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EXISTENCE AND UNIQUENESS OF NONTRIVIAL SOLUTIONS FOR NONLINEAR HIGHER-ORDER THREE-POINT EIGENVALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we study a nonlinear higher-order three-point eigenvalue problems with first-order derivative on time scales. Under certain growth conditions on the nonlinearity, sufficient conditions for existence and uniqueness of nontrivial solutions, which are easily verifiable, are obtained by using the Leray-Schauder nonlinear alternative. The conditions used in the paper are different from those in [4, 10, 21]. To show applications of our main results, we present some examples.

1. Introduction

In recent years, there has been much attention paid to the existence of positive solution for second-order three-point and higher-order two-point boundary value problem on time scales. On the other hand, p-Laplacian problems on time scales have also been studied extensively, for details, see [3, 6, 9, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23] and references therein. However, to the best of our knowledge, there are not many results concerning three-point eigenvalue problems of higher-order on time scales.

A time scale $\mathbf{T}$ is a nonempty closed subset of $\mathbb{R}$. We make the blanket assumption that $0, T$ are points in $\mathbf{T}$. By an interval $(0,T)$, we always mean the intersection of the real interval $(0,T)$ with the given time scale; that is $(0,T)\cap\mathbf{T}$.

Anderson [3] studied the following dynamic equation on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in (0,T), \tag{1.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(T). \tag{1.2}$$

He obtained one positive solution based on the limits $f_0 = \lim_{u\to 0^+}\frac{f(u)}{u}$ and $f_\infty = \lim_{u\to\infty}\frac{f(u)}{u}$. He also obtained at least three positive solutions. Kaufmann [15] also studied (1.1)–(1.2) and obtained finitely many positive solutions and then countably many positive solutions. Luo and Ma [17], discussed the following dynamic equation

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on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in (0,T), \tag{1.3}$$

$$u(0) = \beta u(\eta), \quad u(T) = \alpha u(\eta). \tag{1.4}$$

They obtained results for the existence of one positive solution and for the existence of at least three positive solutions by using a fixed point theorem and the Leggett-Williams fixed point theorem, respectively.

We would also like to mention the results of Boey and Wong [6], Zhao-Cai Hao [12] and Sun [23]. Boey and Wong [6] studied the following two-point right focal boundary-value problems on time scales:

$$(-1)^{n-1}y^{\Delta^n}(t) = (-1)^{p+1}F(t,y,(\sigma^{n-1}(t))), \quad t \in [a,b] \cap \mathbf{T}. \tag{1.5}$$

$$y^{\Delta^i}(a) = 0, \quad 0 \le i \le p-1, \tag{1.6}$$

$$y^{\Delta^i}(\sigma(b)) = 0, \quad p \le i \le n-1, \tag{1.7}$$

where $n \ge 2$, $1 \le p \le n-1$ is fixed and $\mathbf{T}$ is a time scale. Existence criteria are developed for triple positive solutions for the problem (1.5)–(1.7) by applying fixed point theorems for operators on a cone. Zhao-Cai Hao [12] considered the following fourth-order singular boundary value problems:

$$x^{(4)}(t) = \lambda f(t, x(t)), \quad t \in (0,1), \tag{1.8}$$

$$x(0) = x(1) = 0, \quad x''(0) = x''(1) = 0, \tag{1.9}$$

where $f \in C((0,1) \times (0,\infty) \times [0,\infty))$, $\lambda > 0$ is a parameter. He determined values of λ for which there exist positive solutions of the above boundary value problems, and for $\lambda = 1$, he gave criteria for the existence of eigenfunctions.

The present work is motivated by a recent paper Sun [23], where the following third-order two-point boundary-value problem on time scales is considered:

$$u^{\Delta\Delta\Delta}(t) + f(t, u(t), u^{\Delta\Delta}(t)) = 0, \quad t \in [a, \sigma(b)], \tag{1.10}$$

$$u(a) = A, \quad u(\sigma(b)) = B, \;\; u^{\Delta\Delta}(a) = C, \tag{1.11}$$

where $a, b \in \mathbf{T}$ and $a < b$. Existence of solutions and positive solutions is established by using the Leray-Schauder fixed point theorem. However, in the existing literature, very few people have considered the case where the nonlinear term contains the first-order derivative.

In this paper, we are concerned with the existence of nontrivial solutions of the following higher-order three-point eigenvalue problems with the first-order derivative on time scales:

$$u^{\Delta^n}(t) + \lambda f(t, u(t), u^{\Delta}(t)) = 0, \quad t \in (0,T), \tag{1.12}$$

$$u(0) = \alpha u(\eta), \quad u(T) = \beta u(\eta), \tag{1.13}$$

$$u^{\Delta^i}(0) = 0 \quad \text{for } i = 1, 2, \dots, n-2, \tag{1.14}$$

where $\lambda > 0$ is a parameter, $\eta \in (0, \rho(T))$ is a constant, $\alpha, \beta \in \mathbb{R}$, $f \in C_{ld}([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\mathbb{R} = (-\infty, +\infty)$, $n \ge 2$.

We want to point out that when $\mathbf{T} = \mathbb{R}$ and $\lambda = 1$, (1.12)–(1.14) becomes a boundary-value problem of differential equations and has been considered in [16].

The aim of this paper is to establish simple criteria for the existence of nontrivial solutions of the problem (1.12)–(1.14). Our results are new and different from those of [3, 9, 15, 17]. Particularly, we do not require any monotonicity and nonnegative on f, which was essential for the technique used in [3, 9, 15, 17].

2. Preliminaries

For convenience, we list the following definitions which can be found in [2, 4, 7, 8].

Definition 2.1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},$$
$$\rho(r) = \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T}.$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense.

Definition 2.2. Fix $t \in \mathbf{T}$. Let $f : \mathbf{T} \longrightarrow \mathbb{R}$. The delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$. Define $f^{\Delta^n}(t)$ to be the delta derivative of $f^{\Delta^{n-1}}(t)$; i.e., $f^{\Delta^n}(t) = (f^{\Delta^{n-1}}(t))^{\Delta}$.

Definition 2.3. A function f is left-dense continuous (i.e. ld-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$. If $F^{\Delta}(t) = f(t)$, then define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

For the rest of this article, we denote the set of left-dense continuous functions from $[0,T] \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and from $[0,T]$ to $\mathbb{R}$ by $C_{ld}([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $C_{ld}([0,T], \mathbb{R})$, respectively.

Let $X = C_{ld}([0,T], \mathbb{R})$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0,T]$, and $\|u\| = \max_{t\in[0,T]} |u(t)|$. Now we introduce the norm in $Y = C^1_{ld}([0,T], \mathbb{R})$ by

$$\|u\|_1 = \|u\| + \|u^{\Delta}\| = \max_{t\in[0,T]} |u(t)| + \max_{t\in[0,T]} |u^{\Delta}(t)|.$$

Clearly, it follows that $(Y, \|u\|_1)$ is a Banach space.

Lemma 2.4. *Suppose that $d = (1-\alpha)T^{n-1} - (\beta - \alpha)\eta^{n-1} \neq 0$. Then for $y \in C_{ld}([0,T], \mathbb{R})$, the problem*

$$u^{\Delta^n}(t) + y(t) = 0, \quad t \in (0,T), \tag{2.1}$$
$$u(0) = \alpha u(\eta), \quad u(T) = \beta u(\eta), \tag{2.2}$$
$$u^{\Delta^i}(0) = 0 \quad for \quad i = 1, 2, \dots, n-2, \tag{2.3}$$

has a unique solution

$$
\begin{aligned}
u(t) =&\frac{1}{d}\left[\alpha\eta^{n-1}+t^{n-1}(1-\alpha)\right]\int_0^T\frac{(T-s)^{n-1}}{(n-1)!}y(s)\Delta s-\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)\Delta s\\
&+\frac{1}{d}\left[-\alpha T^{n-1}+(\alpha-\beta)t^{n-1}\right]\int_0^\eta\frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\Delta s.
\end{aligned}\tag{2.4}
$$

Proof. From (2.1), we have

$$
u(t)=-\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)\Delta s+\sum_{i=1}^{n-1}A_it^i+B.\tag{2.5}
$$

Since $u^{\Delta^i}(0)=0$ for $i=1,2,\dots,n-2$, one gets $A_i=0$ for $i=1,2,\dots,n-2$. Now, we solve for A_{n-1} and B. By $u(0)=\alpha u(\eta)$ and $u(T)=\beta u(\eta)$, it follows that

$$
B=-\alpha\int_0^\eta\frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\Delta s+\alpha A_{n-1}\eta^{n-1}+\alpha B,\tag{2.6}
$$

and

$$
\begin{aligned}
&-\int_0^T\frac{(T-s)^{n-1}}{(n-1)!}y(s)\Delta s+A_{n-1}T^{n-1}+B\\
&=-\beta\int_0^\eta\frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\Delta s+\beta A_{n-1}\eta^{n-1}+\beta B.
\end{aligned}\tag{2.7}
$$

Solving the above equations (2.6) and (2.7), we get

$$
A_{n-1}=\frac{1}{d}\Big[(\alpha-\beta)\int_0^\eta\frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\Delta s+(1-\alpha)\int_0^T\frac{(T-s)^{n-1}}{(n-1)!}y(s)\Delta s\Big],\tag{2.8}
$$

$$
B=\frac{\alpha}{d}\Big[\eta^{n-1}\int_0^T\frac{(T-s)^{n-1}}{(n-1)!}y(s)\Delta s-T^{n-1}\int_0^\eta\frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\Delta s\Big].\tag{2.9}
$$

Substituting (2.8) and (2.9) in (2.5), one has

$$
\begin{aligned}
u(t) =&\frac{1}{d}[\alpha\eta^{n-1}+t^{n-1}(1-\alpha)]\int_0^T\frac{(T-s)^{n-1}}{(n-1)!}y(s)\Delta s-\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)\Delta s\\
&+\frac{1}{d}[-\alpha T^{n-1}+(\alpha-\beta)t^{n-1}]\int_0^\eta\frac{(\eta-s)^{n-1}}{(n-1)!}y(s)\Delta s.
\end{aligned}
$$

It is easy to see that BVP $u^{\Delta^n}(t)=0$, $u(0)=\alpha u(\eta)$, $u(T)=\beta u(\eta)$, $u^{\Delta^i}(0)=0$, for $i=1,2,\dots,n-2$, has only the trivial solution. Thus u in (2.4) is the unique solution of (2.1), (2.2) and (2.3). The proof is complete. □

To prove our main result, we need a useful lemma which can be found in [11].

Lemma 2.5 ([11])**.** *Let X be a real Banach space and Ω be a bounded open subset of X, $0\in\Omega$, $F:\overline{\Omega}\longrightarrow X$ be a completely continuous operator. Then either there exist $x\in\partial\Omega$, $\lambda>1$ such that $F(x)=\lambda x$, or there exists a fixed point $x^*\in\overline{\Omega}$.*

3. Main results

For convenience, we introduce the following notation. Let

$$\varphi(t,s)=\frac{(t-s)^{n-1}}{(n-1)!}(p(s)+q(s)),\quad \psi(t,s)=\frac{(t-s)^{n-1}}{(n-1)!}r(s),$$

$$\begin{aligned}
M&=\big[1+\frac{(2|\alpha|+1)T^{n-1}}{|d|}\big]\int_0^T\varphi(T,s)\Delta s+\frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta\varphi(\eta,s)\Delta s\\
&\quad+\frac{(|\alpha|+1)T^{n-2}}{|d|}\int_0^T(n-1)\varphi(T,s)\Delta s+\int_0^T\frac{n-1}{T-s}\varphi(T,s)\Delta s\\
&\quad+\frac{(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta(n-1)\varphi(\eta,s)\Delta s,
\end{aligned}$$

$$\begin{aligned}
N&=\big[1+\frac{(2|\alpha|+1)T^{n-1}}{|d|}\big]\int_0^T\psi(T,s)\Delta s+\frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta\psi(\eta,s)\Delta s\\
&\quad+\frac{(|\alpha|+1)T^{n-2}}{|d|}\int_0^T(n-1)\psi(T,s)\Delta s+\int_0^T\frac{n-1}{T-s}\psi(T,s)\Delta s\\
&\quad+\frac{(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta(n-1)\psi(\eta,s)\Delta s.
\end{aligned}$$

Our main result is stated as follows.

Theorem 3.1. *Suppose that $f(t,0,0)\not\equiv 0$, $t\in[0,T]$, $d\neq 0$ and there exist non-negative functions $p,q,r\in L^1[0,T]$ such that*

$$|f(t,u,v)|\le p(t)|u|+q(t)|v|+r(t),\quad a.\ e.(t,u,v)\in[0,T]\times\mathbb{R}\times\mathbb{R},\tag{3.1}$$

and there exists $t_0\in[0,T]$ such that $p(t_0)\neq 0$ or $q(t_0)\neq 0$. Then there exists a constant $\lambda^*>0$ such that for any $0<\lambda\le\lambda^*$, the problem (1.12)–(1.14) *has at least one nontrivial solution $u^*\in C^1_{ld}([0,T],\mathbb{R})$.*

Proof. By Lemma 2.4, the problem (1.12)–(1.14) has a solution $u=u(t)$ if and only if u is a solution of the operator equation

$$\begin{aligned}
u(t)&=\frac{\lambda}{d}\left[\alpha\eta^{n-1}+t^{n-1}(1-\alpha)\right]\int_0^T\frac{(T-s)^{n-1}}{(n-1)!}f(s,u(s),u^\Delta(s))\Delta s\\
&\quad-\lambda\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}f(s,u(s),u^\Delta(s))\Delta s\\
&\quad+\frac{\lambda}{d}\left[-\alpha T^{n-1}+(\alpha-\beta)t^{n-1}\right]\int_0^\eta\frac{(\eta-s)^{n-1}}{(n-1)!}f(s,u(s),u^\Delta(s))\Delta s\\
&=:Fu(t)\,.
\end{aligned}$$

in Y. So we need only to seek for a fixed point of F in Y. Applying Arzela-Ascoli theorem on time scales [1] and the Lebesgue's dominated convergence theorem on time scales [5], we can conclude that this operator $F:Y\to Y$ is a completely continuous operator [22].

Since $|f(t,0,0)|\le r(t)$, a.e. $t\in[0,T]$, we know $\int_0^T\psi(T,s)\Delta s>0$, from $p(t_0)\neq 0$ or $q(t_0)\neq 0$, we easily obtain $\int_0^T\varphi(T,s)\Delta s>0$, so $M>0$, $N>0$. Let

$$m=\frac{N}{M},\quad \Omega=\{u\in C^1_{ld}[0,T]:\|u\|_1<m\}.$$

Suppose $u \in \partial\Omega$, $\mu > 1$ are such that $Fu = \mu u$. Then

$$\mu m = \mu \|u\|_1 = \|Fu\|_1 = \|Fu\| + \|(Fu)^\Delta\|.$$

Since

$$\begin{aligned}
\|Fu\| &= \max_{t\in[0,T]} |Fu(t)| \\
&\leq \Big|\frac{\lambda}{d}\left[\alpha\eta^{n-1} + t^{n-1}(1-\alpha)\right]\Big| \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\quad + \max_{0\leq t\leq T} \Big\{ \lambda \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\quad + \Big|\frac{\lambda}{d}[-\alpha T^{n-1} + (\alpha-\beta)t^{n-1}]\Big| \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \Big\} \\
&\leq \frac{\lambda(2|\alpha|+1)T^{n-1}}{|d|} \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\quad + \lambda \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\quad + \frac{\lambda(2|\alpha|+|\beta|)T^{n-1}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&= \lambda\Big[1 + \frac{(2|\alpha|+1)T^{n-1}}{|d|}\Big] \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\quad + \frac{\lambda(2|\alpha|+|\beta|)T^{n-1}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\leq \lambda\Big[1 + \frac{(2|\alpha|+1)T^{n-1}}{|d|}\Big] \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} [p(s)|u(s)| + q(s)|u^\Delta(s)| + r(s)]\Delta s \\
&\quad + \frac{\lambda(2|\alpha|+|\beta|)T^{n-1}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} [p(s)|u(s)| + q(s)|u^\Delta(s)| + r(s)]\Delta s \\
&\leq \lambda\|u\|_1 \Big\{ \Big[1 + \frac{(2|\alpha|+1)T^{n-1}}{|d|}\Big] \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} (p(s)+q(s))\Delta s \\
&\quad + \frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} (p(s)+q(s))\Delta s \Big\} \\
&\quad + \lambda\Big\{ \Big[1 + \frac{(2|\alpha|+1)T^{n-1}}{|d|}\Big] \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} r(s)\Delta s \\
&\quad + \frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!} r(s)\Delta s \Big\} \\
&\leq \lambda\|u\|_1 \Big\{ \Big[1 + \frac{(2|\alpha|+1)T^{n-1}}{|d|}\Big] \int_0^T \varphi(T,s)\Delta s \\
&\quad + \frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|} \int_0^\eta \varphi(\eta,s)\Delta s \Big\} \\
&\quad + \lambda\Big\{ \Big[1 + \frac{(2|\alpha|+1)T^{n-1}}{|d|}\Big] \int_0^T \psi(T,s)\Delta s \\
&\quad + \frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|} \int_0^\eta \psi(\eta,s)\Delta s \Big\}
\end{aligned}$$

and

$$
\begin{aligned}
&\|(Fu)^\Delta\| \\
&= \max_{t\in[0,T]} |(Fu)^\Delta(t)| \\
&= \max_{t\in[0,T]} \Big| \frac{\lambda}{d}(1-\alpha)t^{n-2} \int_0^T \frac{(T-s)^{n-1}}{(n-2)!} f(s,u(s),u^\Delta(s))\Delta s \\
&\quad - \lambda \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} f(s,u(s),u^\Delta(s))\Delta s \\
&\quad + \frac{\lambda}{d}(\alpha-\beta)t^{n-2} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!} f(s,u(s),u^\Delta(s))\Delta s \Big| \\
&\le \frac{\lambda(|\alpha|+1)T^{n-2}}{|d|} \int_0^T \frac{(T-s)^{n-1}}{(n-2)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\quad + \lambda \int_0^T \frac{(T-s)^{n-2}}{(n-2)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\quad + \frac{\lambda(|\alpha|+|\beta|)T^{n-2}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!} |f(s,u(s),u^\Delta(s))|\Delta s \\
&\le \frac{\lambda(|\alpha|+1)T^{n-2}}{|d|} \int_0^T \frac{(T-s)^{n-1}}{(n-2)!} \big[p(s)|u(s)| + q(s)|u^\Delta(s)| + r(s)\big]\,\Delta s \\
&\quad + \lambda \int_0^T \frac{(T-s)^{n-2}}{(n-2)!} \big[p(s)|u(s)| + q(s)|u^\Delta(s)| + r(s)\big]\,\Delta s \\
&\quad + \frac{\lambda(|\alpha|+|\beta|)T^{n-2}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!} \big[p(s)|u(s)| + q(s)|u^\Delta(s)| + r(s)\big]\,\Delta s \\
&\le \lambda\|u\|_1 \Big\{ \frac{(|\alpha|+1)T^{n-2}}{|d|} \int_0^T \frac{(T-s)^{n-1}}{(n-2)!} (p(s)+q(s))\Delta s \\
&\quad + \int_0^T \frac{(T-s)^{n-2}}{(n-2)!} (p(s)+q(s))\Delta s \\
&\quad + \frac{(|\alpha|+|\beta|)T^{n-2}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!} (p(s)+q(s))\Delta s \Big\} \\
&\quad + \lambda \Big\{ \frac{(|\alpha|+1)T^{n-2}}{|d|} \int_0^T \frac{(T-s)^{n-1}}{(n-2)!} r(s)\Delta s + \int_0^T \frac{(T-s)^{n-2}}{(n-2)!} r(s)\Delta s \\
&\quad + \frac{(|\alpha|+|\beta|)T^{n-2}}{|d|} \int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!} r(s)\Delta s \Big\} \\
&= \lambda\|u\|_1 \Big\{ \frac{(|\alpha|+1)T^{n-2}}{|d|} \int_0^T (n-1)\varphi(T,s)\Delta s + \int_0^T \frac{n-1}{T-s}\varphi(T,s)\Delta s \\
&\quad + \frac{(|\alpha|+|\beta|)T^{n-2}}{|d|} \int_0^\eta (n-1)\varphi(\eta,s)\Delta s \Big\} \\
&\quad + \lambda \Big\{ \frac{(|\alpha|+1)T^{n-2}}{|d|} \int_0^T (n-1)\psi(T,s)\Delta s + \int_0^T \frac{n-1}{T-s}\psi(T,s)\Delta s \\
&\quad + \frac{(|\alpha|+|\beta|)T^{n-2}}{|d|} \int_0^\eta (n-1)\psi(\eta,s)\Delta s \Big\},
\end{aligned}
$$

then

$$\|Fu\|_1 \le \lambda \|u\|_1 M + \lambda N.$$

Choose $\lambda^* = \frac{1}{2M}$. Then when $0 < \lambda \le \lambda^*$, we have

$$\mu m = \mu \|u\|_1 = \|Fu\|_1 \le \frac{1}{2M} M \|u\|_1 + \frac{N}{2M}.$$

Consequently,

$$\mu \le \frac{1}{2} + \frac{N}{2mM} = 1.$$

This contradicts $\mu > 1$. By Lemma 2.5, F has a fixed point $u^* \in \overline{\Omega}$. Since $f(t,0,0) \not\equiv 0$, then when $0 < \lambda \le \lambda^*$, the problem (1.12)–(1.14) has a nontrivial solution $u^* \in C^1_{ld}([0,T], \mathbb{R})$. This completes the proof. $\square$

If we use the following stronger condition than (3.1) to substitute (3.1), we obtain the following Theorem.

Theorem 3.2. *Suppose that $f(t,0,0) \not\equiv 0$, $t \in [0,T]$, $d \neq 0$ and there exist non-negative functions $p, q \in L^1[0,T]$ such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le p(t)|u_1 - u_2| + q(t)|v_1 - v_2|, \tag{3.2}$$

a.e. $(t,u_i,v_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ ($i = 1,2$), and there exists $t_0 \in [0,T]$ such that $p(t_0) \neq 0$ or $q(t_0) \neq 0$. Then there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$, the problem (1.12)–(1.14) has unique nontrivial solution $u^* \in C^1_{ld}([0,T], \mathbb{R})$.

Proof. In fact, if $u_2 = v_2 = 0$, then we have

$$|f(t,u_1,v_1)| \le p(t)|u_1| + q(t)|v_1| + |f(t,0,0)|, \quad \text{a. e. } (t,u_1,v_1) \in [0,T] \times \mathbb{R} \times \mathbb{R}.$$

From Theorem 3.1, we know the problem (1.12)–(1.14) has a nontrivial solution $u^* \in C^1_{ld}([0,T], \mathbb{R})$. But in this case, we prefer to concentrate on the uniqueness of the nontrivial solution for the problem (1.12)–(1.14). Let F be given in Theorem 3.1. We shall show that F is a contraction. On the one hand,

$$\begin{aligned}
&\|Fu_1 - Fu_2\| \\
&= \max_{t \in [0,T]} |Fu_1(t) - Fu_2(t)| \\
&= \max_{t \in [0,T]} \Big| \frac{\lambda}{d} \big[\alpha \eta^{n-1} + t^{n-1}(1-\alpha)\big] \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} \big[f(s,u_1(s),u_1^\Delta(s)) \\
&\quad - f(s,u_2(s),u_2^\Delta(s))\big] \Delta s \\
&\quad - \lambda \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \big[f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\big] \Delta s \\
&\quad + \frac{\lambda}{d} \big[-\alpha T^{n-1} + (\alpha - \beta) t^{n-1}\big] \int_0^\eta \frac{(\eta - s)^{n-1}}{(n-1)!} \Big[f(s,u_1(s),u_1^\Delta(s)) \\
&\quad - f(s,u_2(s),u_2^\Delta(s))\Big\} \Delta s \Big| \\
&\le \frac{\lambda(2|\alpha| + 1) T^{n-1}}{|d|} \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} \big|f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\big| \Delta s \\
&\quad + \lambda \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} \big|f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\big| \Delta s
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}\left|f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\right|\Delta s\\
&\le \frac{\lambda(2|\alpha|+1)T^{n-1}}{|d|}\int_0^T \frac{(T-s)^{n-1}}{(n-1)!}\left[p(s)|u_1(s)-u_2(s)| + q(s)|u_1^\Delta(s)-u_2^\Delta(s)|\right]\Delta s\\
&+ \lambda\int_0^T \frac{(T-s)^{n-1}}{(n-1)!}\left[p(s)|u_1(s)-u_2(s)| + q(s)|u_1^\Delta(s)-u_2^\Delta(s)|\right]\Delta s\\
&+ \frac{\lambda(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}\big[p(s)|u_1(s)-u_2(s)|\\
&+ q(s)|u_1^\Delta(s)-u_2^\Delta(s)|\big]\Delta s\\
&\le \lambda\|u_1-u_2\|_1\Big\{\big[\frac{(2|\alpha|+1)T^{n-1}}{|d|}+1\big]\int_0^T \frac{(T-s)^{n-1}}{(n-1)!}(p(s)+q(s))\Delta s\\
&+ \frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}(p(s)+q(s))\Delta s\Big\}\\
&= \lambda\|u_1-u_2\|_1\Big\{\big[\frac{(2|\alpha|+1)T^{n-1}}{|d|}+1\big]\int_0^T \varphi(T,s)\Delta s\\
&+ \frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta \varphi(\eta,s)\Delta s\Big\}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\|(Fu_1)^\Delta - (Fu_2)^\Delta\|\\
&= \max_{t\in[0,T]} |(Fu_1)^\Delta(t) - (Fu_2)^\Delta(t)|\\
&= \max_{t\in[0,T]} \Big|\frac{\lambda}{d}(1-\alpha)t^{n-2}\int_0^T \frac{(T-s)^{n-1}}{(n-2)!}\big[f(s,u_1(s),u_1^\Delta(s))\\
&- f(s,u_2(s),u_2^\Delta(s))\big]\Delta s\\
&- \lambda\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}\left[f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\right]\Delta s\\
&+ \frac{\lambda}{d}(\alpha-\beta)t^{n-2}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!}\left[f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\right]\Delta s\Big|\\
&\le \frac{\lambda(|\alpha|+1)T^{n-2}}{|d|}\int_0^T \frac{(T-s)^{n-1}}{(n-2)!}\left|f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\right|\Delta s\\
&+ \lambda\int_0^T \frac{(T-s)^{n-2}}{(n-2)!}\left|f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\right|\Delta s\\
&+ \frac{\lambda(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!}\left|f(s,u_1(s),u_1^\Delta(s)) - f(s,u_2(s),u_2^\Delta(s))\right|\Delta s\\
&\le \frac{\lambda(|\alpha|+1)T^{n-2}}{|d|}\int_0^T \frac{(T-s)^{n-1}}{(n-2)!}\left[p(s)|u_1(s)-u_2(s)| + q(s)|u_1^\Delta(s)-u_2^\Delta(s)|\right]\Delta s\\
&+ \lambda\int_0^T \frac{(T-s)^{n-2}}{(n-2)!}\left[p(s)|u_1(s)-u_2(s)| + q(s)|u_1^\Delta(s)-u_2^\Delta(s)|\right]\Delta s\\
&+ \frac{\lambda(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!}\big[p(s)|u_1(s)-u_2(s)|
\end{aligned}
$$

$$
\begin{aligned}
&+ q(s)|u_1^{\Delta}(s) - u_2^{\Delta}(s)|\big]\Delta s \\
&\le \lambda\|u_1 - u_2\|_1\Big\{\frac{(|\alpha|+1)T^{n-2}}{|d|}\int_0^T \frac{(T-s)^{n-1}}{(n-2)!}(p(s)+q(s))\Delta s \\
&+ \int_0^T \frac{(T-s)^{n-2}}{(n-2)!}(p(s)+q(s))\Delta s \\
&+ \frac{(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-2)!}(p(s)+q(s))\Delta s\Big\} \\
&= \lambda\|u_1 - u_2\|_1\Big\{\frac{(|\alpha|+1)T^{n-2}}{|d|}\int_0^T (n-1)\varphi(T,s)\Delta s + \int_0^T \frac{n-1}{T-s}\varphi(T,s)\Delta s \\
&+ \frac{(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta (n-1)\varphi(\eta,s)\Delta s\Big\}.
\end{aligned}
$$

Then

$$\|Fu_1 - Fu_2\|_1 \le \lambda\|u_1 - u_2\|_1 M.$$

If we choose $\lambda^* = \frac{1}{2M}$. Then, when $0 < \lambda \le \lambda^*$, we have

$$\|Fu_1 - Fu_2\|_1 \le \frac{1}{2}\|u_1 - u_2\|_1.$$

So F is indeed a contraction. Finally, we use the Banach fixed point theorem to deduce the existence of unique solution to the problem (1.12)–(1.14). $\square$

Corollary 3.3. *Suppose that $f(t,0,0) \not\equiv 0$, $t \in [0,T]$, $d \neq 0$ and*

$$0 \le L = \limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|u|+|v|} < +\infty. \tag{3.3}$$

Then there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$, the problem (1.12)–(1.14) has at least one nontrivial solution $u^* \in C_{ld}^1([0,T],\mathbb{R})$

Proof. Let $\varepsilon > 0$ such that $L + 1 - \varepsilon > 0$. By (3.3), there exists $H > 0$ such that

$$|f(t,u,v)| \le (L+1-\varepsilon)(|u|+|v|), \quad |u|+|v| \ge H, \quad 0 \le t \le T.$$

Let $K = \max_{t\in[0,T],\ |u|+|v|\le H} |f(t,u,v)|$. Then for any $(t,u,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}$, we have

$$|f(t,u,v)| \le (L+1-\varepsilon)(|u|+|v|) + K.$$

From Theorem 3.1, we know the problem (1.12)–(1.14) has at least one nontrivial solution $u^* \in C_{ld}^1([0,T],\mathbb{R})$. $\square$

Corollary 3.4. *Suppose that $f(t,0,0) \not\equiv 0$, $t \in [0,T]$, $d \neq 0$ and*

$$0 \le L = \limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|u|} < +\infty,$$

or

$$0 \le L = \limsup_{|u|+|v|\to+\infty} \max_{t\in[0,T]} \frac{|f(t,u,v)|}{|v|} < +\infty.$$

Then there exists a constant $\lambda^* > 0$, such that for any $0 < \lambda \le \lambda^*$, problem (1.12)–(1.14) has at least one nontrivial solution $u^* \in C_{ld}^1([0,T],\mathbb{R})$.

We remark that Corollaries 3.3 and 3.4 include the case that f is jointly sublinear at $(-\infty,+\infty)$; that is,

$$\limsup_{|u|+|v|\to+\infty}\max_{t\in[0,T]}\frac{|f(t,u,v)|}{|u|+|v|}=0 \quad\text{or}\quad \limsup_{|u|+|v|\to+\infty}\max_{t\in[0,T]}\frac{|f(t,u,v)|}{|u|}=0$$
$$\text{or}\quad \limsup_{|u|+|v|\to+\infty}\max_{t\in[0,T]}\frac{|f(t,u,v)|}{|v|}=0.$$

4. Some examples

In the section, we illustrate our results, with some examples. We only study the case $\mathbf{T}=\mathbb{R}$ and $(0,T)=(0,1)$.

Example 4.1. Consider the forth-order eigenvalue problem

$$u^{(4)}+\lambda\Big(\frac{ut\sin t}{t^2+1}-t(\cos u')^2+t(1+t)\Big)=0,\quad t\in(0,1), \tag{4.1}$$
$$u(0)=-u(\frac{1}{2}),\quad u(1)=u(\frac{1}{2}),\quad u'(0)=0,\quad u''(0)=0. \tag{4.2}$$

Set $\alpha=-1$, $\beta=1$, $\eta=\frac{1}{2}$, $n=4$, $f(t,u,u')=\frac{ut\sin t}{t^2+1}-t(\cos u')^2+t(1+t)$,

$$d=(1-\alpha)T^{n-1}-(\beta-\alpha)\eta^{n-1}=(1+1)\cdot 1^{4-1}-(1+1)\cdot(\frac{1}{2})^{4-1}=\frac{7}{4}>0,$$
$$p(t)=\frac{t}{t^2+1},\quad q(t)=t,\quad r(t)=t^2.$$

Noticing that

$$\Big|\frac{ut\sin t}{t^2+1}-t(\cos u')^2+t(1+t)\Big|\le p(t)|u|+q(t)|u'|+r(t),$$

it follows from a direct calculation that

$$\varphi(t,s)=\frac{(t-s)^{n-1}}{(n-1)!}(p(s)+q(s))=\frac{(t-s)^{4-1}}{(4-1)!}(\frac{s}{s^2+1}+s)=\frac{(t-s)^3}{6}(\frac{s}{s^2+1}+s),$$

$$\begin{aligned}
M=&\big[1+\frac{(2|\alpha|+1)T^{n-1}}{|d|}\big]\int_0^T\varphi(T,s)ds+\frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta\varphi(\eta,s)ds\\
&+\frac{(|\alpha|+1)T^{n-2}}{|d|}\int_0^T(n-1)\varphi(T,s)ds+\int_0^T\frac{n-1}{T-s}\varphi(T,s)ds\\
&+\frac{(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta(n-1)\varphi(\eta,s)ds\\
=&\Big[1+\frac{(2\times1+1)\cdot 1^{4-1}}{\frac{7}{4}}\Big]\int_0^1\frac{(1-s)^3}{6}(\frac{s}{s^2+1}+s)ds\\
&+\frac{(2\times1+1)\cdot 1^{4-1}}{\frac{7}{4}}\int_0^{\frac{1}{2}}\frac{(\frac{1}{2}-s)^3}{6}(\frac{s}{s^2+1}+s)ds\\
&+\frac{(1+1)\cdot 1^{4-2}}{\frac{7}{4}}\int_0^1\frac{(1-s)^3}{2}(\frac{s}{s^2+1}+s)ds+\int_0^1\frac{(1-s)^2}{2}(\frac{s}{s^2+1}+s)ds\\
&+\frac{(1+1)\cdot 1^{4-2}}{\frac{7}{4}}\int_0^{\frac{1}{2}}\frac{(\frac{1}{2}-s)^3}{2}(\frac{s}{s^2+1}+s)ds\\
=&\,0.1763.
\end{aligned}$$

Choose $\lambda^* = \frac{1}{2M} = 2.8367$. Then by Theorem 3.2, we know the problem (4.1)–(4.2) has a unique nontrivial solution $u^* \in C^1([0,T],\mathbb{R})$ for any $\lambda \in (0, 2.8367]$.

Example 4.2. Consider the third-order eigenvalue problem

$$u''' + \lambda\Big(\frac{u}{2} - \cos u'\Big) = 0, \quad t \in (0,1), \tag{4.3}$$

$$u(0) = \frac{1}{4}u(\frac{1}{2}), \quad u(1) = \frac{3}{4}u(\frac{1}{2}), \quad u'(0) = 0. \tag{4.4}$$

Set $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$, $\eta = \frac{1}{2}$, $n = 3$, $f(t,u,u') = \frac{u}{2} - \cos u'$,

$$d = (1-\alpha)T^{n-1} - (\beta-\alpha)\eta^{n-1} = (1-\frac{1}{4})\cdot 1^{3-1} - (\frac{3}{4} - \frac{1}{4})(\frac{1}{2})^{3-1} = \frac{5}{8} > 0,$$

$p(t) = \frac{1}{2}$, $q(t) = 1$. Noticing that

$$\big|\frac{u_1}{2} - \cos u_1' - \frac{u_2}{2} + \cos u_2'\big| \le p(t)|u_1 - u_2| + q(t)|u_1' - u_2'|,$$

it follows from a direct calculation that

$$\varphi(t,s) = \frac{(t-s)^{n-1}}{(n-1)!}(p(s)+q(s)) = \frac{(t-s)^{3-1}}{(3-1)!}(\frac{1}{2}+1) = \frac{3(t-s)^2}{4},$$

$$\begin{aligned}
M =& \big[1 + \frac{(2|\alpha|+1)T^{n-1}}{|d|}\big]\int_0^T \varphi(T,s)ds + \frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta \varphi(\eta,s)ds \\
&+ \frac{(|\alpha|+1)T^{n-2}}{|d|}\int_0^T (n-1)\varphi(T,s)ds + \int_0^T \frac{n-1}{T-s}\varphi(T,s)ds \\
&+ \frac{(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta (n-1)\varphi(\eta,s)ds \\
=& \big[1 + \frac{(2\times\frac{1}{4}+1)\cdot 1^{3-1}}{\frac{5}{8}}\big]\int_0^1 \frac{3(1-s)^2}{4}ds \\
&+ \frac{(2\times\frac{1}{4}+\frac{3}{4})\cdot 1^{3-1}}{\frac{5}{8}}\int_0^{\frac{1}{2}} \frac{3(\frac{1}{2}-s)^2}{4}ds \\
&+ \frac{(\frac{1}{4}+1)\cdot 1^{3-2}}{\frac{5}{8}}\int_0^1 \frac{3(1-s)^2}{2}ds + \int_0^1 \frac{3(1-s)}{2}ds \\
&+ \frac{(\frac{1}{4}+\frac{3}{4})\cdot 1^{3-2}}{\frac{5}{8}}\int_0^{\frac{1}{2}} \frac{3(\frac{1}{2}-s)^2}{2}ds \\
=& 2.7625.
\end{aligned}$$

Choose $\lambda^* = \frac{1}{2M} = 0.1810$. Then by Theorem 3.2, problem (4.3)-(4.4) has a unique nontrivial solution $u^* \in C^1([0,T],\mathbb{R})$ for any $\lambda \in (0, 0.1810]$.

Example 4.3. Consider the third-order eigenvalue problem

$$u''' + \lambda\Big(-u^{\frac{1}{2}} + t^2\sin\sqrt{u^4 + u'^2} + t^3(1-t)e^{\cos t}\Big) = 0, \quad t \in (0,1), \tag{4.5}$$

$$u(0) = 2u(\frac{1}{4}), \quad u(1) = u(\frac{1}{4}), \quad u'(0) = 0. \tag{4.6}$$

Set, $\alpha = 2$, $\beta = 1$, $\eta = \frac{1}{4}$, $n = 3$,

$$f(t,u,u') = -u^{\frac{1}{2}} + t^2\sin\sqrt{u^4 + u'^2} + t^3(1-t)e^{\cos t},$$

$$d=(1-\alpha)T^{n-1}-(\beta-\alpha)\eta^{n-1}=(1-2)\cdot 1^{3-1}-(1-2)(\frac{1}{4})^{3-1}=-\frac{15}{16}<0\,.$$

It is obvious that

$$\limsup_{|u|+|u'|\to+\infty}\max_{t\in[0,T]}\frac{|-u^{\frac{1}{2}}+t^2\sin\sqrt{u^4+u'^2}+t^3(1-t)e^{\cos t}|}{|u|+|u'|}=0.$$

Choose $\varepsilon=\frac{1}{2}$. In this case, $p(t)=\frac{1}{2}$, $q(t)=\frac{1}{2}$. It follows from a direct calculation that

$$\varphi(t,s)=\frac{(t-s)^{n-1}}{(n-1)!}(p(s)+q(s))=\frac{(t-s)^{3-1}}{(3-1)!}(\frac{1}{2}+\frac{1}{2})=\frac{(t-s)^2}{2},$$

$$\begin{aligned}
M =&\big[1+\frac{(2|\alpha|+1)T^{n-1}}{|d|}\big]\int_0^T\varphi(T,s)ds+\frac{(2|\alpha|+|\beta|)T^{n-1}}{|d|}\int_0^\eta\varphi(\eta,s)ds\\
&+\frac{(|\alpha|+1)T^{n-2}}{|d|}\int_0^T(n-1)\varphi(T,s)ds+\int_0^T\frac{n-1}{T-s}\varphi(T,s)ds\\
&+\frac{(|\alpha|+|\beta|)T^{n-2}}{|d|}\int_0^\eta(n-1)\varphi(\eta,s)ds\\
=&\big[1+\frac{(2\times 2+1)\cdot 1^{3-1}}{\frac{15}{16}}\big]\int_0^1\frac{(1-s)^2}{2}ds+\frac{(2\times 2+1)\cdot 1^{3-1}}{\frac{15}{16}}\int_0^{\frac{1}{4}}\frac{(\frac{1}{4}-s)^2}{2}ds\\
&+\frac{(2+1)\cdot 1^{3-2}}{\frac{15}{16}}\int_0^1(1-s)^2ds+\int_0^1(1-s)ds\\
&+\frac{(2+1)\cdot 1^{3-2}}{\frac{15}{16}}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^2ds\\
=&\ 2.6528.
\end{aligned}$$

Choose $\lambda^*=\frac{1}{2M}=0.1885$. Then by Corollary 3.3, we know the problem (4.5)–(4.6) has a unique nontrivial solution $u^*\in C^1([0,T],\mathbb{R})$ for any $\lambda\in(0,0.1885]$.

Remark 4.4. The boundary-value problem (1.12)–(1.14) includes (BVP) (1.1)-(1.2) of [3, 15], (1.3)-(1.4) of [17].

For the case where $\alpha=\beta=0$, $\mathbf{T}=\mathbb{R}$, $\lambda=1$, (BVP) (1.12)–(1.14) becomes

$$\begin{gathered}
u^{(n)}+a(t)f(u)=0,\quad t\in(0,1),\\
u^{(i)}(0)=u(1)=0,\quad i=0,1,2,\dots,n-2,
\end{gathered}$$

The above problem was studied by Eloe and Henderson [10].

As usual we write

$$\begin{gathered}
\max f_\infty:=\lim_{u\to\infty}\max_{t\in[0,T]}\frac{f(t,u)}{u},\quad \min f_\infty:=\lim_{u\to\infty}\min_{t\in[0,T]}\frac{f(t,u)}{u},\\
\max f_0:=\lim_{u\to 0^+}\max_{t\in[0,T]}\frac{f(t,u)}{u},\quad \min f_0:=\lim_{u\to 0^+}\min_{t\in[0,T]}\frac{f(t,u)}{u}.
\end{gathered}$$

Function f in [3, 17, 18] is assumed to be superlinear ($\max f_0=0$ and $\max f_\infty=\infty$) or sublinear ($\max f_\infty=0$ and $\max f_0=\infty$).

The condition:

$$0\le\overline{f}_0=\limsup_{u\to0}\max_{t\in[0,T]}\frac{f(t,u)}{u}<L,\quad l<\underline{f}_\infty=\liminf_{u\to\infty}\min_{t\in[0,T]}\frac{f(t,u)}{u}\le\infty,\quad(4.7)$$

or

$$0 \le \overline{f}_\infty = \limsup_{u\to\infty} \max_{t\in[0,T]} \frac{f(t,u)}{u} < L, \quad l < \underline{f}_0 = \liminf_{u\to 0} \min_{t\in[0,T]} \frac{f(t,u)}{u} \le \infty, \quad (4.8)$$

is required in [22, 24], where L and l are given. In this paper, we do not assume that nonlinear term f satisfy either superlinear (sublinear) conditions, or the conditions (4.7) and (4.8). Consequently, in view of different aspect, we can say that main results in [3, 10, 17, 18, 22, 24] do not apply to (4.1)–(4.3). The sufficient conditions in this paper ,which are easily verifiable, have a wider adaptive range. These have an important of leadings significance in both theory and application of boundary value problems.

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