m-2}a_j(\xi_j-s)\big]}{d}y(s)ds\\
&+\int_{\xi_{m-2}}^{1}\frac{(\beta t+\gamma)(1-s)}{d}y(s)ds.
\end{aligned}
$$

If $\xi_{m-2}\le t\le 1$, the unique solution (2.3) can be given in the form

$$
\begin{aligned}
u(t) &= \int_0^{\xi_1}\frac{(\beta s+\gamma)\big[(1-t)-\sum_{j=1}^{m-2}a_j(\xi_j-t)\big]}{d}y(s)ds\\
&+\sum_{i=2}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\Big[(\beta s+\gamma)(1-t)-\sum_{j=i}^{m-2}a_j(\xi_j-t)(\beta s+\gamma)\\
&+\sum_{j=1}^{i-1}a_j(\beta\xi_j+\gamma)(t-s)\Big]\frac{y(s)}{d_1}ds\\
&+\int_{\xi_{m-2}}^{t}\frac{(\beta s+\gamma)(1-t)+\sum_{j=1}^{i-1}a_j(\beta\xi_j+\gamma)(t-s)}{d}y(s)ds\\
&+\int_t^1\frac{(\beta t+\gamma)(1-s)}{d}y(s)ds.
\end{aligned}
$$

The lemma is proved. □

Now let $X=C[0,1]$, $K=\{u\in X: u(t)\ge 0,\forall t\in[0,1]\}$, $K'=\{u\in X:$ u is nonnegative, concave, and nonincreasing$\}$. Equip X with the supremum norm

$\|u\| := \sup_{t\in[0,1]} |u(t)|$. Clearly, $K, K' \subset X$ are cones with $K' \subset K$. For $\forall u \in K$, define

$$\alpha(u) = \min_{\xi_{m-2}\le t\le 1} u(t),$$

$$(Tu)(t) = \Big(\int_0^1 G(t,s)f(s,u(s))ds\Big)^+, \quad t \in [0,1],$$

where $(B)^+ = \max\{B, 0\}$.

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds, t \in [0,1],$$

For $x \in X$, define $\theta : X \to K$ by $(\theta u)(t) = \max\{u(t), 0\}$, then $T = \theta \circ A$. For $u \in K'$, define

$$(T'u)(t) = \int_0^1 G(t,s)f^+(s,u(s))ds, \quad t \in [0,1],$$

where $f^+(t,s) = \max\{f(t,s), 0\}$.

Lemma 2.6. *Let $X = C[0,1], K = \{u \in X : u \ge 0\}$. Suppose $T : X \to X$ is completely continuous. Define $\theta : TX \to K$ by*

$$(\theta y) = \max\{y(t), \omega(t)\}, \quad \textit{for } y \in TX,$$

where $\omega \in C^1[0,1], \omega(t) \ge 0$ is given function. Then $\theta \circ T : X \to K$ is also a completely continuous operator.

Proof. The complete continuity of T implies that T is continuous and maps each bounded subset in X to a relatively compact set. Denote θy by $\overline{y}$.

Given a function $h \in C[0,1]$, for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|Th - Tg\| < \varepsilon, \quad \text{for } g \in X, \|g - h\| < \delta.$$

Since

$$\begin{aligned} |(\theta Th)(t) - (\theta Tg)(t)| &= |\max\{(Th)(t), \omega(t)\} - \max\{(Tg)(t), \omega(t)\}| \\ &\le |(Th)(t) - (Tg)(t)| < \varepsilon, \end{aligned}$$

we have

$$\|(\theta T)h - (\theta T)g\| < \varepsilon, \quad \text{for } g \in X, \|g - h\| < \delta,$$

and so θT is continuous.

For any arbitrary bounded set $D \subset X$ and for all $\varepsilon > 0$, there are $y_i, i = 1, 2, \dots, m$ such that

$$TD \subset \bigcup_{i=1}^m B(y_i, \varepsilon),$$

where $B(y_i, \varepsilon) := \{u \in X : \|u - y_i\| < \varepsilon\}$. Then, for for all $\overline{y} \in (\theta \circ T)D$, there is a $y \in TD$ such that $\overline{y}(t) = \max\{y(t), \omega(t)\}$. We choose $i \in \{1, 2, \dots, m\}$ such that $\|y - y_i\| < \varepsilon$. The fact

$$\max_{t\in[0,1]} |\overline{y}(t) - \overline{y}_i(t)| \le \max_{t\in[0,1]} |y(t) - y_i(t)|,$$

which implies $\overline{y} \in B(\overline{y}_i, \varepsilon)$. Hence $(\theta\circ T)D$ has a finite $\varepsilon-net$ and therefore $(\theta\circ T)D$ is relatively compact. $\square$

3. Main results

In this section, we present the existence of two positive solutions for boundary value problem (1.1)-(1.2) by applying a new fixed-point theorem in double cones.

Obviously, $G(t,s) \geq 0$. In the following, we denote

$$M = \max_{t\in[0,1]} \int_0^1 G(t,s)ds, \quad n = \min_{t\in[\xi_{m-2},1]} \int_{\xi_{m-2}}^1 G(t,s)ds.$$

For $t \in [\xi_{m-2}, 1]$, by computing we have

$$\begin{aligned}\int_{\xi_{m-2}}^1 G(t,s)ds &= \int_{\xi_{m-2}}^t \frac{(\beta_1 s+\gamma_1)(1-t)+\sum_{j=1}^{i-1} a_j(\beta_1\xi_j+\gamma_1)(t-s)}{d_1} ds \\ &\quad + \int_t^1 \frac{(\beta_1 t+\gamma_1)(1-s)}{d_1} ds > 0.\end{aligned}$$

So $0 < n < M$.

In the rest of the paper, we use the following assumptions:

(H1) $\beta \geq 0$, $\gamma > 0$, $\alpha_i \geq 0$, $i = 1, 2, \dots, m-3, \alpha_{m-2} > 0$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \Sigma_{i=1}^{m-2}\alpha_i\xi_i < 1$, $d_1 = \beta(1-\Sigma_{i=1}^{m-2}\alpha_i\xi_i) + \gamma(1-\Sigma_{i=1}^{m-2}\alpha_i) > 0$;
(H2) $f : [0,1] \times [0,+\infty) \to R$ is continuous and $f(t,0) \geq (\not\equiv 0), t \in [0,1]$;
(H3) $h : [0,1] \to R^+$ is continuous.

Theorem 3.1. *Suppose that conditions* (H1)–(H3) *hold. Assume that there exist positive numbers a, b, d such that*

$$0 < \big(1+\frac{\beta}{\gamma}\big) \max\Big\{1, \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}\Big\} d < a < \sigma b < b$$

such that

(H4) $f(t,u) \geq 0$ *for* $(t,u) \in [0,1] \times [d,b]$;
(H5) $f(t,u) < \frac{a}{M}$ *for* $(t,u) \in [0,1] \times [0,a]$;
(H6) $f(t,u) \geq \frac{\sigma b}{n}$ *for* $(t,u) \in [0,1] \times [\sigma b, b]$.

Then, (1.1)-(1.2) *has at least two positive solutions u_1 and u_2 such that* $0 \leq \|u_1\| < a < \|u_2\|$, $\alpha(u_2) < b$

Proof. Firstly we prove that T has a fixed point $u_1 \in K$ with $0 < \|u_1\| \leq a$. In fact, for all $u \in \partial K_a$, from (H5)we have

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]} \Big(\int_0^1 G(t,s)f(s,u(s))ds\Big)^+ \\ &\leq \max_{t\in[0,1]} \max\Big\{\int_0^1 G(t,s)f(s,u(s))ds, 0\Big\} \\ &< \frac{a}{M} \max_{t\in[0,1]} \int_0^1 G(t,s)ds = a.\end{aligned}$$

The existence of u_1 is proved by using (C1) Theorem 1.1.

Obviously, u_1 is a solution of (1.1)-(1.2) if and only if u_1 is a fixed point of A. Next, we need to prove that u_1 is a solution of (1.1)-(1.2). Suppose the contrary; i.e., there exists $t_0 \in (0,1)$ such that $u_1(t_0) \neq (Au_1)(t_0)$. It must be $(Au_1)(t_0) < 0 = u_(t_0)$. Let (t_1,t_2) be the maximal interval and contains t_0 such that $(Au_1)(t) < 0$ for all $t \in (t_1,t_2)$. Obviously, $(t_1,t_2) \neq [0,1]$ by (H2). If $t_2 < 1$, then $u_1(t) \equiv 0$ for

$t \in [t_1, t_2]$, and $(Au_1)(t) < 0$ for $t \in (t_1, t_2)$, and $(Au_1)(t_2) = 0$. Thus, $(Au_1)'(t_2) = 0$. From (H2) we get $(Au_1)''(t) = -f(t, 0) \le 0$ for $t \in [t_1, t_2]$. So $(Au_1)'(t) \ge 0$ for $t \in [t_1, t_2]$. We obtain $t_1 = 0$. On the other hand, $\beta(Au_1)(0) - \gamma(Au_1)'(0) = 0$ implies $(Au_1)'(0) \le 0$ a contradiction. If $t_1 > 0$, we have $u_1(t) \equiv 0$ for $t \in [t_1, t_2]$ and $(Au_1)(t) < 0$ for $t \in (t_1, t_2)$, $(Au_1)(t_1) = 0$. Thus, $(Au_1)'(t_1) \le 0$. (H2) implies $(Au_1)''(t) = -f(t, 0) \le 0$ for $t \in [t_1, t_2]$. So $t_2 = 1$. From $(Au_1)(1) = \sum_{i=1}^{m-2} \alpha_i (Au_1)(\xi_i) < 0$, there exists $i_0 \in \{1, 2, \dots, m-2\}$ such that $(Au_1)(\xi_i) < 0$ for $i_0 \le j \le m-2$ and $(Au_1)(\xi_j) \ge 0$ for $0 \le j \le i_0 - 1$. So $\xi_j \in (t_1, 1)$ for $i_0 \le j \le m-2$. From the concavity $(Au_1)(t)$ on $[t_1, 1]$, we have

$$\frac{|(Au_1)(\xi_j)|}{\xi_j - 1} \le \frac{|(Au_1)(1)|}{1 - t_1}, \quad \text{for } i_0 \le j \le m-2;$$

i.e.,

$$|(Au_1)(\xi_j)| \le \frac{\xi_j - t_1}{1 - t_1} |(Au_1)(1)| < \xi_j |(Au_1)(1)|, \quad \text{for } i_0 \le j \le m-2.$$

From the above inequalities, we have

$$\sum_{j=i_0}^{m-2} \alpha_j |(Au_1)(\xi_j)| \le \sum_{j=i_0}^{m-2} \alpha_j \xi_j |(Au_1)(1)| < |(Au_1)(1)|.$$

On the other hand, from $(Au_1)(1) < 0$, we have

$$|(Au_1)(1)| = |\sum_{j=1}^{m-2} \alpha_j (Au_1)(\xi_j)| \le \sum_{j=i_0}^{m-2} \alpha_j |(Au_1)(\xi_j)|,$$

a contraction. Therefore u_1 is a solution of (1.1)-(1.2) with $0 < \|u_1\| < a$.

We now show that (C2) of Theorem 1.1 is satisfied. For $u \in \partial K'_a$; i.e., $\|u\| = a$. From (H5) we have

$$\begin{aligned}
\|T'u\| &= \max_{t \in [0,1]} \int_0^1 G(t, s) f^+(s, u(s)) ds \\
&< \frac{a}{M} \max_{t \in [0,1]} \int_0^1 G(t, s) ds = a.
\end{aligned}$$

Whereas for $u \in \partial K'(\sigma b)$; i.e., $\alpha(u) = \sigma b$. For $t \in [\xi_{m-2}, 1]$ we have $\sigma b \le u(t) \le b$. We may use condition (H6) to obtain

$$\begin{aligned}
\alpha(T'u) &= \min_{t \in [\xi_{m-2}, 1]} \int_0^1 G(t, s) f^+(s, u(s)) ds \\
&\ge \min_{t \in [\xi_{m-2}, 1]} \int_{\xi_{m-2}}^1 G(t, s) f^+(s, u(s)) ds \\
&\ge \frac{\sigma b}{n} \min_{t \in [\xi_{m-2}, 1]} \int_{\xi_{m-2}}^1 G(t, s) ds \\
&= \sigma b.
\end{aligned}$$

Finally, we show that (C3) of Theorem 1.1 is also satisfied. Let $u \in \partial K'_a(\sigma b) \cap \{u : T'u = u\}$, then

$$\|u\| > a > \big(1 + \frac{\beta}{\gamma}\big) \max\Big\{1, \frac{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i}{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)}\Big\} d.$$

We will prove

$$u(0) \geq \max\Big\{1, \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}\Big\}d. \tag{3.1}$$

Suppose this is not true, then there exists $t_0 \in (0,1)$ such that

$$u'(t_0) > \frac{\beta}{\gamma}\max\Big\{1, \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}\Big\}d.$$

It follows from the concavity of $u(t)$ that

$$u'(0) \geq u'(t_0) > \frac{\beta}{\gamma}\max\Big\{1, \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}\Big\}d.$$

So we have

$$\begin{aligned} 0 &= \beta u(0) - \gamma u'(0) \\ &< \beta\max\Big\{1, \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}\Big\}d - \gamma\frac{\beta}{\gamma}\max\Big\{1, \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}\Big\}d = 0, \end{aligned}$$

which is a contradiction.

Next we claim that $u(1) \geq d$. If not, by the concavity of $u(t)$ we have

$$\frac{u(\xi_i)-u(1)}{1-\xi_i} \geq \frac{u(0)-u(1)}{1-0}, \quad \text{for } i = 1,2,\dots,m-2;$$

i.e., $u(0)(1-\xi_i) \leq u(\xi_i) - \xi_i u(1)$. By $u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i)$ we get

$$u(0) \leq \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}u(1) < \frac{1-\sum_{i=1}^{m-2}\alpha_i\xi_i}{\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)}d,$$

which contradicts to (3.1) . Thus, $d \leq u(t) \leq b$ for $t \in [0,1]$. From (H4) we know that $f^+(s,u(s)) = f(s,u(s))$. This implies that $Tu = T'u$ for $u \in \partial K'_a(\sigma b) \cap \{u : T'u = u\}$. The proof is complete. □

4. Applications

Consider the second-order third-point boundary value problem

$$u''(t) + f(t,u) = 0, \quad 0 < t < 1, \tag{4.1}$$

$$u(0) - \frac{1}{4}u'(0) = 0, \quad u(1) = 2u(\frac{1}{4}), \tag{4.2}$$

where $\beta = 1$, $\gamma = \frac{1}{4}$, $m = 3$, $\alpha_1 = 2$, $\xi_1 = \frac{1}{4}$,

$$f(t,u) = \begin{cases} 1 - 16u^2, & 0 \leq t \leq 1, 0 \leq u < \frac{1}{2}, \\ -7 + 8u, & 0 < t < 1, \frac{1}{2} \leq u < 1, \\ 1 + \frac{2}{25}(u-1)^2, & 0 \leq t \leq 1,\ 1 \leq u < 6, \\ \frac{40}{11} + 2(u-6)^2, & 0 \leq t \leq 1,\ 6 \leq u < 32, \\ 2727 - 5(u-32)^2, & 0 \leq t \leq 1,\ u \geq 32. \end{cases}$$

Then (4.1)-(4.2) has at least two positive solutions.

Proof. Let $\xi_1 = \frac{1}{4}$, $d = 1$, $a = 6$, $b = 32$. By Lemma 2.5 we can get

$$\int_0^1 G(t,s)ds = -\frac{1}{2}t^2 + \frac{7}{4}t + \frac{7}{16}, \quad \int_{1/4}^1 G(t,s)ds = -\frac{1}{2}t^2 + \frac{7}{8}t + \frac{1}{2}.$$

So, $M = \frac{27}{16}$, $m = \frac{11}{16}$, $\sigma = \frac{1}{4}$. It is easy see by calculating that

$$\begin{gathered} f(t,u) \geq 0, \quad \text{for } (t,u) \in [0,1] \times [1,32], \\ f(t,u) \leq 3, \quad \text{for } (t,u) \in [0,1] \times [0,6], \\ f(t,u) \geq \frac{128}{11}, \quad \text{for } (t,u) \in [\frac{1}{4},1] \times [8,32]. \end{gathered}$$

So the conditions of Theorem 3.1 hold. Then (4.1)-(4.2) has at least two positive solutions. □

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