ALMOST AUTOMORPHY OF SEMILINEAR PARABOLIC EVOLUTION EQUATIONS

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Abstract. This paper studies the existence and uniqueness of almost automorphic mild solutions to the semilinear parabolic evolution equation
\[ u'(t) = A(t)u(t) + f(t, u(t)), \]
assuming that the linear operators \( A(\cdot) \) satisfy the 'Acquistapace–Terreni' conditions, the evolution family generated by \( A(\cdot) \) has an exponential dichotomy, and the resolvent \( R(\omega, A(\cdot)) \), and \( f \) are almost automorphic.

1. Introduction

In this work we investigate the almost automorphy of the solutions to the parabolic evolution equations
\begin{align*}
  u'(t) &= A(t)u(t) + g(t), \quad t \in \mathbb{R}, \\
  u'(t) &= A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R},
\end{align*}
in a Banach space \( X \), where the linear operators \( A(t) \) satisfy the 'Acquistapace–Terreni' conditions and that the evolution family \( U \) generated by \( A(\cdot) \) has an exponential dichotomy. The asymptotic behavior of these equations was studied by several authors. The most extensively studied cases are the autonomous case \( A(t) = A \) and the periodic case \( A(t + T) = A(t) \), see \[3, 4, 7, 13, 14, 22, 26\] for almost periodicity and \[6, 10, 12, 16, 20, 21\] for almost automorphy. Maniar and Schnaubelt \[19\] studied the general case, where some resolvent \( R(\omega, A(\cdot)) \) of \( A(\cdot) \) is only almost periodic.

In this paper, we follow the idea of \[19\] and assume that the function \( t \mapsto R(\omega, A(t)) \in \mathcal{L}(X) \), for \( \omega \geq 0 \), is almost automorphic. We show first the almost automorphy of the Green's function corresponding to \( U \), following the strategy of \[19\] which consists in using Yosida-approximations of \( A(\cdot) \). This result will yield the existence of a unique almost automorphic mild solution \( u : \mathbb{R} \to X \) of \[1.1\] given by
\[ u(t) = \int_{\mathbb{R}} \Gamma(t, \tau)g(\tau) \, d\tau, \quad t \in \mathbb{R}, \]

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for every almost automorphic function \(g\). Using an interpolation argument, as in [3], we show that the solution \(u\) of (1.1) given by (1.3) is also almost automorphic in every time invariant interpolation space \(X_\alpha, 0 \leq \alpha < 1\).

Finally, by a fixed point technique, if the semilinear term \(f : \mathbb{R} \times X_\alpha \to X\) is almost automorphic and globally small Lipschitzian; i.e., the Lipschitz constant is small, we show that there is a unique almost automorphic mild solution on \(X_\alpha\) to the semilinear parabolic evolution problem (1.2). This is an extension of [20] Theorem 3.1.

To illustrate our results, we also study an example of a reaction diffusion equation with time-varying coefficients. If the coefficients and the semilinear term \(f\) are almost automorphic, we show that the solutions are almost automorphic.

2. Prerequisites

A set \(U = \{U(t, s) : t \geq s, t, s \in \mathbb{R}\}\) of bounded linear operators on a Banach space \(X\) is called an evolution family if

(E1) \(U(t, s) = U(t, r)U(r, s)\) and \(U(s, s) = I\) for \(t \geq r \geq s\) and

(E2) \((t, s) \mapsto U(t, s)\) is strongly continuous for \(t > s\).

We say that an evolution family \(U\) has an exponential dichotomy if there are projections \(P(t)\), \(t \in \mathbb{R}\), being uniformly bounded and strongly continuous in \(t\) and constants \(\delta > 0\) and \(N \geq 1\) such that

(1) \(U(t, s)P(s) = P(t)U(t, s)\),

(2) the restriction \(U_Q(t, s) : Q(s)X \to Q(t)X\) of \(U(t, s)\) is invertible (and we set \(U_Q(t, s) := U(t, s)^{-1}\)),

(3) \(\|U(t, s)P(s)\| \leq Ne^{-\delta(t-s)}\) and \(\|U_Q(s, t)Q(t)\| \leq Ne^{-\delta(t-s)}\)

for \(t \geq s\) and \(t, s \in \mathbb{R}\). Here and below we let \(Q(\cdot) = I - P(\cdot)\). Exponential dichotomy is a classical concept in the study of the long–term behaviour of evolution equations; see e.g., [8, 9, 11, 15, 17, 23, 25]. If \(U\) has an exponential dichotomy, then the operator family

\[
\Gamma(t, s) := \begin{cases} U(t, s)P(s), & t \geq s, t, s \in \mathbb{R}, \\ -U_Q(s, t)Q(s), & t < s, t, s \in \mathbb{R}, \end{cases}
\]

is called the Green’s function corresponding to \(U\) and \(P(\cdot)\). If \(P(t) = I\) for \(t \in \mathbb{R}\), then \(U\) is exponentially stable. The evolution family is called exponentially bounded if there are constants \(M > 0\) and \(\gamma \in \mathbb{R}\) such that \(\|U(t, s)\| \leq Me^{\gamma(t-s)}\) for \(t \geq s\).

In the present work, we study operators \(A(t), t \in \mathbb{R}\), on \(X\) subject to the following hypothesis introduced by P. Acquistapace and B. Terreni in [2].

(H1) There is an \(\omega \geq 0\) such that the operators \(A(t), t \in \mathbb{R}\), satisfy \(\Sigma_\phi \cup \{0\} \subseteq \rho(A(t) - \omega), \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1+|\lambda|}\), and

\[
\|(A(t) - \omega)R(\lambda, A(t) - \omega) - R(\omega, A(t))\| \leq L|t-s|^\mu|\lambda|^{-\nu}
\]

for \(t, s \in \mathbb{R}\), \(\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \phi\}\), and constants \(\phi \in (\frac{\pi}{2}, \pi), L, K \geq 0, \mu, \nu \in (0, 1]\) with \(\mu + \nu > 1\).

This assumption implies that there exists a unique evolution family \(U\) on \(X\) such that \((t, s) \mapsto U(t, s) \in \mathcal{L}(X)\) is continuous for \(t > s\), \(U(\cdot, s) \in C^1((s, \infty), \mathcal{L}(X))\), \(\partial_t U(t, s) = A(t)U(t, s)\), and

\[
\|A(t)^kU(t, s)\| \leq C(t-s)^{-k} \tag{2.1}
\]
for $0 < t - s \leq 1$, $k = 0, 1$, $0 \leq \alpha < \mu$, $x \in D((\omega - A(s))^{\alpha})$, and a constant $C$ depending only on the constants in (H1). Moreover, \( \partial_s^k U(t, s)x = -U(t, s)A(s)x \) for $t > s$ and $x \in D(A(s))$ with $A(s)x \in D(A(s))$. We say that $A(\cdot)$ generates $U$. Note that $U$ is exponentially bounded by (2.1) with $k = 0$.

We further suppose that

(H2) the evolution family $U$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green’s function $\Gamma$.

For the sequel, we need the following estimates, see [5] for the proof.

**Proposition 2.1.** For every $0 \leq \alpha \leq 1$, we have the following assertions:

(i) There is a constant $c(\alpha)$, such that 
\[
\|U(t, s)P(s)x\|_n^\alpha \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|
\]  

(ii) there is a constant $m(\alpha)$, such that 
\[
\|\tilde{U}_Q(s, t)Q_x^m x\|_n^\alpha \leq m(\alpha)e^{-\delta(t-s)}\|x\|
\]  

for every $x \in X$ and $t > s$.

We need to introduce the following definition, and we refer to [21] for more information.

**Definition 2.2** (S. Bochner). (i) A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every sequence $(\sigma_n)_{n \in \mathbb{N}}$ there exists a subsequence $(s_n)_{n \in \mathbb{N}} \subset (\sigma_n)_{n \in \mathbb{N}}$ such that 
\[
\lim_{n, m \to +\infty} f(t + s_n - s_m) = f(t) \quad \text{for each } t \in \mathbb{R}.
\]

This is equivalent to 
\[
g(t) := \lim_{n \to +\infty} f(t + s_n) \quad \text{and} \quad f(t) = \lim_{n \to +\infty} g(t - s_n)
\]

are well defined for each $t \in \mathbb{R}$. We note that $f \in AA(\mathbb{R}, X)$.

(ii) A function $f : \mathbb{R} \times Y \to X$ is said to be almost automorphic if it satisfies the following conditions: $f(\cdot, y)$ is almost automorphic for every $y \in Y$ and $f$ is continuous jointly in $(t, x)$. We note $f \in AA(\mathbb{R} \times Y, X)$.

The function $g$ in the definition above is measurable, but not necessarily continuous. It is well-known that $AA(\mathbb{R}, X)$ is a Banach space under the sup-norm $\|f\|_{AA(\mathbb{R}, X)} = \sup_{t \in \mathbb{R}} \|f(t)\|$.

3. Main results

In this section, we study the existence of almost automorphic solutions to the semilinear evolution equations 
\[
u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R},
\]

where $A(t), t \in \mathbb{R}$, satisfy (H1) and (H2), and the following assumptions hold:

(H3) $R(\omega, A(\cdot)) \in AA(\mathbb{R}, L(X))$;

(H4) there are $0 \leq \alpha < \beta < 1$ such $X_\alpha^t = X_\alpha$, $t \in \mathbb{R}$, $X_\beta^t = X_\beta$, $t \in \mathbb{R}$, with uniform equivalent norms;
(H5) the function $f : \mathbb{R} \times X_\alpha \to X$ belongs to $AA(\mathbb{R} \times X_\alpha, X)$ and is globally small Lipschitzian; i.e., there is a small $K_f > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq K_f \|u - v\|_\alpha \quad \text{for all } t \in \mathbb{R} \text{ and } u, v \in X_\alpha.$$ 

By a mild solution of (3.1) we understand a continuous function $u : \mathbb{R} \to X_\alpha$, which satisfies the following variation of constants formula

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \sigma)f(\sigma, u(\sigma))d\sigma \quad \text{for all } t \geq s, t, s \in \mathbb{R}. \quad (3.2)$$

To achieve the goal of this section, we show some intermediate results. Let us define the Yosida approximations $A_n(t) = nA(t)R(n, A(t))$ of $A(t)$ for $n > \omega$ and $t \in \mathbb{R}$. These operators generate an evolution family $U_n$ on $X$. It has been shown in [19] Lemma 3.1, Proposition 3.3, Corollary 3.4 that assumptions (H1) and (H2) are satisfied by $A_n(\cdot)$ with the same constants for every $n \geq n_0$.

In the following lemma, we show that the Yosida approximations $A_n(\cdot)$ satisfy also assumption (H3) for large $n$. The formulas on the resolvent used in the proof are taken from [19].

**Lemma 3.1.** If (H1) and (H3) hold, then there is a number $n_1 \geq n_0$ such that $R(\omega, A_n(\cdot)) \in AA(\mathbb{R}, L(X))$ for $n \geq n_1$.

**Proof.** Let $(s_l)_{l \in \mathbb{N}}$ be a sequence of real numbers, as $R(\omega, A(\cdot))$ is almost automorphic, there is a subsequence $(s_l)_{l \in \mathbb{N}}$ such that

$$\lim_{t, k \to +\infty} \|R(\omega, A(t + s_l - s_k)) - R(\omega, A(t))\| = 0, \quad (3.3)$$

for each $t \in \mathbb{R}$ If $n \geq n_0$ and $|\arg(\lambda - \omega)| \leq \phi$, we have

$$R(\omega, A_n(t + s_l - s_k)) - R(\omega, A_n(t)) = \frac{n^2}{(\omega + n)^2} \left[ R\left( \frac{\omega n}{\omega + n}, A(t + s_l - s_k) \right) - R\left( \frac{\omega n}{\omega + n}, A(t) \right) \right]$$

$$= \frac{n^2}{(\omega + n)^2} R(\omega, A(t + s_l - s_k)) \left[ 1 - \frac{\omega^2}{\omega + n} R(\omega, A(t + s_l - s_k)) \right]^{-1} \quad (3.4)$$

$$- \frac{n^2}{(\omega + n)^2} R(\omega, A(t)) \left[ 1 - \frac{\omega^2}{\omega + n} R(\omega, A(t)) \right]^{-1}.$$ 

We can also see that

$$\left\| \frac{\omega^2}{\omega + n} R(\omega, A(s)) \right\| \leq \frac{\omega^2}{\omega + n} \frac{K}{1 + n} \leq \frac{\omega K}{n} \leq \frac{1}{2}$$

for $n \geq n_1 := \max\{n_0, 2\omega K\}$ and $s \in \mathbb{R}$. In particular,

$$\left\| \left[ 1 - \frac{\omega^2}{\omega + n} R(\omega, A(s)) \right]^{-1} \right\| \leq 2. \quad (3.5)$$

Hence, (3.4) implies

$$\|R(\omega, A_n(t + s_l - s_k)) - R(\omega, A_n(t))\| \leq 2\|R(\omega, A(t + s_l - s_k)) - R(\omega, A(t))\|$$

$$+ \frac{K}{1 + \omega} \left\| \left[ 1 - \frac{\omega^2}{\omega + n} R(\omega, A(t + s_l - s_k)) \right]^{-1} - \left[ 1 - \frac{\omega^2}{(\omega + n)^2} R(\omega, A(t)) \right]^{-1} \right\|.$$
Employing (3.5) again, we obtain
\[
\left\| \left[1 - \frac{\omega^2}{\omega + n} R(\omega, A(t + s_l - s_k)) \right]^{-1} - \left[1 - \frac{\omega^2}{\omega + n} R(\omega, A(t)) \right]^{-1} \right\|
\leq 4 \left\| \left[1 - \frac{\omega^2}{\omega + n} R(\omega, A(t + s_l - s_k)) \right] - \left[1 - \frac{\omega^2}{\omega + n} R(\omega, A(t)) \right] \right\|
\leq 4\omega \| R(\omega, A(t + s_l - s_k)) - R(\omega, A(t)) \|
\]
Therefore,
\[
\| R(\omega, A_n(t + s_l - s_k)) - R(\omega, A_n(t)) \|
\leq (2 + 4K) \| R(\omega, A(t + s_l - s_k)) - R(\omega, A(t)) \|
\] (3.6)
for \( n \geq n_1 \) and \( t \in \mathbb{R} \). The assertion thus follows from (3.3). \( \square \)

The following technical lemma is also needed.

**Lemma 3.2.** Assume that (H1)–(H3) hold. For every sequence \((s'_t)_{t \in \mathbb{N}} \in \mathbb{R} \), there is a subsequence \((s_t)_{t \in \mathbb{N}} \) such that for every \( \eta > 0 \), and \( t, s \in \mathbb{R} \) there is \( l(\eta, t, s) > 0 \) such that
\[
\| \Gamma_n(t + s_l - s_k, s + s_l - s_k) - \Gamma_n(t, s) \| \leq c n^2 \eta
\] (3.7)
for a large \( n \) and \( l, k \geq l(\eta, t, s) \).

**Proof.** Let a sequence \((s'_t)_{t \in \mathbb{N}} \in \mathbb{R} \). Since \( R(\omega, A(\cdot)) \in AA(\mathbb{R}, X) \), then we can extract a subsequence \((s_t)_{t \in \mathbb{N}} \) such that
\[
\| R(\omega, A(\sigma + s_l - s_k)) - R(\omega, A(\sigma)) \| \to 0, \quad k, l \to \infty,
\] (3.8)
for all \( \sigma \in \mathbb{R} \). As in [19], we have
\[
\Gamma_n(t + s_l - s_k, s + s_l - s_k) - \Gamma_n(t, s)
= \int_{\mathbb{R}} \Gamma_n(t, \sigma)(A_n(\sigma) - \omega)[R(\omega, A_n(\sigma + s_l - s_k)) - R(\omega, A_n(\sigma))]
\times (A_n(\sigma + s_l - s_k) - \omega)\Gamma_n(\sigma + s_l - s_k, s + s_l - s_k) \, d\sigma
\]
for \( s, t \in \mathbb{R} \) and \( l, k, n \in \mathbb{N} \) and large \( n \). This formula, the estimate (3.6) and [19 Corollary 3.4] imply
\[
\| \Gamma_n(t + s_l - s_k, s + s_l - s_k) - \Gamma_n(t, s) \|
\leq c n^2 \int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|} e^{-\frac{3\delta}{4}l|\sigma-s|} \| R(\omega, A_n(\sigma + s_l - s_k)) - R(\omega, A_n(\sigma)) \| \, d\sigma
\leq c n^2 (2 + 4K) \int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|} e^{-\frac{3\delta}{4}l|\sigma-s|} \| R(\omega, A(\sigma + s_l - s_k)) - R(\omega, A(\sigma)) \| \, d\sigma \to 0,
\]
as \( k, l \to \infty \), by (3.8) and the Lebesgue’s Dominated Convergence Theorem. Hence, for \( \eta > 0 \) there is \( l(\eta, t, s) > 0 \) such that
\[
\| \Gamma_n(t + s_l - s_k, s + s_l - s_k) - \Gamma_n(t, s) \| < c n^2 \eta
\]
for large \( n \) and \( l, k \geq l(\eta, t, s) \). \( \square \)

The almost automorphy of the Green function \( \Gamma \) is proved in the next proposition. An analogous result for the almost periodicity is shown in [19].
Proposition 3.3. Assume that (H1)–(H2) hold. Let a sequence \((s_l')_{l \in \mathbb{N}} \in \mathbb{R}\) there is a subsequence \((s_l)_{l \in \mathbb{N}}\) such that for every \(h > 0\)
\[
\| \Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s) \| \to 0, \quad k, l \to \infty
\]
for \(|t - s| \geq h\).

Proof. Let \((s_l')_{l \in \mathbb{N}}\) be a sequence in \(\mathbb{R}\), and consider the subsequence \((s_l)\) given by Lemma 3.2. Let \(\varepsilon > 0\) and \(h > 0\). There is \(t_\varepsilon > h\) such that
\[
\| \Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s) \| \leq \varepsilon
\]
for \(|t - s| \geq t_\varepsilon\) and \(k, l \in \mathbb{N}\). For \(h \leq |t - s| \leq t_\varepsilon\), by [19, Lemma 4.2] we have
\[
\| \Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma_n(t + s_l - s_k, s + s_l - s_k) \| \leq c(t_\varepsilon) n^{-\theta}, \quad (3.9)
\]
\[
\| \Gamma(t, s) - \Gamma_n(t, s) \| \leq c(t_\varepsilon) n^{-\theta} \quad (3.10)
\]
for all \(k, l\) and large \(n\). Let \(n_\varepsilon > 0\) large enough such that \(n^{-\theta} < \frac{\varepsilon}{2c(t_\varepsilon)}\) for \(n \geq n_\varepsilon\). Take \(0 < \eta < \frac{\varepsilon}{2c(t_\varepsilon)}\). Hence, by (3.9), (3.10) and Lemma 3.2 one has
\[
\| \Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s) \| \leq 2c(t_\varepsilon) n_\varepsilon^{-\theta} + cn_\varepsilon^2 \eta \leq \varepsilon
\]
for all \(k, l \geq l(\varepsilon, t, s)\). Consequently, \(\| \Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s) \| \to 0\) as \(l, k \to +\infty\) for \(|t - s| > h > 0\).

Using Proposition 3.3, we show the existence of a unique almost automorphic solution to the inhomogeneous evolution equation
\[
u'(t) = A(t) u(t) + g(t), \quad t \in \mathbb{R}. \quad (3.11)
\]

More precisely, we state the following main result.

Theorem 3.4. Assume (H1)–(H4). Then, for every \(g \in AA(\mathbb{R}, X)\), the unique bounded mild solution \(u(\cdot) = \int_\mathbb{R} \Gamma(\cdot, s) g(s) ds\) of (3.11) belongs to \(AA(\mathbb{R}, X_\alpha)\).

Proof. First we prove that the mild solution \(u\) is almost automorphic in \(X\). Let a sequence \((s_l')_{l \in \mathbb{N}}\) and \(h > 0\). As \(g \in AA(\mathbb{R}, X)\) there exists a subsequence \((s_l)_{l \in \mathbb{N}}\) such that \(\lim_{k, l \to +\infty} \|g(t + s_l - s_k) - g(t)\| \to 0\). Now, we write
\[
u(t + s_l - s_k) - u(t)
\]
\[
= \int_\mathbb{R} \Gamma(t + s_l - s_k, s + s_l - s_k) g(s + s_l - s_k) ds - \int_\mathbb{R} \Gamma(t, s) g(s) ds
\]
\[
= \int_\mathbb{R} \Gamma(t + s_l - s_k, s + s_l - s_k) (g(s + s_l - s_k) - g(s)) ds
\]
\[
+ \int_{|t - s| \geq h} (\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)) g(s) ds
\]
\[
+ \int_{|t - s| \leq h} (\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)) g(s) ds.
\]

For \(\varepsilon > 0\), we deduce from Proposition 3.3 and (H2) that
\[
\|u(t + s_l - s_k) - u(t)\| \leq 2N \int_\mathbb{R} e^{-\delta|t - s|} \|g(s + s_l - s_k) - g(s)\| ds + \left(\frac{1}{2} \varepsilon^2 + 4Nh\right)\|g\|_{\infty}
\]
for \(t \in \mathbb{R}\) and \(k, l \geq l(\varepsilon, h) > 0\). Now, for \(\varepsilon > 0\), take \(h\) small and then \(\varepsilon > 0\) small such that
\[
\|u(t + s_l - s_k) - u(t)\| \leq 2N \int_\mathbb{R} e^{-\delta|t - s|} \|g(s + s_l - s_k) - g(s)\| ds + \frac{\varepsilon}{2}
\]
for $t \in \mathbb{R}$ and $l, k > l(\varepsilon) > 0$. Finally, by the Lebesgue’s Dominated Convergence Theorem, $u$ is almost automorphic in $X$.

Using the reiteration theorem, we obtain $X_\alpha = (X, X_\beta)_\theta$, with $\theta = \alpha/\beta$. By the property of interpolation, we have
\[
\|u(t + s_l - s_k) - u(t)\|_\alpha \\
\leq c(\alpha, \beta)\|u(t + s_l - s_k) - u(t)\|_{\frac{\beta}{\beta-\alpha}}^\frac{\beta}{\alpha} \|u(t + s_l - s_k) - u(t)\|_\beta.
\]

Using estimates in Proposition 2.1, we can show that $u$ is bounded in $X_\beta$. Hence,
\[
\|u(t + s_l - s_k) - u(t)\|_\alpha \leq c(\alpha, \beta)c^\theta \|u(t + s_l - s_k) - u(t)\|_{\frac{\beta}{\beta-\alpha}}^\frac{\beta}{\alpha} \\
\leq c'\|u(t + s_l - s_k) - u(t)\|_{\frac{\beta}{\beta-\alpha}}.
\]

Since $u$ is almost automorphic in $X$, $u(t + s_l - s_k) \to u(t)$, as $l, k \to \infty$, for $t \in \mathbb{R}$, and thus $x \in AA(\mathbb{R}, X_\alpha)$.

As a consequence of Theorem 3.4 and a fixed point technique, we achieve the aim of the paper.

**Theorem 3.5.** Assume that (H1)–(H5) hold. Then (3.1) admits a unique mild solution $u$ in $AA(\mathbb{R}, X_\alpha)$.

**Proof.** Consider $v \in AA(\mathbb{R}, X_\alpha)$ and $f \in AA(\mathbb{R} \times X_\alpha, X)$. Then, by [21] Theorem 2.2.4, p. 21, the function $g(\cdot) := f(\cdot, v(\cdot)) \in AA(\mathbb{R}, X)$, and from Theorem 3.4 the inhomogeneous evolution equation
\[
u'(t) = A(t)u(t) + g(t), \quad t \in \mathbb{R},
\]

admits a unique mild solution $u \in AA(\mathbb{R}, X)$ given by
\[
u(t) = \int_\mathbb{R} \Gamma(t, s)f(s, v(s))ds, \quad t \in \mathbb{R}.
\]

Let the operator $F : AA(\mathbb{R}, X_\alpha) \to AA(\mathbb{R}, X_\alpha)$ be defined by
\[
(Fv)(t) := \int_\mathbb{R} \Gamma(t, s)f(s, v(s))ds \quad \text{for all } t \in \mathbb{R}.
\]

Now we prove that $F$ has a unique fixed point. The estimates (2.2) and (2.3) yield
\[
\|Fx(t) - Fy(t)\|_\alpha \leq c(\alpha)\int_{-\infty}^t e^{-\delta(t-s)}(t-s)^{-\alpha}\|f(s, y(s)) - f(s, x(s))\|ds \\
+ c(\alpha)\int_t^{+\infty} e^{-\delta(t-s)}\|f(s, y(s)) - f(s, x(s))\|ds.
\]

\[
\leq K_f \delta'(\alpha)\|x - y\|_\infty
\]

for all $t \in \mathbb{R}$ and $x, y \in AA(\mathbb{R}, X_\alpha)$. If we assume that $K_f \delta'(\alpha) < 1$, then $F$ has a unique fixed point $u \in AA(\mathbb{R}, X_\alpha)$. Thus $u$ is the unique almost automorphic solution to the equation (3.1).

**Example 3.6.** Consider the parabolic problem
\[
\partial_t u(t, x) = A(t, x, D)u(t, x) + h(t, \nabla u(t, x)), \quad t \in \mathbb{R}, \ x \in \Omega,
\]

\[
B(x, D)u(t, x) = 0, \quad t \in \mathbb{R}, \ x \in \partial\Omega,
\]

(3.13)
on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial \Omega$ of class $C^2$ and outer unit normal vector $\nu(x)$, employing the differential expressions

$$A(t, x, D) = \sum_{k,l} a_{kl}(t, x) \partial_k \partial_l + \sum_k a_k(t, x) \partial_k + a_0(t, x),$$
$$B(x, D) = \sum_k b_k(x) \partial_k + b_0(x).$$

We require that $a_{kl} = a_{lk}$ and $b_k$ are real–valued, $a_{kl}, a_k, a_0 \in C^0_b(\mathbb{R}, C(\overline{\Omega}))$, $b_k, b_0 \in C^1(\partial \Omega)$,

$$\sum_{k,l=1}^n a_{kl}(t, x) \xi_k \xi_l \geq \eta|\xi|^2, \quad \text{and} \quad \sum_{k=1}^n b_k(x) \nu_k(x) \geq \beta$$

for constants $\mu \in (1/2, 1)$, $\beta, \eta > 0$ and all $\xi \in \mathbb{R}^n$, $k, l = 1, \cdots, n$, $t \in \mathbb{R}$, $x \in \overline{\Omega}$ resp. $x \in \partial \Omega$. ($C^0_b$ is the space of bounded, globally Hölder continuous functions.)

We set $X = C^2(\Omega)$,

$$D(A(t)) = \{ u \in \bigcap_{p>1} W^2_p(\Omega) : A(t, \cdot, D)u \in C(\overline{\Omega}) \cap D(\cdot, D)u = 0 \text{ on } \partial \Omega \}$$

for $t \in \mathbb{R}$. It is known that the operators $A(t), t \in \mathbb{R}$, satisfy (H1), see [18], or [24] Exa.2.9]. Thus $A(\cdot)$ generates an evolution family $U(\cdot, \cdot)$ on $X$. Let $\alpha \in (1/2, 1)$ and $p > \frac{n}{2(1-\alpha)}$. Then $X_\alpha = X_\alpha = \{ f \in C^{2\alpha}(\overline{\Omega}) : B(\cdot, D)u = 0 \}$ with uniformly equivalent constants due to [18] Theorem 3.1.30, and $X_\alpha \hookrightarrow W^2_p(\Omega)$. It is clear that the function $f(t, u)(x) := h(t, \nabla u(x))$, $x \in \Omega$, is continuous from $\mathbb{R} \times X_\alpha$ to $X$, and if $h$ is small Lipschitzian and almost automorphic then $f$ is. Under the exponential dichotomy of $U(\cdot, \cdot)$ and almost automorphy of $R(\omega, A(\cdot))$, the parabolic equation [3.13] has a unique almost automorphic solution.

References


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