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ELLIPTIC EQUATIONS WITH MEASURE DATA IN ORLICZ SPACES

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Abstract. This article shows the existence of solutions to the nonlinear elliptic problem $A(u) = f$ in Orlicz-Sobolev spaces with a measure valued right-hand side, where $A(u) = -\operatorname{div} a(x, u, \nabla u)$ is a Leray-Lions operator defined on a subset of $W_0^1 L_M(\Omega)$, with general M.

1. Introduction

Let $M : \mathbb{R} \to \mathbb{R}$ be an N-function; i.e. M is continuous, convex, with $M(u) > 0$ for $u > 0$, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to \infty$ as $t \to \infty$. Equivalently, M admits the representation $M(u) = \int_0^u \phi(t)dt$, where ϕ is the derivative of M, with ϕ non-decreasing, right continuous, $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, and $\phi(t) \to \infty$ as $t \to \infty$.

The N-function $\bar{M}$ conjugate to M is defined by $\bar{M}(v) = \int_0^t \psi(s)ds$, where ψ is given by $\psi(s) = \sup\{t : \phi(t) \le s\}$.

The N-function M is said to satisfy the Δ_2 condition, if for some $k > 0$ and $u_0 > 0$,

$$M(2u) \le kM(u), \quad \forall u \ge u_0.$$

Let P, Q be two N-functions, $P \ll Q$ means that P grows essentially less rapidly than Q; i.e. for each $\varepsilon > 0$, $P(t)/Q(\varepsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} Q^{-1}(t)/P^{-1}(t) = 0$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the segment property. The class $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) consists of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$).

Orlicz spaces $L_M(\Omega)$ are endowed with the Luxemburg norm

$$\|u\|_{(M)} = \inf\big\{\lambda > 0 : \int_\Omega M\big(\frac{|u(x)|}{\lambda}\big)dx \le 1\big\}.$$

The classes $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ of such functions may be given the norm

$$\|u\|_{\Omega,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{(M)}.$$

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These classes will be Banach spaces under this norm. I refer to spaces of the forms $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ as Orlicz-Sobolev spaces. Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ and $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. If M satisfies Δ_2 condition, then $L_M(\Omega) = E_M(\Omega)$ and $W^1L_M(\Omega) = W^1E_M(\Omega)$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of $C_0^\infty(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ closure of $C_0^\infty(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\bar{M}}(\Omega)$ (resp. $W^{-1}E_{\bar{M}}(\Omega)$) denote the space of distributions on which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\bar{M}}(\Omega)$ (resp. $E_{\bar{M}}(\Omega)$). It is a Banach space under the usual quotient norm (see [12]).

If the open set Ω has the segment property, then the space $C_0^\infty(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\bar{M}})$ (cf. [12, 13]).

Let $A(u) = -\operatorname{div} a(x, u, \nabla u)$ be a Leray-Lions operator defined on $W^{1,p}(\Omega)$, $1 < p < \infty$. Boccardo-Gallouet [7] proved the existence of solutions for the Dirichlet problem for equations of the form

$$A(u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where the right hand f is a bounded Radon measure on Ω (i.e. $f \in \mathcal{M}_b(\Omega)$). The function a is supposed to satisfy a polynomial growth condition with respect to u and ∇u.

Benkirane [4, 5] proved the existence of solutions to

$$A(u) + g(x, u, \nabla u) = f, \tag{1.3}$$

in Orlicz-Sobolev spaces where

$$A(u) = -\operatorname{div}(a(x, u, \nabla u)) \tag{1.4}$$

is a Leray-Lions operator defined on $D(A) \subset W_0^1L_M(\Omega)$, g is supposed to satisfy a *natural* growth condition with $f \in W^{-1}E_{\bar{M}}(\Omega)$ and $f \in L^1(\Omega)$, respectively, but the result is restricted to N-functions M satisfying a Δ_2 condition. Elmahi extend the results of [4, 5] to general N-functions (i.e. without assuming a Δ_2-condition on M) in [8, 9], respectively.

The purpose of this paper is to solve (1.1) when f is a bounded Radon measure, and the Leray-Lions operator A in (1.4) is defined on $D(A) \subset W_0^1L_M(\Omega)$, with general M. We show that the solutions to (1.1) belong to the Orlicz-Sobolev space $W_0^1L_B(\Omega)$ for any $B \in \mathcal{P}_M$, where $\mathcal{P}_M$ is a special class of N-function (see below). Specific examples to which our results apply include the following:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu \quad \text{in } \Omega,$$
$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u \log^\beta(1 + |\nabla u|)) = \mu \quad \text{in } \Omega$$
$$-\operatorname{div} \frac{M(|\nabla u|)\nabla u}{|\nabla u|^2} = \mu \quad \text{in}\Omega$$

where $p > 1$ and μ is a given Radon measure on Ω.

For some classical and recent results on elliptic and parabolic problems in Orlicz spaces, I refer the reader to [2, 3, 6, 10, 11, 12, 14, 16, 18].

2. Preliminaries

We define a subset of N-functions as follows.

$$\mathcal{P}_M = \Big\{ B : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is an } N\text{-function, } B''/B' \le M''/M' \\ \text{and } \int_0^1 B \circ H^{-1}(1/t^{1-1/N})dt < \infty \Big\}$$

where $H(r) = M(r)/r$. Assume that

$$\mathcal{P}_M \neq \emptyset \tag{2.1}$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the segment property, M, P be two N-functions such that $P \ll M$, $\bar{M}, \bar{P}$ be the complementary functions of M, P, respectively, $A : D(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\bar{M}}(\Omega)$ be a mapping given by $A(u) = -\operatorname{div} a(x, u, \nabla u)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$:

$$|a(x, s, \xi)| \le \beta M(|\xi|)/|\xi| \tag{2.2}$$

$$[a(x, s, \xi) - a(x, s, \eta)][\xi - \eta] > 0 \tag{2.3}$$

$$a(x, s, \xi)\xi \ge \alpha M(|\xi|) \tag{2.4}$$

where $\alpha, \beta, \gamma > 0$.

Furthermore, assume that there exists $D \in \mathcal{P}_M$ such that

$$D \circ H^{-1} \text{ is an } N\text{-function.} \tag{2.5}$$

Set $T_k(s) = \max(-k, \min(k, s))$, $\forall s \in \mathbb{R}$, for all $k \ge 0$. Define by $\mathcal{M}_b(\Omega)$ as the set of all bounded Radon measure defined on Ω and by $T_0^{1,M}(\Omega)$ as the set of measurable functions $\Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^1 L_M(\Omega) \cap D(A)$.

Assume that $f \in \mathcal{M}_b(\Omega)$, and consider the following nonlinear elliptic problem with Dirichlet boundary

$$A(u) = f \quad \text{in } \Omega. \tag{2.6}$$

The following lemmas can be found in [4].

Lemma 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with$F(0) = 0$. Let M be an N-function, $u \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$ (resp.* $W^1 E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' *is finite, then*

$$\frac{\partial (F \circ u)}{\partial x_i} = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & a.e. \ in \ \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \ in \ \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. I suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1 L_M(\Omega) \to W^1 L_M(\Omega)$ is sequentially continuous with respect to the weak∗ topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$.*

3. Existence theorem

Theorem 3.1. *Assume that* (2.1)-(2.5) *hold and $f \in \mathcal{M}_b(\Omega)$. Then there exists at least one weak solution of the problem*

$$u \in T_0^{1,M}(\Omega) \cap W_0^1 L_B(\Omega), \quad \forall B \in \mathcal{P}_M$$

$$\int_\Omega a(x, u, \nabla u) \nabla \phi dx = \langle f, \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega)$$

Proof. Denote $V = W_0^1 L_M(\Omega)$. (1) Consider the approximate equations

$$\begin{gathered} u_n \in V \\ -\operatorname{div} a(x, u_n, \nabla u_n) = f_n \end{gathered} \tag{3.1}$$

where f_n is a smooth function which converges to f in the distributional sense that such that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{\mathcal{M}_b(\Omega)}$. By [4, Theorem 3.1] or [8], there exists at least one solution $\{u_n\}$ to (3.1).

For $k > 0$, by taking $T_k(u_n)$ as test function in (3.1), one has

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq Ck.$$

In view of (2.4), we get

$$\int_\Omega M(|\nabla T_k(u_n)|) dx \leq Ck. \tag{3.2}$$

Hence $\nabla T_k(u_n)$ is bounded in $(L_M(\Omega))^N$. By [9] there exists u such that $u_n \to u$ almost everywhere in Ω and

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma\,(\Pi L_M, \Pi E_{\bar{M}}). \tag{3.3}$$

For $t > 0$, by taking $T_h(u_n - T_t(u_n))$ as test function, we deduce that

$$\int_{t < |u_n| \leq t+h} a(x, u_n, \nabla u_n) \nabla u_n dx \leq h \|f\|_{M_b(\Omega)}$$

which gives

$$\frac{1}{h} \int_{t < |u_n| \leq t+h} M(|\nabla u_n|) dx \leq \|f\|_{M_b(\Omega)}$$

and by letting $h \to 0$,

$$-\frac{d}{dt} \int_{|u_n| > t} M(|\nabla u_n|) dx \leq \|f\|_{M_b(\Omega)}.$$

Let now $B \in \mathcal{P}_M$. Following the lines of [17], it is easy to deduce that

$$\int_\Omega B(|\nabla u_n|) dx \leq C, \quad \forall n. \tag{3.4}$$

We shall show that $a(x, T_k(u_n), \nabla T_k(u_n))$ is bounded in $(L_{\bar{M}}(\Omega))^N$. Let $\varphi \in (E_M(\Omega))^N$ with $\|\varphi\|_{(M)} = 1$. By (2.2) and Young inequality, one has

$$\begin{aligned} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \varphi dx &\leq \beta \int_\Omega \bar{M}\Big(\frac{M(|\nabla T_k(u_n)|)}{|\nabla T_k(u_n)|}\Big) dx + \beta \int_\Omega M(|\varphi|) dx \\ &\leq \beta \int_\Omega M(|\nabla T_k(u_n)|) dx + \beta \end{aligned}$$

This last inequality is deduced from $\bar{M}(M(u)/u) \leq M(u)$, for all $u > 0$, and $\int_\Omega M(|\varphi|) dx \leq 1$. In view of (3.2),

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \varphi dx \leq Ck + \beta,$$

which implies $\{a(x, T_k(u_n), \nabla T_k(u_n))\}_n$ being a bounded sequence in $(L_{\bar{M}}(\Omega))^N$.

(2) For the rest of this article, χ_r, χ_s and $\chi_{j,s}$ will denoted respectively the characteristic functions of the sets $\Omega_r = \{x \in \Omega; |\nabla T_k(u(x))| \leq r\}$, $\Omega_s = \{x \in \Omega; |\nabla T_k(u(x))| \leq s\}$ and $\Omega_{j,s} = \{x \in \Omega; |\nabla T_k(v_j(x))| \leq s\}$. For the sake of

simplicity, I will write only $\varepsilon(n,j,s)$ to mean all quantities (possibly different) such that

$$\lim_{s\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon(n,j,s)=0.$$

Take a sequence $(v_j)\subset\mathcal{D}(\Omega)$ which converges to u for the modular convergence in V (cf. [13]). Taking $T_\eta(u_n-T_k(v_j))$ as test function in (3.1), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))dx\le C\eta \tag{3.5}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))dx\\
&=\int_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|\le k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))dx\\
&\quad+\int_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla T_k(v_j))dx\\
&=\int_{\{|T_ku_n-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))dx\\
&\quad+\int_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla u_n dx\\
&\quad-\int_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)dx
\end{aligned}$$

By (2.4) the second term of the right-hand side satisfies

$$\int_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla u_n dx\ge 0.$$

Since $a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\bar M}(\Omega))^N$, there exists some $h_{k+\eta}\in(L_{\bar M}(\Omega))^N$ such that

$$a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\rightharpoonup h_{k+\eta}$$

weakly in $(L_{\bar M}(\Omega))^N$ for $\sigma(\Pi L_{\bar M},\Pi E_M)$. Consequently the third term of the right-hand side satisfies

$$\begin{aligned}
&\int_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)dx\\
&=\int_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|>k\}} a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\nabla T_k(v_j)dx\\
&=\int_{\{|u-T_k(v_j)|\le\eta\}\cap\{|u|>k\}} h_{k+\eta}\nabla T_k(v_j)dx+\varepsilon(n)
\end{aligned}$$

since

$$\nabla T_k(v_j)\chi_{\{|u_n-T_k(v_j)|\le\eta\}\cap\{|u_n|>k\}}\to\nabla T_k(v_j)\chi_{\{|u-T_k(v_j)|\le\eta\}\cap\{|u|>k\}}$$

strongly in $(E_M(\Omega))^N$ as $n\to\infty$. Hence

$$\int_{\{|T_ku_n-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]dx$$

$$\leq C\eta + \varepsilon(n) + \int_{\{|u-T_k(v_j)|\leq\eta\}\cap\{|u|>k\}} h_{k+\eta}\nabla T_k(v_j)dx$$

Let $0<\theta<1$. Define

$$\Phi_{n,k} = [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)].$$

For $r>0$, I have

$$\begin{aligned}
0 &\leq \int_{\Omega_r} \{[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\}^{\theta}dx \\
&= \int_{\Omega_r} \Phi_{n,k}^{\theta}\chi_{\{|T_k(u_n)-T_k(v_j)|>\eta\}}dx + \int_{\Omega_r} \Phi_{n,k}^{\theta}\chi_{\{|T_k(u_n)-T_k(v_j)|\leq\eta\}}dx
\end{aligned}$$

Using the Hölder Inequality (with exponents $1/\theta$ and $1/(1-\theta)$), the first term of the right-side hand is less than

$$\Big(\int_{\Omega_r}\Phi_{n,k}dx\Big)^{\theta}\Big(\int_{\Omega_r}\chi_{\{|T_k(u_n)-T_k(v_j)|>\eta\}}dx\Big)^{1-\theta}.$$

Noting that

$$\begin{aligned}
&\int_{\Omega_r}\Phi_{n,k}dx \\
&= \int_{\Omega_r} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx - \int_{\Omega_r} a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u_n)dx \\
&\quad - \int_{\Omega_r} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)dx + \int_{\Omega_r} a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)dx \\
&\leq Ck + \beta\int_{\Omega_r}\bar{M}\Big(\frac{M(|\nabla T_k(u)|)}{|\nabla T_k(u)|}\Big)dx + \beta\int_{\Omega_r} M(|\nabla T_k(u_n)|)dx \\
&\quad + \beta\int_{\Omega_r}\bar{M}\Big(\frac{M(|\nabla T_k(u_n)|)}{|\nabla T_k(u_n)|}\Big)dx + \beta\int_{\Omega_r} M(|\nabla T_k(u)|)dx \\
&\quad + \beta\int_{\Omega_r} M(|\nabla T_k(u)|)dx \\
&\leq Ck + \beta\int_{\Omega_r} M(|\nabla T_k(u)|)dx + \beta\int_{\Omega} M(|\nabla T_k(u_n)|)dx \\
&\quad + \beta\int_{\Omega} M(|\nabla T_k(u_n)|)dx + \beta\int_{\Omega_r} M(|\nabla T_k(u)|)dx + \beta\int_{\Omega_r} M(|\nabla T_k(u)|)dx \\
&\leq (2\beta+1)Ck + 3M(r)\operatorname{meas}\Omega
\end{aligned}$$

it follows that

$$\int_{\Omega_r}\Phi_{n,k}^{\theta}\chi_{\{|T_k(u_n)-T_k(v_j)|>\eta\}}dx \leq \tilde{C}(\operatorname{meas}\{|T_k(u_n)-T_k(v_j)|>\eta\})^{1-\theta},$$

where $\tilde{C} = [(2\beta+1)Ck + 3M(r)\operatorname{meas}\Omega]^{\theta}$.

Using the Hölder Inequality (with exponents $1/\theta$ and $1/(1-\theta)$),

$$\begin{aligned}
&\int_{\Omega_r}\Phi_{n,k}^{\theta}\chi_{\{|T_k(u_n)-T_k(v_j)|\leq\eta\}}dx \\
&\leq \Big(\int_{\Omega_r}\Phi_{n,k}\chi_{\{|T_k(u_n)-T_k(v_j)|\leq\eta\}}dx\Big)^{\theta}\Big(\int_{\Omega_r}dx\Big)^{1-\theta}
\end{aligned}$$

$$\le \Big(\int_{\Omega_r} \Phi_{n,k}\chi_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}}dx\Big)^{\theta}\big(\operatorname{meas}\Omega\big)^{1-\theta}$$

Hence

$$\begin{aligned}
0 &\le \int_{\Omega_r} \{[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\}^{\theta}dx \\
&\le \tilde{C}\big(\operatorname{meas}\{|T_k(u_n)-T_k(v_j)|>\eta\}\big)^{1-\theta} \\
&\quad + \Big(\int_{\Omega_r} \Phi_{n,k}\chi_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}}dx\Big)^{\theta}\big(\operatorname{meas}\Omega\big)^{1-\theta} \\
&= \tilde{C}\big(\operatorname{meas}\{|T_k(u_n)-T_k(v_j)|>\eta\}\big)^{1-\theta} \\
&\quad + \Big(\int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big] \\
&\quad \times \big[\nabla T_k(u_n)-\nabla T_k(u)\big]dx\Big)^{\theta}\big(\operatorname{meas}\Omega\big)^{1-\theta}
\end{aligned}$$

For each $s\ge r$ one has

$$\begin{aligned}
0 &\le \int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big] \\
&\quad \times \big[\nabla T_k(u_n)-\nabla T_k(u)\big]dx \\
&\le \int_{\Omega_s\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big] \\
&\quad \times \big[\nabla T_k(u_n)-\nabla T_k(u)\big]dx \\
&= \int_{\Omega_s\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big] \\
&\quad \times \big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big]dx \\
&\le \int_{\Omega\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big] \\
&\quad \times \big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\big]dx \\
&= \int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_{j,s})\big] \\
&\quad \times \big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_{j,s}\big]dx \\
&\quad + \int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\big[\nabla T_k(v_j)\chi_{j,s}-\nabla T_k(u)\chi_s\big]dx \\
&\quad + \int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(v_j)\chi_{j,s}) \\
&\quad - a(x,T_k(u_n),\nabla T_k(u)\chi_s)\big]\nabla T_k(u_n)dx \\
&\quad - \int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(v_j)\chi_{j,s})\nabla T_k(v_j)\chi_{j,s}dx \\
&\quad + \int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s dx \\
&= I_1(n,j,s)+I_2(n,j,s)+I_3(n,j,s)+I_4(n,j,s)+I_5(n,j,s)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]dx\\
&=\int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_{j,s})\big]\\
&\quad\times\big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_{j,s}\big]dx\\
&\quad+\int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(v_j)\chi_{j,s})\big[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_{j,s}\big]dx\\
&\quad-\int_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_{\{|\nabla T_k(v_j)|>s\}}dx
\end{aligned}$$

The second term of the right-hand side tends to

$$\int_{\{|T_k(u)-T_k(v_j)|\le\eta\}} a(x,T_k(u),\nabla T_k(u)\chi_s)[\nabla T_k(u)-\nabla T_k(v_j)\chi_s]dx$$

since $a(x,T_k(u_n),\nabla T_k(u)\chi_s)\chi_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}}$ tends to

$$a(x,T_k(u),\nabla T_k(u)\chi_s)\chi_{\{|T_k(u)-T_k(v_j)|\le\eta\}}$$

in $(E_{\bar M}(\Omega))^N$ while $\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s$ tends weakly to $\nabla T_k(u)-\nabla T_k(v_j)\chi_s$ in $(L_M(\Omega))^N$ for $\sigma(\Pi L_M,\Pi E_{\bar M})$.

Since $a(x,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\bar M}(\Omega))^N$ there exists some $h_k\in(L_{\bar M}(\Omega))^N$ such that (for a subsequence still denoted by u_n)

$$a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup h_k \quad \text{weakly in } (L_{\bar M}(\Omega))^N \text{ for } \sigma(\Pi L_{\bar M},\Pi E_M).$$

In view of the fact that $\nabla T_k(v_j)\chi_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}}\to\nabla T_k(v_j)\chi_{\{|T_k(u)-T_k(v_j)|\le\eta\}}$ strongly in $(E_M(\Omega))^N$ as $n\to\infty$ the third term of the right-hand side tends to

$$-\int_{\{|T_k(u)-T_k(v_j)|\le\eta\}} h_k\nabla T_k(v_j)\chi_{\{|\nabla T_k(v_j)|>s\}}dx.$$

Hence in view of the modular convergence of (v_j) in V, one has

$$\begin{aligned}
I_1(n,j,s)&\le C\eta+\varepsilon(n)+\int_{\{|u-T_k(v_j)|\le\eta\}\cap\{|u|>k\}} h_{k+\eta}\nabla T_k(v_j)dx\\
&\quad+\int_{\{|T_k(u)-T_k(v_j)|\le\eta\}} h_k\nabla T_k(v_j)\chi_{\{|\nabla T_k(v_j)|>s\}}dx\\
&\quad-\int_{\{|T_k(u)-T_k(v_j)|\le\eta\}} a(x,T_k(u),\nabla T_k(u)\chi_s)[\nabla T_k(u)-\nabla T_k(v_j)\chi_s]dx\\
&=C\eta+\varepsilon(n)+\varepsilon(j)+\int_\Omega h_k\nabla T_k(u)\chi_{\{|\nabla T_k(u)|>s\}}dx\\
&\quad-\int_\Omega a(x,T_k(u),0)\chi_{\{|\nabla T_k(u)|>s\}}dx
\end{aligned}$$

Therefore,

$$I_1(n,j,s)=C\eta+\varepsilon(n,j,s) \tag{3.6}$$

For what concerns I_2, by letting $n\to\infty$, one has

$$I_2(n,j,s)=\int_{\{|T_k(u)-T_k(v_j)|\le\eta\}} h_k[\nabla T_k(v_j)\chi_{j,s}-\nabla T_k(u)\chi_s]dx+\varepsilon(n)$$

since

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } (L_{\bar{M}})^N \text{ for } \sigma(\Pi L_{\bar{M}},\Pi E_M)$$

while $\chi_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}}[\nabla T_k(v_j)\chi_{j,s}-\nabla T_k(u)\chi_s]$ approaches

$$\chi_{\{|T_k(u)-T_k(v_j)|\le\eta\}}[\nabla T_k(v_j)\chi_{j,s}-\nabla T_k(u)\chi_s]$$

strongly in $(E_M)^N$. By letting $j\to\infty$, and using Lebesgue theorem, then

$$I_2(n,j,s)=\varepsilon(n,j). \tag{3.7}$$

Similar tools as above, give

$$I_3(n,j,s)=-\int_\Omega a(x,T_k(u),\nabla T_k(u)\chi_s)\nabla T_k(u)\chi_s dx+\varepsilon(n,j) \tag{3.8}$$

Combining (3.6), (3.7), and (3.8), we have

$$\begin{aligned}&\int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}}\big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\big]\\&\times\big[\nabla T_k(u_n)-\nabla T_k(u)\big]dx\\&\le\varepsilon(n,j,s).\end{aligned}$$

Therefore,

$$\begin{aligned}0&\le\int_{\Omega_r}\{[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\}^\theta dx\\&\le\tilde{C}(\operatorname{meas}\{|T_k(u_n)-T_k(v_j)|>\eta\})^{1-\theta}+(\operatorname{meas}\Omega)^{1-\theta}(\varepsilon(n,j,s))^\theta\end{aligned}$$

Which yields, by passing to the limit superior over n,j,s and η,

$$\begin{aligned}&\lim_{n\to\infty}\int_{\Omega_r}\big\{\big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\big]\\&\times\big[\nabla T_k(u_n)-\nabla T_k(u)\big]\big\}^\theta dx=0\,.\end{aligned}$$

Thus, passing to a subsequence if necessary, $\nabla u_n\to\nabla u$ a.e. in Ω_r, and since r is arbitrary,

$$\nabla u_n\to\nabla u \quad \text{a.e. in } \Omega.$$

By (2.2) and (2.5),

$$\int_\Omega D\circ H^{-1}\Big(\frac{|a(x,u_n,\nabla u_n)|}{\beta}\Big)dx\le\int_\Omega D(|\nabla u_n|)dx\le C$$

Hence

$$a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u) \quad \text{weakly for } \sigma(\Pi L_{D\circ H^{-1}}\Pi E_{\overline{D\circ H^{-1}}}).$$

Going back to approximate equations (3.1), and using $\phi\in\mathcal{D}(\Omega)$ as the test function, one has

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\phi dx=\langle f_n,\phi\rangle$$

in which I can pass to the limit. This completes the proof. $\square$

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