SOME NONLINEAR INTEGRAL INEQUALITIES ARISING IN DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to obtain estimates for functions satisfying some nonlinear integral inequalities. Using ideas from Pachpatte [3], we generalize the estimates presented in [2, 4].

1. Introduction and main results

Integral inequalities are a necessary tools in the study of properties of the solutions of linear and nonlinear differential equations, such as boundedness, stability, uniqueness, etc. This justifies the intensive investigation on integral inequalities; see for example [1] [3] [5]. The aim of this paper is to establish some new generalizations of integral inequalities that have a wide applications in the study of differential equations. More precisely, using some ideas from [3], we give further generalizations of the results presented in [2, 4].

We begin by giving some material necessary for our study. We denote by \( \mathbb{R} \) the set of real numbers, and by \( \mathbb{R}_+ \) the nonnegative real numbers.

Lemma 1.1. For \( x, y \in \mathbb{R}_+ \), \( 1/p + 1/q = 1 \), we have \( x^{1/p}y^{1/q} \leq x/p + y/q \).

Now we state the main results of this work.

Theorem 1.2. Let \( u, a, b, g \) and \( h \) be real valued nonnegative continuous functions defined on \( \mathbb{R}_+ \), \( p, r, q \) be real non negative constants. Assume that the functions

\[
\frac{a(t) + p/r}{b(t)}, \quad \frac{a(t) + r/p}{b(t)}, \quad \frac{a(t) + \min(r/p, q/p)}{b(t)}
\]

are nondecreasing and that

\[
u^p(t) \leq a(t) + b(t) \int_0^t [g(s)u^q(s) + h(s)u^r(s)]ds. \tag{1.1}
\]

(1) If \( 0 < r < p < q \), then

\[
u(t) \leq \left( a(t) + \frac{p}{r} \right)^{1/p} \left( 1 - \left( \frac{q}{p} - 1 \right) \int_0^t \left( b(s)g(s) + \frac{r}{p}h(s) \right)^{\frac{r}{p} - 1}ds \right)^{1/\left( \frac{r}{p} - 1 \right)} \tag{1.2}
\]
for \( t \leq \beta_{p,q,r} \), where

\[
\beta_{p,q,r} = \sup \{ t \in \mathbb{R}_+ : (q - 1) \int_0^t b(s)(g(s) + \frac{r}{p}) \frac{g}{s}^{-1} ds < 1 \}.
\]

(2) If \( 0 < p < r < q \), then

\[
u(t) \leq (a(t) + \frac{r}{p})^{1/p} \left( 1 - (q - 1) \int_0^t b(s)(g(s) + h(s))(a(s) + \frac{r}{p}) \frac{g}{s}^{-1} ds \right)^{-\frac{1}{q-1}}
\]

for \( t \leq \beta_{p,q,r} \), where

\[
\beta_{p,q,r} = \sup \{ t \in \mathbb{R}_+ : (q - 1) \int_0^t b(s)(g(s) + h(s))(a(s) + \frac{r}{p}) \frac{g}{s}^{-1} ds < 1 \}.
\]

(3) If \( 0 < p < q \) and \( p < r \), then

\[
u(t) \leq (a(t) + \min\left(\frac{r}{p}, \frac{q}{p}\right))^{1/p} \left( 1 - (\max\left(\frac{q}{p}, \frac{r}{p}\right) - 1) \int_0^t b(s)(g(s) + h(s))(a(s) + \frac{r}{p}) \frac{g}{s}^{-1} ds \right)^{\frac{1}{p(1-\max\left(\frac{q}{p}, \frac{r}{p}\right))}}
\]

for \( t \leq \beta_{p,q,r} \), where

\[
\beta_{p,q,r} = \sup \{ t \in \mathbb{R}_+ : \left( \max\left(\frac{q}{p}, \frac{r}{p}\right) - 1 \right) \int_0^t b(s)(g(s) + h(s))(a(s) + \frac{r}{p}) \frac{g}{s}^{-1} ds < 1 \}.
\]

Theorem 1.3. Suppose that the hypothesis of Theorem 1.2 hold and the function \( b(t) \) is decreasing. Let \( c \) be a real valued nonnegative continuous and nondecreasing function for \( t \in \mathbb{R}_+ \). Also assume that

\[
u^p(t) \leq c^p(t) + b(t) \int_0^t [g(s)u^q(s) + h(s)u^r(s)] ds.
\]

(1) If \( 0 < r < p < q \), then

\[
u(t) \leq c(t)(1 + \frac{p}{r})^{1/p} \left( 1 - (q - 1) \int_0^t (1 + \frac{p}{r}) \frac{g}{s}^{-1} b(s)K(s) ds \right)^{\frac{1}{q-1}}
\]

for \( t \leq \beta_{p,q,r} \), where

\[
\beta_{p,q,r} = \sup \{ t \in \mathbb{R}_+ : \left( \frac{q}{p} - 1 \right) \int_0^t (1 + \frac{p}{r}) \frac{g}{s}^{-1} b(s)K(s) ds < 1 \},
\]

\[K(s) = g(s)c(s)^{q-p} + \frac{r}{p} h(s)c(s)^{r-p}.
\]

(2) If \( t \in \mathbb{R}_+ \) and \( 0 < p < r < q \), then

\[
u(t) \leq c(t)(1 + \frac{r}{p})^{1/p} \left( 1 - (q - 1) \int_0^t b(s)K(s)(1 + \frac{r}{p}) \frac{g}{s}^{-1} ds \right)^{\frac{1}{q-1}}
\]

for \( t \leq \beta_{p,q,r} \), where

\[
\beta_{p,q,r} = \sup \{ t \in \mathbb{R}_+ : \left( \frac{q}{p} - 1 \right) \int_0^t b(s)K(s)(1 + \frac{r}{p}) \frac{g}{s}^{-1} ds < 1 \},
\]

\[K(s) = (g(s)c(s)^{q-p} + h(s)c(s)^{r-p}).
\]
(3) If \( t \in \mathbb{R}^+ \) and \( 0 < p < q, \ p < r, \) then

\[
\begin{align*}
u(t) & \leq (1 + \min\left(\frac{r}{p}, \frac{q}{p}\right))^{1/p} \left(1 - (\max\left(\frac{q}{p}, \frac{r}{p}\right) - 1) \int_0^t b(s)K(s)\,ds\right) \\
& \quad \times (1 + \min\left(\frac{r}{p}, \frac{q}{p}\right))^{\max\left(\frac{q}{p}, \frac{r}{p}\right)-1} \int_0^t b(s)K(s)\,ds < 1
\end{align*}
\]

for \( t \leq \beta_{p,q,r} \), where

\[ \beta_{p,q,r} = \sup\left\{ t \in R_+ : \left(\max\left(\frac{q}{p}, \frac{r}{p}\right) - 1\right) \int_0^t b(s)K(s)\,ds < 1 \right\} \]

and \( K(s) = (g(s)c(s)^q + h(s)c(s)^r)^{-1} \).

Note that in Theorems 1.2 and 1.3, we have studied the case \( p < q \). For the case \( p > q \), similar results are given in [2].

**Proof of Theorem 1.2.** (1) Define a function

\[ v(t) = \int_0^t \left[g(s)a^q(s) + h(s)a^r(s)\right]\,ds. \]

then from inequality (1.1) and Lemma 1.1 we deduce that

\[
\begin{align*}
u^q(t) & \leq (a(t) + b(t)v(t))^{q/p}, \\
u^r(t) & \leq (a(t) + b(t)v(t))^{r/p},
\end{align*}
\]

(1.5) (1.6)

\[
\begin{align*}
u^r(t) & \leq \frac{r}{p}(a(t) + b(t)v(t)) + \frac{p - r}{p}, \\
u^r(t) & \leq \frac{r}{p}(a(t) + b(t)v(t) + \frac{p - r}{r}),
\end{align*}
\]

(1.7) (1.8)

Since \( \frac{q}{p} > 1 \), which implies

\[
\begin{align*}
v'(t) & \leq \left[g(t) + \frac{r}{p}h(t)\right][a(t) + b(t)v(t)]^{q/p}.
\end{align*}
\]

(1.10)

Taking into account that the function \( \frac{a(t) + \frac{p}{r}}{b(t)} \) is nondecreasing for \( 0 \leq t \leq \tau \), we have

\[ v'(t) \leq M(t)\left(\frac{a(\tau) + \frac{p}{r}}{b(\tau)} + v(t)\right), \]

where

\[ M(t) = b(t)(g(t) + \frac{r}{p}h(t))(a(t) + b(t)v(t) + \frac{p}{r})^{\frac{q-1}{r}}, \]

consequently

\[ v(t) + \frac{a(\tau) + \frac{p}{r}}{b(\tau)} \leq \frac{a(\tau) + \frac{p}{r}}{b(\tau)} \exp \int_0^t M(s)\,ds. \]

For \( \tau = t \), we can see that

\[ a(t) + b(t)v(t) + \frac{p}{r} \leq (a(t) + \frac{p}{r}) \exp \int_0^t M(s)\,ds, \]

(1.11)
then the function $M(t)$ can be estimated as

$$M(t) \leq b(t)(g(t) + \frac{r}{p} h(t))(a(t) + \frac{p}{r})^{\frac{q}{p} - 1} \exp \int_0^t (\frac{q}{p} - 1)M(s)ds. \quad (1.12)$$

Let

$$L(t) = (\frac{q}{p} - 1)M(t). \quad (1.13)$$

Now we estimate the expression $L(t) \exp(- \int_0^t L(s)ds)$ by using (1.12) to obtain

$$L(t) \exp(\int_0^t -L(s)ds) \leq (\frac{q}{p} - 1)b(t)(g(t) + \frac{r}{p} h(t))(a(t) + \frac{p}{r})^{\frac{q}{p} - 1}.$$

Observing that

$$L(t) \exp(\int_0^t -L(s)ds) = \frac{d}{dt}(- \exp(\int_0^t -L(s)ds)),$$

$$\leq (\frac{q}{p} - 1)b(t)(g(t) + \frac{r}{p} h(t))(a(t) + \frac{p}{r})^{\frac{q}{p} - 1}.$$

Then integrate from 0 to $t$ to obtain

$$(1 - \exp(\int_0^t -L(s)ds)) \leq \exp(\int_0^t (\frac{q}{p} - 1)b(s)(g(s) + \frac{r}{p} h(s))(a(s) + \frac{p}{r})^{\frac{q}{p} - 1} ds).$$

Replacing $L(t)$ by its value in (1.13), we obtain

$$(1 - \exp(\int_0^t (1 - \frac{q}{p})M(s)ds) \leq \int_0^t (\frac{q}{p} - 1)b(s)(g(s) + \frac{r}{p} h(s))(a(s) + \frac{p}{r})^{\frac{q}{p} - 1} ds,$$

then

$$\exp(\int_0^t M(s)ds \leq \left\{ 1 - \left[ \int_0^t (\frac{q}{p} - 1)b(s)(g(s) + \frac{r}{p} h(s))(a(s) + \frac{p}{r})^{\frac{q}{p} - 1} ds \right] \right\}^{\frac{p}{q}}.$$

Using this inequality, (1.11), and (1.1) we obtain (1.2). This completes the proof of statement (1).

(2) for $t \in \mathbb{R}^+$ and $0 < p < r < q$, from (1.3) we have

$$u^{\alpha}(t) \leq (a(t) + b(t)v(t)) + \frac{r}{p}^{\alpha/p},$$

$$v^{\alpha}(t) \leq (g(t) + h(t))(a(t) + b(t)v(t)) + \frac{r}{p}^{\alpha/p}.$$
(3) For \( t \in \mathbb{R}^+ \) and \( p < r, p < q \), we have
\[
u^q(t) \leq (a(t) + b(t)v(t) + \min\left(\frac{r}{p}, \frac{q}{p}\right)\max(r/p, q/p),
\]
which gives
\[
v'(t) \leq (g(t) + h(t))(a(t) + b(t)v(t) + \min\left(\frac{r}{p}, \frac{q}{p}\right)\max(r/p, q/p)
\]
\[
v'(t) \leq M(t)(a(\tau) + \min\left(\frac{r}{p}, \frac{q}{p}\right) + v(t),
\]
where
\[
M(t) = b(t)(g(t) + h(t))(a(t) + b(t)v(t) + \min\left(\frac{r}{p}, \frac{q}{p}\right)\max(r/p, q/p)-1.
\]
Using the proof of the first part of Theorem 1.2, we get the desired result. \( \square \)

**Proof of Theorem 1.3.** Since \( c(t) \) is nonnegative, continuous and nondecreasing, it follows that (1.3) can be written as
\[
\left(\frac{u(t)}{c(t)}\right)^p \leq 1 + b(t)\int_0^t \left[g(s)\left(\frac{u(s)}{c(s)}\right)^q c(s)^{q-p} + h(s)\left(\frac{u(s)}{c(s)}\right)^r c(s)^{r-p}\right] ds.
\]  
Then a direct application of the inequalities established in Theorem 1.2 gives the required results. \( \square \)

## 2. Application

As an application of Theorem 1.2, consider the nonlinear differential equation
\[
u^{p-1}(t)u'(t) + g(t)u^q(t) = l(t, u(t)). \tag{2.1}
\]
Assume that \( p < q, u : \mathbb{R}^+ \to \mathbb{R}, g : \mathbb{R}^+ \to \mathbb{R}^+, l : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \)
\[
|l(t, u(t))| \leq \alpha(t) + h(t)|u(t)|^r, \tag{2.2}
\]
\( \alpha : \mathbb{R}^+ \to \mathbb{R}^+, h : \mathbb{R}^+ \to \mathbb{R}^+ \) are continuous functions.

Integrating (2.1) from 0 to \( t \), we have
\[
\frac{u^p(t)}{p} - \frac{u^p(0)}{p} + \int_0^t g(s)u^q(s)ds = \int_0^t l(s)ds.
\]
From this equality and (2.2), we obtain
\[
|u(t)|^p \leq \alpha(t) + p\int_0^t [g(s)|u(s)|^q + h(s)|u(s)|^r]ds,
\]
where \( \alpha(t) = |u_0|^p + p\int_0^t \alpha(s)ds \). Applying Theorem 1.2, we find explicit bounds of the solution \( u(t) \) of the equation (2.1) in different cases where \( p < r \) and \( p > r \).

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