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SOME NONLINEAR INTEGRAL INEQUALITIES ARISING IN DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to obtain estimates for functions satisfying some nonlinear integral inequalities. Using ideas from Pachpatte [3], we generalize the estimates presented in [2, 4].

1. Introduction and main results

Integral inequalities are a necessary tools in the study of properties of the solutions of linear and nonlinear differential equations, such as boundness, stability, uniqueness, etc. This justifies the intensive investigation on integral inequalities; see for example [1, 5, 6]. The aim of this paper is to establish some new generalizations of integral inequalities that have a wide applications in the study of differential equations. More precisely, using some ideas from [3], we give further generalizations of the results presented in [2, 4].

We begin by giving some material necessary for our study. We denote by $\mathbb{R}$ the set of real numbers, and by $\mathbb{R}_+$ the nonnegative real numbers

Lemma 1.1. *For $x \in \mathbb{R}_+$, $y \in \mathbb{R}_+$, $1/p + 1/q = 1$, we have $x^{1/p}y^{1/q} \le x/p + y/q$.*

Now we state the main results of this work.

Theorem 1.2. *Let u, a, b, g and h be real valued nonnegative continuous functions defined on $\mathbb{R}_+$, p, r, q be real non negative constants. Assume that the functions*

$$\frac{a(t)+p/r}{b(t)}, \quad \frac{a(t)+r/p}{b(t)}, \quad \frac{a(t)+\min(r/p,q/p)}{b(t)}$$

are nondecreasing and that

$$u^p(t) \le a(t) + b(t)\int_0^t [g(s)u^q(s) + h(s)u^r(s)]ds. \tag{1.1}$$

(1) If $0 < r < p < q$, then

$$u(t) \le (a(t)+\frac{p}{r})^{1/p}\Big(1-(\frac{q}{p}-1)\int_0^t b(s)(g(s)+\frac{r}{p}h(s))\big(a(s)+\frac{p}{r}\big)^{\frac{q}{p}-1}ds\Big)^{\frac{1}{p-q}} \tag{1.2}$$

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for $t \le \beta_{p,q,r}$*, where*

$$\beta_{p,q,r} = \sup\big\{t \in \mathbb{R}_+ : (\frac{q}{p} - 1)\int_0^t b(s)(g(s) + \frac{r}{p}h(s))\big(a(s) + \frac{p}{r}\big)^{\frac{q}{p}-1} ds < 1\big\}.$$

(2) If $0 < p < r < q$*, then*

$$u(t) \le (a(t) + \frac{r}{p})^{1/p}\Big(1 - (\frac{q}{p} - 1)\int_0^t b(s)(g(s) + h(s))\big(a(s) + \frac{r}{p}\big)^{\frac{q}{p}-1} ds\Big)^{\frac{1}{p-q}}$$

for $t \le \beta_{p,q,r}$*, where*

$$\beta_{p,q,r} = \sup\big\{t \in R_+ : (\frac{q}{p} - 1)\int_0^t b(s)(g(s) + h(s))\big(a(s) + \frac{r}{p}\big)^{\frac{q}{p}-1} ds < 1\big\}.$$

(3) If $0 < p < q$ *and* $p < r$*, then*

$$\begin{aligned} u(t) \le (a(t) + \min(\frac{r}{p}, \frac{q}{p}))^{1/p}\Big(1 - (\max(\frac{q}{p}, \frac{r}{p}) - 1)\int_0^t b(s)(g(s) + h(s))\big(a(s) \\ + \min(\frac{r}{p}, \frac{q}{p})\big)^{\max(\frac{q}{p}, \frac{r}{p})-1} ds\Big)^{\frac{1}{p(1-\max\{q/p, r/p\})}} \end{aligned}$$

for $t \le \beta_{p,q,r}$*, where*

$$\begin{aligned} \beta_{p,q,r} = \sup\big\{t \in R_+ : \Big(\max(\frac{q}{p}, \frac{r}{p}) - 1\Big)\int_0^t b(s)(g(s) + h(s))\big(a(s) \\ + \min(\frac{r}{p}, \frac{q}{p})\big)^{\max(\frac{q}{p}, \frac{r}{p})-1} ds < 1\big\}. \end{aligned}$$

Theorem 1.3. *Suppose that the hypothesis of Theorem 1.2 hold and the function* $b(t)$ *is decreasing. Let* c *be a real valued nonnegative continuous and nondecreasing function for* $t \in \mathbb{R}_+$*. Also assume that*

$$u^p(t) \le c^p(t) + b(t)\int_0^t [g(s)u^q(s) + h(s)u^r(s)] ds\,. \tag{1.3}$$

(1) If $0 < r < p < q$*, then*

$$u(t) \le c(t)(1 + \frac{p}{r})^{1/p}\Big\{1 - (\frac{q}{p} - 1)\int_0^t (1 + \frac{p}{r})^{\frac{q}{p}-1} b(s)K(s) ds\Big\}^{\frac{1}{p-q}} \tag{1.4}$$

for $t \le \beta_{p,q,r}$*, where*

$$\beta_{p,q,r} = \sup\big\{t \in R_+ : (\frac{q}{p} - 1)\int_0^t (1 + \frac{p}{r})^{\frac{q}{p}-1} b(s)K(s) ds < 1\big\},$$
$$K(s) = g(s)c(s)^{q-p} + \frac{r}{p}h(s)c(s)^{r-p}.$$

(2) If $t \in \mathbb{R}+$ *and* $0 < p < r < q$*, then*

$$u(t) \le c(t)(1 + \frac{r}{p})^{1/p}\Big(1 - (\frac{q}{p} - 1)\int_0^t b(s)K(s)(1 + \frac{r}{p})^{\frac{q}{p}-1} ds\Big)^{\frac{1}{p-q}}$$

for $t \le \beta_{p,q,r}$*, where*

$$\beta_{p,q,r} = \sup\big\{t \in R_+ : (\frac{q}{p} - 1)\int_0^t b(s)K(s)(1 + \frac{r}{p})^{\frac{q}{p}-1} ds < 1\big\},$$
$$K(s) = (g(s)c(s)^{q-p} + h(s)c(s)^{r-p}).$$

(3) If $t \in \mathbb{R}+$ and $0 < p < q$, $p < r$, then

$$u(t) \le (1+\min(\frac{r}{p},\frac{q}{p}))^{1/p}\Big(1-(\max(\frac{q}{p},\frac{r}{p})-1)\int_0^t b(s)K(s) \times (1+\min(\frac{r}{p},\frac{q}{p}))^{\max(\frac{q}{p},\frac{r}{p})-1}ds\Big)^{\frac{1}{p(1-\max(\frac{q}{p},\frac{r}{p}))}}$$

for $t \le \beta_{p,q,r}$, where

$$\beta_{p,q,r} = \sup\Big\{t \in R_+ : \Big(\max(\frac{q}{p},\frac{r}{p})-1\Big)\int_0^t b(s)K(s)(1+\min(\frac{r}{p},\frac{q}{p})\Big)^{\max(q/p,r/p)-1}ds < 1$$

and $K(s) = (g(s)c(s)^{q-p} + h(s)c(s)^{r-p})$.

Note that in Theorems 1.2 and 1.3, we have studied the case $p < q$. For the case $p > q$, similar results are given in [2].

Proof of Theorem 1.2. (1) Define a function

$$v(t) = \int_0^t [g(s)u^q(s) + h(s)u^r(s)]\,ds\,.$$

then from inequality (1.1) and Lemma 1.1, we deduce that

$$u^q(t) \le (a(t)+b(t)v(t))^{q/p}, \tag{1.5}$$

$$u^r(t) \le (a(t)+b(t)v(t))^{r/p}, \tag{1.6}$$

$$u^r(t) \le \frac{r}{p}(a(t)+b(t)v(t)) + \frac{p-r}{p}, \tag{1.7}$$

$$u^r(t) \le \frac{r}{p}(a(t)+b(t)v(t)+\frac{p-r}{r}), \tag{1.8}$$

$$u^r(t) \le \frac{r}{p}(a(t)+b(t)v(t)+\frac{p}{r}). \tag{1.9}$$

Since $\frac{q}{p} > 1$, which implies

$$v'(t) \le \big[g(t)+\frac{r}{p}h(t)\big]\big[a(t)+b(t)v(t)+\frac{p}{r}\big]^{q/p}. \tag{1.10}$$

Taking into account that the function $\frac{a(t)+\frac{p}{r}}{b(t)}$ is nondecreasing for $0 \le t \le \tau$, we have

$$v'(t) \le M(t)(\frac{a(\tau)+\frac{p}{r}}{b(\tau)} + v(t)),$$

where

$$M(t) = b(t)(g(t)+\frac{r}{p}h(t))(a(t)+b(t)v(t)+\frac{p}{r})^{\frac{q}{p}-1},$$

consequently

$$v(t) + \frac{a(\tau)+\frac{p}{r}}{b(\tau)} \le \frac{a(\tau)+\frac{p}{r}}{b(\tau)}\exp\int_0^t M(s)ds.$$

For $\tau = t$, we can see that

$$a(t)+b(t)v(t)+\frac{p}{r} \le (a(t)+\frac{p}{r})\exp\int_0^t M(s)ds, \tag{1.11}$$

then the function $M(t)$ can be estimated as

$$M(t) \leq b(t)(g(t) + \frac{r}{p}h(t))(a(t) + \frac{p}{r})^{\frac{q}{p}-1}. \exp \int_0^t (\frac{q}{p} - 1)M(s)ds. \tag{1.12}$$

Let

$$L(t) = (\frac{q}{p} - 1)M(t). \tag{1.13}$$

Now we estimate the expression $L(t)\exp(-\int_0^t L(s)ds)$ by using (1.12) to obtain

$$L(t)\exp(\int_0^t -L(s)ds) \leq (\frac{q}{p} - 1)b(t)(g(t) + \frac{r}{p}h(t))(a(t) + \frac{p}{r})^{\frac{q}{p}-1}.$$

Observing that

$$\begin{aligned} L(t)\exp(\int_0^t -L(s)ds) &= \frac{d}{dt}(-\exp(\int_0^t -L(s)ds)), \\ &\leq (\frac{q}{p} - 1)b(t)(g(t) + \frac{r}{p}h(t))(a(t) + \frac{p}{r})^{\frac{q}{p}-1}. \end{aligned}$$

Then integrate from 0 to t to obtain

$$(1 - \exp \int_0^t -L(s)ds) \leq \int_0^t (\frac{q}{p} - 1)b(s)(g(s) + \frac{r}{p}h(s))\big(a(s) + \frac{p}{r}\big)^{\frac{q}{p}-1}ds.$$

Replacing $L(t)$ by its value in (1.13), we obtain

$$(1 - \exp \int_0^t (1 - \frac{q}{p})M(s)ds) \leq \int_0^t (\frac{q}{p} - 1)b(s)(g(s) + \frac{r}{p}h(s))\big(a(s) + \frac{p}{r}\big)^{\frac{q}{p}-1}ds,$$

then

$$\exp \int_0^t M(s)ds \leq \Big\{1 - \Big[\int_0^t (\frac{q}{p} - 1)b(s)(g(s) + \frac{r}{p}h(s))\big(a(s) + \frac{p}{r}\big)^{\frac{q}{p}-1}ds\Big]\Big\}^{\frac{p}{p-q}}.$$

Using this inequality, (1.11), and (1.1) we obtain (1.2). This completes the proof of stament (1).

(2) for $t \in \mathbb{R}+$ and $0 < p < r < q$, from (1.5) we have

$$\begin{gathered} u^q(t) \leq \big(a(t) + b(t)v(t) + \frac{r}{p}\big)^{q/p}, \\ v'(t) \leq (g(t) + h(t))\big(a(t) + b(t)v(t) + \frac{r}{p}\big)^{q/p}. \end{gathered}$$

Since $\frac{a(t)+\frac{r}{p}}{b(t)}$ is nondecreasing for $0 \leq t \leq \tau$,

$$v'(t) \leq M(t)(\frac{a(\tau) + \frac{r}{p}}{b(\tau)} + v(t)),$$

where

$$M(t) = b(t)(g(t) + h(t))\big(a(t) + b(t)v(t) + \frac{r}{p}\big)^{\frac{q}{p}-1}.$$

By the same method as in the proof of the first part, we have

$$u(t) \leq (a(t) + \frac{r}{p})^{1/p}\Big(1 - (\frac{q}{p} - 1)\int_0^t b(s)(g(s) + h(s))\big(a(s) + \frac{r}{p}\big)^{\frac{q}{p}-1}ds\Big)^{\frac{1}{p-q}},$$

where

$$\beta_{p,q,r} = \sup\big\{t \in R_+ : (\frac{q}{p} - 1)\int_0^t b(s)(g(s) + h(s))\big(a(s) + \frac{r}{p}\big)^{\frac{q}{p}-1}ds < 1\big\}.$$

(3) For $t \in \mathbb{R}+$ and $p < r$, $p < q$, we have

$$u^q(t) \le (a(t) + b(t)v(t) + \min(\frac{r}{p}, \frac{q}{p}))^{\max(r/p, q/p)},$$

which gives

$$v'(t) \le (g(t) + h(t))(a(t) + b(t)v(t) + \min(\frac{r}{p}, \frac{q}{p}))^{\max(r/p, q/p)}$$

$$v'(t) \le M(t)(\frac{a(\tau) + \min(\frac{r}{p}, \frac{q}{p})}{b(\tau)} + v(t)),$$

where

$$M(t) = b(t)(g(t) + h(t))(a(t) + b(t)v(t) + \min(\frac{r}{p}, \frac{q}{p}))^{\max(\frac{r}{p}, \frac{q}{p}) - 1}.$$

Using the proof of the first part of Theorem 1.2, we get the desired result. □

Proof of Theorem 1.3. Since $c(t)$ is nonnegative, continuous and nondecreasing, it follows that (1.3) can be written as

$$(\frac{u(t)}{c(t)})^p \le 1 + b(t) \int_0^t \big[g(s)(\frac{u(s)}{c(s)})^q . c(s)^{q-p} + h(s)(\frac{u(s)}{c(s)})^r c(s)^{r-p}\big] ds. \tag{1.14}$$

Then a direct application of the inequalities established in Theorem 1.2 gives the required results. □

2. Application

As an application of Theorem 1.2, consider the nonlinear differential equation

$$u^{p-1}(t)u'(t) + g(t)u^q(t) = l(t, u(t)). \tag{2.1}$$

Assume that $p < q$, $u : \mathbb{R}_+ \to \mathbb{R}$, $g : \mathbb{R}_+ \to \mathbb{R}_+$, $l : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$|l(t, u(t))| \le \alpha(t) + h(t)|u(t)|^r, \tag{2.2}$$

$\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous functions.

Integrating (2.1) from 0 to t, we have

$$\frac{u^p(t)}{p} - \frac{u_0^p}{p} + \int_0^t g(s)u^q(s) ds = \int_0^t l(s) ds.$$

From this equality and (2.2), we obtain

$$|u(t)|^p \le a(t) + p \int_0^t [g(s)|u(s)|^q + h(s)|u(s)|^r] ds,$$

where $a(t) = |u_0|^p + p \int_0^t \alpha(s) ds$. Applying Theorem 1.2, we find explicit bounds of the solution $u(t)$ of the equation (2.1) in different cases where $p < r$ and $p > r$.

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