Electronic Journal of Differential Equations, Vol. 2008(2008), No. 85, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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COMPLETE CLASSIFICATION AND STABILITY OF EQUILIBRIA IN A DELAYED RING NETWORK

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Abstract. In this paper, we consider a neural network model consisting of four neurons with delayed self and nearest-neighbor connections. We provide a full classification of all equilibria and their stability in the connection weighter parameter space. Such a classification is essential for the description of spatio-temporal patterns of the model system and for the applications to dynamic memory storage and retrieval.

1. Introduction

Golubitsky *et al* [7] showed that systems with symmetry can lead to many interesting patterns of oscillation, which are predictable based on the theory of equivariant bifurcations. In a series of papers [14, 15, 19], the theory of equivariant Hopf bifurcation has been extended to systems with time delays (functional differential equations). It should be noted that this theory predicts the possible patterns of oscillation in a system solely on the symmetry structure of the system. To understand which patterns occur in a particular system and whether they are stable, one needs to consider a specific model for a system.

With this in mind, there is interest in applying these results to models related to the Hopfield-Cohen-Grossberg neural networks [6, 8, 12, 13], and to models with time delays [16, 17]. Such models make an ideal test bed for this theory, as the models for the individual elements are quite simple (one variable for each element); yet with the introduction of time delays the behaviour can be quite complex. The focus of this work is on networks with a ring structure with nearest-neighbour (bi-directional) coupling between the elements. This leads to a system with $\mathbb{D}_n$ symmetry; i.e., a system which has the symmetries of a polygon with n sides of equal length. Most of these studies have concerned lower dimensional systems [3, 11, 18] and/or systems with a single time delay [9, 10]. There is also work on Hopfield-Cohen-Grossberg networks with a ring structure and uni-directional coupling [2, 4].

Recurrent neural networks have found successful applications in many areas such as associative memory, image processing, signal processing, and optimization. The qualitative study of such networks have been a subject of current interest and has benefited very much from the desired neural network applications. For example, the

2000 *Mathematics Subject Classification.* 34K18, 92B20.
Key words and phrases. Delay; neural network; equilibrium; stability.

Submitted April 16, 2008. Published June 9, 2008.

global asymptotical stability of a unique equilibrium is necessary for optimization, but the coexistence of multiple stable equilibria is critical for a network's capacity to store and retrieve memories stored as stable equilibria. When designing an associative memory neural network, we should make as many stable equilibrium states as possible to provide a memory system with large information capacity, an attractive region of each stable equilibrium as large as possible to provide the robustness and fault tolerance for information processing.

In this paper, we consider the locations and stability of all equilibria existing in a ring network modelled by the system of delay differential equations

$$\dot{u}_i(t) = -u_i(t) + \alpha f(u_i(t-\tau)) + \beta[f(u_{i-1}(t-\tau)) + f(u_{i+1}(t-\tau))], \tag{1.1}$$

with $i \bmod 4$, the strength of the self and nearest-neighbour coupling are denoted by α and β, and respectively, τ denote the corresponding delay. In addition, f is adequately smooth, and satisfies the following normalization, monotonicity, concavity, and boundedness conditions:

- (C1) $f(0) = f''(0) = 0$, $f'(0) = 1$, and $f'(x) > 0$ for all $x \in \mathbb{R}$;
- (C2) $xf''(x) < 0$ for all $x \neq 0$;
- (C3) There exists $M > 0$ such that $|f(x)| < M$ for all $x \in \mathbb{R}$.
- (C4) $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Networks of the form (1.1) have been studied extensively. Wu et al ([20]) consider the synchronization of solutions of a three-neuron network for the connection weights α and β in a certain region of parameter plane; that is, each solution of the network is convergent to the set of synchronous states in the phase state, dependently of the size of the delay. However, they only consider the existence and stability of synchronous equilibria and did not study the other equillibria of different patterns. Besides, in the paper by Bungay and Campbell ([1]), they also consider a ring network with three neurons, but they just investigated the existence and bifurcation of two classes of nontrivial equilibria. In fact, much remains to be done in order to fully understand their global dynamics. For example, in the case where $\tau = 0$, every solution of system (1.1) tends to the set of equilibria (see Hopfield [12, 13]). But to apply this convergence result to associative memory and pattern recognition, one must know the location of all equilibria and their stability. When $\tau > 0$, it will cause nonlinear oscillation (see Marcus and Westervelt [17]) and desynchronization induced by the large scale of networks (see Chen *et al* [5]). For instance, 8 branches of periodic solutions may bifurcate simultaneously from the trivial equilibrium. For detail, we refer to Guo and Huang [9, 10]. To see how these periodic solutions and equilibria are connected together to form the global attractor, it is again critical to have full classification of the equilibrium structure. Unfortunately, this issue has not been addressed in the literature.

In this paper, the discussion of synchronous equillibria is only a small part of the paper: We clearly state in Theorems 2.1 and 2.2 that this is only a triviality that is stated for completeness, which can be generalized to a ring network of n neurons, and of course available coincides with the paper by Wu et al ([20]) is for 3 neurons in this aspect. The main result of this paper is another one (see Theorems 2.3–2.7) and we describe the branching pattern of those equillibria that can not be found from the paper by Wu et al ([20]). More importantly, we consider the stabilities of these equillibria of different patterns. In the paper by Wu et al ([20]), they only consider the stabilities of synchronous equillibria. The outline of the paper is as follows. In Section 2 we will completely locate all the equilibria of system (1.1) and find the conditions ensuring the existence of equilibria with different patterns. In Section 3 we will discuss the stability and un-stability of the equilibria in Section 2.

2. Patterns of Equilibria

First of all, we consider the patterns of the equilibria of system (1.1). To study the steady state bifurcations of the trivial solution of (1.1), one must consider the characteristic equation of the linearization of (1.1) about the trivial solution. The characteristic matrix of the linearization of (1.1) at this equilibrium 0 is:

$$\mathcal{M}_4(0,\lambda) = (\lambda + 1 - \alpha e^{-\lambda\tau})I - \beta e^{-\lambda\tau}M.$$

Let

$$\chi = e^{i\frac{\pi}{2}}, \quad v_j = (1, \chi^j, \chi^{2j}, \chi^{3j})^T, \quad (j = 0, 1, 2, 3).$$

Since

$$\begin{gathered} \chi^{4j} = e^{2\pi j i} = 1, \\ \chi^{3j} = e^{\frac{3}{2}\pi j i} = e^{-\frac{\pi j}{2} i} = \chi^{-j}, \\ \chi^{2j} = e^{\pi j i} = e^{(2\pi - \pi) j i} = chi^{-2j}, \end{gathered}$$

it follows that

$$Mv^j = \begin{bmatrix} \chi^j + \chi^{3j} \\ 1 + \chi^{2j} \\ \chi^j + \chi^{3j} \\ 1 + \chi^{2j} \end{bmatrix} = (\chi^j + \chi^{-j})v_j\,.$$

Hence,

$$\begin{aligned} \mathcal{M}_4(0,\lambda)v_j &= (\lambda + 1 - \alpha e^{-\lambda\tau} - \beta e^{-\lambda\tau}(\chi^j + \chi^{-j}))v_j \\ &= (\lambda + 1 - \alpha e^{-\lambda\tau} - 2\beta e^{-\lambda\tau}\cos\frac{2\pi j}{4})v_j. \end{aligned}$$

and the characteristic equation is

$$\begin{aligned} \det \mathcal{M}_4(0,\lambda) &= \prod j = 0^3(\lambda + 1 - \alpha e^{-\lambda\tau} - 2\beta e^{-\lambda\tau}\cos\frac{2\pi j}{4}) \\ &= (a - 2b)\prod j = 1^3(a - 2b\cos\frac{2\pi j}{4}) = 0 \end{aligned}$$

where $a = \lambda + 1 - \alpha e^{-\lambda\tau}$ and $b = \beta e^{-\lambda\tau}$. Thus, we can rewrite

$$\begin{aligned} &\det \mathcal{M}_4(0,\lambda) \\ &= (\lambda + 1 - \alpha e^{-\lambda\tau})^2(\lambda + 1 - \alpha e^{-\lambda\tau} - 2\beta e^{-\lambda\tau})(\lambda + 1 - \alpha e^{-\lambda\tau} + 2\beta e^{-\lambda\tau})\,. \end{aligned}$$

Clearly the characteristic equation has a simple zero if either $1 - \alpha - 2\beta = 0$ or $1 - \alpha + 2\beta = 0$, and a double zero root if $1 - \alpha = 0$.

Thus $\alpha + 2\beta = 1$ and $\alpha - 2\beta = 1$ are potential standard steady state bifurcation curves,while the line $\alpha = 1$ is a potential equivariant state bifurcation line. Since the nonlinearities in the model (1.1) are odd functions these will be pitchfork bifurcations.

The simplest nontrivial equilibria are of the form (x^*, x^*, x^*, x^*) where x^* satisfies:

$$-x^* + (\alpha + 2\beta)f(x^*) = 0 \tag{2.1}$$

We call these *synchronous* equilibria since all the four components are the same.

Theorem 2.1. *If $\alpha + 2\beta > 1$ then there are at least two nontrivial synchronous equilibria (x^*, x^*, x^*, x^*) and $(-x^*, -x^*, -x^*, -x^*)$. If $\alpha + 2\beta < 1$, there are no nontrivial synchronous equilibria.*

Proof. Let $F(x) = -x + (\alpha + 2\beta)f(x)$. It follows that F is odd, $F(0) = 0$, $F'(x) = -1 + (\alpha + 2\beta)f'(x)$ and $F''(x) = (\alpha + 2\beta)f''(x)$.

Let $\alpha + 2\beta < 1$. Then $F'(0) = -1 + \alpha + 2\beta < 0$. Since $f''(x) < 0$ for $x > 0$ then $f'(x) < f'(0) = 1$. Similarly, we have $f'(x) < 1$ for $x < 0$. Therefore, $0 < f'(x) < 1$ for all $x \in \mathbb{R}$. It follows from $xf''(x) < 0$ that $F''(x) < 0$ for all $x > 0$, and hence $F'(x)$ is strictly decreasing in $x \in [0, +\infty)$. So $F'(x) < F'(0) < 0$ for $x > 0$. Thus F is strictly decreasing for $x > 0$ and so $F(x) < F(0) = 0$ for all $x > 0$, then $F(x)$ has no zero in $(0, +\infty)$. Hence system (1.1) has no nontrivial equilibria.

On the other hand, let $\alpha + 2\beta > 1$, then $F'(0) > 0$ and $\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty}(-x + (\alpha + 2\beta)f(x))$. It follows from $(C3)$ that $\lim_{x\to+\infty} F(x) < 0$. Thus there is an $x^* > 0$ such that $F(x^*) = 0$. By symmetry $F(-x^*) = 0$. □

This result about the existence of synchronous equilibria allows us to make a conclusion about bifurcation of the trivial solution.

Theorem 2.2. *There is a supercritical pitchfork bifurcation of the trivial solution of system* (1.1) *at $\alpha + 2\beta = 1$, leading to two branches of synchronous equilibria given by $(\pm x^*, \pm x^*, \pm x^*, \pm x^*)$ where x^* satisfies* (2.1).

Proof. It is clear from the characteristic equation (2) that the trivial solution gains a real eigenvalue with positive real part as $\alpha + 2\beta$ increases and passes through 1. It follows from Theorem 2.1 that the synchronous equilibria $(\pm x^*, \pm x^*, \pm x^*, \pm x^*)$ exist only for $\alpha + 2\beta > 1$. Finally, it can be shown that $x^* \to 0$ as $\alpha + 2\beta \to 1^+$, the result follows. □

As we know, system (1.1) is $\mathbb{D}_4$-equivariant. Moreover, since the function f is odd, then system (1.1) is $\mathbb{D}_4 \oplus \mathbb{Z}_2$-equivariant. Define the action of some generators of $\mathbb{D}_4$ on $\mathbb{R}^4$:

$$(\rho x)_i = x_{i+1}, \quad (\kappa x)_i = x_{2-i}, \quad \zeta x = -x \quad \text{for all } x \in \mathbb{R}^4.$$

Then, we can classify system (1.1)'s equilibria into five cases.

Firstly, the equilibria characterized by $\mathbb{D}_4$'s subgroup $\mathbb{Z}(\rho\kappa)$, take the forms of (x_1, x_2, x_2, x_1) or (x_1, x_1, x_2, x_2) with $x_1, x_2 \in \mathbb{R}$. Here,we only consider the existence of equilibrium (x_1, x_2, x_2, x_1) because the other can be discussed analogously. By (1.1), components $x_1, x_2 \in \mathbb{R}$ satisfy

$$\begin{aligned} x_1 - \alpha f(x_1) &= \beta f(x_1) + \beta f(x_2), \\ x_2 - \alpha f(x_2) &= \beta f(x_1) + \beta f(x_2). \end{aligned} \tag{2.2}$$

Theorem 2.3. *If $\alpha > 1$, there are at least four equilibria: $(\pm x_1, \pm x_2, \pm x_2, \pm x_1)$ or $(\pm x_1, \pm x_1, \pm x_2, \pm x_2)$, where x_1, x_2 satisfy* (2.2) *and $x_1 \neq x_2$.*

Proof. Rewriting the equations

$$\begin{aligned} -x_1 + (\alpha + \beta)f(x_1) + \beta f(x_2) &= 0, \\ -x_2 + (\alpha + \beta)f(x_2) + \beta f(x_1) &= 0. \end{aligned}$$

and solving for x_2 gives

$$x_2 = \frac{\alpha+\beta}{\beta}x_1 - \frac{\alpha^2+2\alpha\beta}{\beta}f(x_1).$$

Putting this in the first equation gives an equation for x_1 alone.

$$F(x_1) = -x_1 + (\alpha+\beta)f(x_1) + \beta f(\frac{\alpha+\beta}{\beta}x_1 - \frac{\alpha^2+2\alpha\beta}{\beta}f(x_1))$$

Defining $g(x) = \frac{\alpha+\beta}{\beta}x - \frac{\alpha^2+2\alpha\beta}{\beta}f(x)$ we can write $F(x) = -x + (\alpha+\beta)f(x) + \beta f(g(x))$ and find

$$F'(x) = -1 + (\alpha+\beta)f'(x) + (\alpha+\beta)f'(g(x)) - (\alpha^2+2\alpha\beta)f'(x)f'(g(x)).$$

In fact, we have $F(0) = 0$, $\lim_{x\to\infty} F(x) < 0$ and

$$F'(0) = -1 + 2(\alpha+\beta) - \alpha^2 - 2\alpha\beta = (1-\alpha)(\alpha+2\beta-1).$$

So if $\alpha + 2\beta < 1$ and $\alpha > 1$ then $F'(0) > 0$. This, together with the fact that $\lim_{x\to+\infty} F(x) = -\infty$ and $F(0) = 0$ implies that there exists a positive constant ξ such that $F(\xi) = 0$ and hence $x_1 = \xi$ and $x_2 = g(\xi)$. In what follows, we show that $x_1 \neq x_2$, that is, $\xi \neq g(\xi)$. In fact, if $\xi = g(\xi)$ then $\xi = (\alpha+2\beta)f(\xi)$, and so $f(\xi) \geq \xi$, where the equality holds if and only $\xi = 0$, and contradicts the fact $\xi > 0$. Hence $\xi \neq g(\xi)$.

If $\alpha+2\beta = 1$ and $\alpha > 1$ then $\beta < 0$, note that $g(0) = 0$ and $\lim_{x\to+\infty} g(x) = -\infty$, so there exist a positive constant ξ_1 such that $g(\xi_1) = 0$, viz., $(\alpha+\beta)\xi_1 = \alpha f(\xi_1)$. It is clear that $F(\xi_1) = ((\alpha+\beta)^2 - \alpha)f(\xi_1)/\alpha = \beta^2 f(\xi_1)/\alpha > 0$. Again since $\lim_{x\to+\infty} F(x) = -\infty$, then there exists a point $x = x_0$ such that $F(x_0) = 0$ and $x_0 \neq g(x_0)$.

If $\alpha + 2\beta > 1$ and $\alpha > 1$, then $F'(0) < 0$. Obviously, $F(x^*) = 0$, where x^* is given in Theorem 2.1 such that $x^* = (\alpha+2\beta)f(x^*)$, that is, $g(x^*) = x^*$. Note that

$$\begin{aligned} F'(x^*) &= -1 + 2(\alpha+\beta)f'(x^*) + (\alpha^2+2\alpha\beta)(f'(x^*))^2 \\ &= -(\alpha f'(x^*) - 1)((\alpha+2\beta)f'(x^*) - 1). \end{aligned}$$

It is clear that $F'(x^*) \neq 0$ if $f'(x^*) \neq 1/\alpha$ and $f'(x^*) \neq 1/(\alpha+2\beta)$.

Again since $x^* = (\alpha+2\beta)f(x^*)$, we have $f'(x^*) < 1/(\alpha+2\beta)$. We distinguish two cases to discuss whether $f'(x^*) = 1/\alpha$.

Case (I). If $f'(x^*) \neq 1/\alpha$, then $F'(x^*) \neq 0$.

Case (II). If $f'(x^*) = 1/\alpha$, then $h(x) = \alpha f(x) - x$ attains the maximal value at the point $x = x^*$, and $h(x^*) > 0$. Note that $\lim_{x\to+\infty} h(x) < 0$, so there is a point $x = \bar{x} \neq x^*$ such that $h(\bar{x}) = 0$. At the point $x = \bar{x}$, we have $F(\bar{x}) = 0$ and $g(\bar{x}) \neq \bar{x}$.

From above, there is at least one point x_0 in interval $(0, +\infty)$ such that $F(x_0) = 0$ and $g(x_0) \neq x_0$. So there are at least four equilibria: $(\pm x_1, \pm x_2, \pm x_2, \pm x_1)$ or $(\pm x_1, \pm x_1, \pm x_2, \pm x_2)$, where x_1, x_2 satisfy Eq.(2.2) and $x_1 \neq x_2$. $\square$

Secondly, the equilibria characterized by the subgroup $\mathbb{Z}(\zeta\kappa)$ of $\mathbb{D}_4 \oplus \mathbb{Z}_2$, take the forms of $(0, x, 0, -x)$ or $(x, 0, -x, 0)$ with $x \in \mathbb{R}$. Here, we only consider the existence of equilibrium $(0, x, 0, -x)$ because the other can be discussed analogously. By (1.1), the component $x \in \mathbb{R}$ satisfies

$$-x + \alpha f(x) = 0. \tag{2.3}$$

Theorem 2.4. *If $\alpha - 1 > 0$, then there are at least four equilibria:* $(0, \pm x, 0, \mp x)$ *or* $(\pm x, 0, \mp x, 0)$*, where x satisfy* (2.3) *and* $x \neq 0$.

Proof. Let $F(x) = -x + \alpha f(x)$, then $F'(x) = -1 + \alpha f'(x)$. Note that $F(0) = 0$, $\lim_{x\to\infty} F(x) < 0$ and $F'(0) = -1 + \alpha > 0$. Since F is a continuous function it follows that there is an $x^* > 0$ such that $F(x^*) = 0$. By symmetry $F(-x^*) = 0$ as well.

If $\alpha \leq 1$, note that $0 < f'(x) < 1$ when $x \neq 0$ then $F'(x) = -1 + \alpha f'(x) < 0$, so $F(x) < F(0) = 0$ for all $x > 0$. We find $F(x)$ have no root in interval $(0, +\infty)$, as $F(x)$ is odd function, we know $F(x)$ have only a root: $x = 0$. Then induce a conflict. So the existing of the four equilibria imply $\alpha > 1$. $\square$

Thirdly, the equilibria characterized by the subgroup $\mathbb{Z}(\zeta\rho\kappa)$ of $\mathbb{D}_4 \bigoplus \mathbb{Z}_2$, take the form of $(x_1, x_2, -x_2, -x_1)$ or $(x_1, -x_1, -x_2, x_2)$ with $x_1, x_2 \in \mathbb{R}$. Here,we only consider the existence of equilibrium $(x_1, x_2, -x_2, -x_1)$ because the other can be discussed analogously. By (1.1), the components $x_1, x_2 \in \mathbb{R}$ satisfy

$$\begin{aligned} x_1 - \alpha f(x_1) &= -\beta f(x_1) + \beta f(x_2), \\ x_2 - \alpha f(x_2) &= \beta f(x_1) - \beta f(x_2). \end{aligned} \tag{2.4}$$

Theorem 2.5. *If $\alpha > 1$, there are at least four equilibria:* $(\pm x_1, \pm x_2, \mp x_2, \mp x_1)$ *or* $(\pm x_1, \mp x_1, \mp x_2, \pm x_2)$*, where x_1, x_2 satisfy* (2.4).

Proof. Rewriting the equations

$$\begin{aligned} -x_1 + (\alpha - \beta) f(x_1) + \beta f(x_2) = 0, \\ -x_2 + (\alpha - \beta) f(x_2) + \beta f(x_1) = 0. \end{aligned}$$

and solving for x_2 gives

$$x_2 = \frac{\alpha - \beta}{\beta} x_1 - \frac{\alpha^2 - 2\alpha\beta}{\beta} f(x_1).$$

Putting this in the first equation gives an equation for x_1 alone.

$$F(x_1) = -x_1 + (\alpha - \beta) f(x_1) + \beta f\left(\frac{\alpha - \beta}{\beta} x_1 - \frac{\alpha^2 - 2\alpha\beta}{\beta} f(x_1)\right)$$

Defining $g(x) = \frac{\alpha-\beta}{\beta} x - \frac{\alpha^2 - 2\alpha\beta}{\beta} f(x)$ and hence $F(x) = -x + (\alpha - \beta) f(x) + \beta f(g(x))$, we have

$$F'(x) = -1 + (\alpha - \beta) f'(x) + (\alpha - \beta) f'(g(x)) - (\alpha^2 - 2\alpha\beta) f'(x) f'(g(x)).$$

In fact, we know $F(0) = 0$, $\lim_{x\to\infty} F(x) < 0$ and

$$F'(0) = -1 + 2(\alpha - \beta) - \alpha^2 + 2\alpha\beta = (1 - \alpha)(\alpha - 2\beta - 1).$$

So if $\alpha - 2\beta < 1$ and $\alpha > 1$ then $F'(0) > 0$. This, together with the fact that $\lim_{x\to+\infty} F(x) = -\infty$ and $F(0) = 0$ implies that there exists a positive constant ξ such that $F(\xi) = 0$ and hence $x_1 = \xi$ and $x_2 = g(\xi)$. In what follows, we show that $x_1 \neq x_2$, viz., $\xi \neq g(\xi)$. In fact, if $\xi = g(\xi)$ then $\xi = (\alpha - 2\beta) f(\xi)$, so $f(\xi) \geq \xi$, where the equality holds if and only $\xi = 0$, and contradicts the fact $\xi > 0$. Hence $\xi \neq g(\xi)$.

If $\alpha - 2\beta = 1$ and $\alpha > 1$ then $g(0) = 0$ and $\lim_{x\to+\infty} g(x) = -\infty$, so there exist a positive constant ξ_1 such that $g(\xi_1) = 0$, viz., $(\alpha - \beta)\xi_1 = \alpha f(\xi_1)$. It is clear that $F(\xi_1) = ((\alpha - \beta)^2 - \alpha) f(\xi_1)/\alpha = \beta^2 f(\xi_1)/\alpha > 0$. Again since $\lim_{x\to+\infty} F(x) = -\infty$, then there exists a point $x = x_0$ such that $F(x_0) = 0$ and $x_0 \neq g(x_0)$.

If $\alpha - 2\beta > 1$ and $\alpha > 1$, then $F'(0) < 0$. Obviously, $F(x^*) = 0$, where x^* is given in Theorem 2.1 such that $x^* = (\alpha - 2\beta)f(x^*)$, viz., $g(x^*) = x^*$. Note that

$$\begin{aligned} F'(x^*) &= -1 + 2(\alpha - \beta)f'(x^*) - (\alpha^2 - 2\alpha\beta)(f'(x^*))^2) \\ &= -(\alpha f'(x^*) - 1)((\alpha - 2\beta)f'(x^*) - 1). \end{aligned}$$

It is clear that $F'(x^*) \neq 0$ if $f'(x^*) \neq 1/\alpha$ and $f'(x^*) \neq 1/(\alpha - 2\beta)$.

Again since $x^* = (\alpha - 2\beta)f(x^*)$, we have $f'(x^*) < 1/(\alpha - 2\beta)$. We can distinguish two cases to discuss whether $f'(x^*) = 1/\alpha$.

Case (I). If $f'(x^*) \neq 1/\alpha$, then $F'(x^*) \neq 0$.

Case (II). If $f'(x^*) = 1/\alpha$, then $h(x) = \alpha f(x) - x$ attains the maximal value at the point $x = x^*$, and $h(x^*) > 0$. Note that $\lim_{x\to+\infty} h(x) < 0$, so there is a point $x = \bar{x} \neq x^*$ such that $h(\bar{x}) = 0$. At the point $x = \bar{x}$, we have $F(\bar{x}) = 0$ and $g(\bar{x}) \neq \bar{x}$.

From above, there is at least one point x_0 in interval $(0, +\infty)$ such that $F(x_0) = 0$ and $g(x_0) \neq x_0$. So there are at least four equilibria: $(\pm x_1, \pm x_2, \mp x_2, \mp x_1)$ or $(\pm x_1, \mp x_1, \mp x_2, \pm x_2)$, where x_1, x_2 satisfy (2.4) and $x_1 \neq x_2$. □

Fourthly, the equilibria $(x, -x, x, -x)$ with $x \in \mathbb{R}$. By (1.1), components $x \in \mathbb{R}$ satisfies

$$-x + (\alpha - 2\beta)f(x) = 0. \tag{2.5}$$

By using a similar arguments as the proof of Theorem 2.4, we have the following result.

Theorem 2.6. *If $\alpha - 2\beta > 1$, there are at least two equilibria: $(\pm x, \mp x, \pm x, \mp x)$, where x satisfy* (2.5).

Fifthly, the equilibrium (x_1, x_2, x_1, x_2) with $x_1, x_2 \in \mathbb{R}$. By (1.1), the components $x_1, x_2 \in \mathbb{R}$ satisfy

$$\begin{aligned} -x_1 + \alpha f(x_1) + 2\beta f(x_2) = 0, \\ -x_2 + \alpha f(x_2) + 2\beta f(x_1) = 0. \end{aligned} \tag{2.6}$$

Theorem 2.7. *If $\alpha - 2\beta > 1$, then there are at least the following two equilibria: $(\pm x_1, \pm x_2, \pm x_1, \pm x_2)$, where x_1, x_2 satisfy* (2.6) *and $x_1 \neq x_2$.*

Proof. From the equations above, we find

$$x_2 = \frac{\alpha}{2\beta}x_1 - \frac{\alpha^2 - 4\beta^2}{2\beta}f(x_1).$$

Putting this in the first equation gives an equation for x_1 alone.

$$F(x_1) = -x_1 + \alpha f(x_1) + 2\beta f(\frac{\alpha}{2\beta}x_1 - \frac{\alpha^2 - 4\beta^2}{2\beta}f(x_1))$$

Defining $g(x) = \frac{\alpha}{2\beta}x - \frac{\alpha^2-4\beta^2}{2\beta}f(x)$ we can write $F(x) = -x + \alpha f(x) + 2\beta f(g(x))$ and find

$$F'(x) = -1 + \alpha f'(x) + \alpha f'(g(x)) - (\alpha^2 - 4\beta^2)f'(x)f'(g(x)).$$

In fact, we know $F(0) = 0$, $\lim_{x\to\infty} F(x) < 0$ and

$$F'(0) = -1 + 2\alpha - \alpha^2 + 4\beta^2 = -(\alpha - 2\beta - 1)(\alpha + 2\beta - 1).$$

So if $\alpha + 2\beta < 1$ and $\alpha - 2\beta > 1$ then $F'(0) > 0$. This, together with the fact that $\lim_{x\to+\infty} F(x) = -\infty$ and $F(0) = 0$ implies that there exists a positive constant ξ such that $F(\xi) = 0$ and hence $x_1 = \xi$ and $x_2 = g(\xi)$. In what follows, we show that

$x_1 \neq x_2$, viz., $\xi \neq g(\xi)$. In fact, if $\xi = g(\xi)$ then $\xi = (\alpha + 2\beta)f(\xi)$, so $f(\xi) \geq \xi$, where the equality holds if and only $\xi = 0$, and contradicts the fact $\xi > 0$. Hence $\xi \neq g(\xi)$.

If $\alpha+2\beta = 1$ and $\alpha-2\beta > 1$ then $\beta < 0$. Note that $g(0) = 0$ and $\lim_{x\to+\infty} g(x) = -\infty$, so there exist a positive constant ξ_1 such that $g(\xi_1) = 0$, viz., $\alpha\xi_1 = (\alpha - x\beta)f(\xi_1)$. It is clear that $F(\xi_1) = (\alpha^2 - \alpha + 2\beta)f(\xi_1)/\alpha = 4\beta^2 f(\xi_1)/\alpha > 0$. Again since $\lim_{x\to+\infty} F(x) = -\infty$, then there exists a point $x = x_0$ such that $F(x_0) = 0$ and $x_0 \neq g(x_0)$.

If $\alpha + 2\beta > 1$ and $\alpha - 2\beta > 1$, then $F'(0) < 0$. Obviously, $F(x^*) = 0$, where x^* is given in Theorem 2.1 such that $x^* = (\alpha + 2\beta)f(x^*)$, viz., $g(x^*) = x^*$. Note that

$$\begin{aligned} F'(x^*) &= -1 + 2\alpha f'(x^*) - (\alpha^2 - 4\beta^2)(f'(x^*))^2 \\ &= -((\alpha - 2\beta)f'(x^*) - 1)((\alpha + 2\beta)f'(x^*) - 1). \end{aligned}$$

It is clear that $F'(x^*) \neq 0$ if $f'(x^*) \neq 1/(\alpha - 2\beta)$ and $f'(x^*) \neq 1/(\alpha + 2\beta)$.

Again since $x^* = (\alpha + 2\beta)f(x^*)$, we have $f'(x^*) < 1/(\alpha + 2\beta)$. We distinguish two cases to discuss whether $f'(x^*) = 1/(\alpha - 2\beta)$.

Case (I). If $f'(x^*) \neq 1/(\alpha - 2\beta)$, then $F'(x^*) \neq 0$.

Case (II). If $f'(x^*) = 1/(\alpha - 2\beta)$, then $h(x) = (\alpha - 2\beta)f(x) - x$ attains the maximal value at the point $x = x^*$, and $h(x^*) > 0$. Note that $\lim_{x\to+\infty} h(x) < 0$, so there is a point $x = \bar{x} \neq x^*$ such that $h(\bar{x}) = 0$. At the point $x = \bar{x}$, we have $F(\bar{x}) = 0$ and $g(\bar{x}) \neq \bar{x}$.

From above, there is at least one point x_0 in interval $(0, +\infty)$ such that $F(x_0) = 0$ and $g(x_0) \neq x_0$. So there are at least two equilibria: $(\pm x_1, \pm x_2, \pm x_1, \pm x_2)$, where x_1, x_2 satisfy (2.6) and $x_1 \neq x_2$. □

3. Linear Stability Analysis

To investigate the local stability of the system, we need to consider the characteristic equation of (1.1) associated with each equilibrium type. To begin, we calculate the linearization about a generic nontrivial equilibrium $(x_1^*, x_2^*, x_3^*, x_4^*)$, viz.,

$$\dot{\mathbf{x}} = -\mathbf{x} + A\mathbf{x}(t - \tau), \tag{3.1}$$

where

$$A = \begin{pmatrix} k_1\alpha & k_2\beta & 0 & k_4\beta \\ k_1\beta & k_2\alpha & k_3\beta & 0 \\ 0 & k_2\beta & k_3\alpha & k_4\beta \\ k_1\beta & 0 & k_3\beta & k_4\alpha \end{pmatrix}$$

and $k_j = f'(x_j^*)$. The corresponding characteristic equation can be found by looking for solutions to (3.1) of the form $\mathbf{x} = e^{\lambda t}\mathbf{u}$, where $\lambda \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^4$. Substituting $\mathbf{x} = e^{\lambda t}\mathbf{u}$ into (3.1) yields an equation for λ and $\mathbf{u} = [u_1, u_2, u_3, u_4]^T$:

$$\begin{bmatrix} -1 + k_1\alpha e^{-\lambda\tau} - \lambda & k_2\beta e^{-\lambda\tau} & 0 & k_4\beta e^{-\lambda\tau} \\ k_1\beta e^{-\lambda\tau} & -1 + k_2\alpha e^{-\lambda\tau} - \lambda & k_3\beta e^{-\lambda\tau} & 0 \\ 0 & k_2\beta e^{-\lambda\tau} & -1 + k_3\alpha e^{-\lambda\tau} - \lambda & k_4\beta e^{-\lambda\tau} \\ k_1\beta e^{-\lambda\tau} & 0 & k_3\beta e^{-\lambda\tau} & -1 + k_4\alpha e^{-\lambda\tau} - \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0. \tag{3.2}$$

Requiring nontrivial solutions ($\mathbf{u} \neq 0$) gives the characteristic equation

$$\det \begin{bmatrix} -1+k_1\alpha e^{-\lambda\tau}-\lambda & k_2\beta e^{-\lambda\tau} & 0 & k_4\beta e^{-\lambda\tau} \\ k_1\beta e^{-\lambda\tau} & -1+k_2\alpha e^{-\lambda\tau}-\lambda & k_3\beta e^{-\lambda\tau} & 0 \\ 0 & k_2\beta e^{-\lambda\tau} & -1+k_3\alpha e^{-\lambda\tau}-\lambda & k_4\beta e^{-\lambda\tau} \\ k_1\beta e^{-\lambda\tau} & 0 & k_3\beta e^{-\lambda\tau} & -1+k_4\alpha e^{-\lambda\tau}-\lambda \end{bmatrix} = 0.$$

At the equilibrium point (x^*, x^*, x^*, x^*), we have $k_j = k = f'(x^*)$, $j = 1, 2, 3, 4$, thus the characteristic equation of the system (1.1) becomes

$$\begin{aligned} &(-1-\lambda+\alpha k e^{-\lambda\tau})^2(-1-\lambda+\alpha k e^{-\lambda\tau}-2\beta k e^{-\lambda\tau}) \\ &\times(-1-\lambda+\alpha k e^{-\lambda\tau}+2\beta k e^{-\lambda\tau}) \\ &=\prod_{j=1}^{4}(\lambda+1-k\alpha e^{-\lambda\tau}-2k\beta e^{-\lambda\tau}\cos\frac{\pi j}{2})=0. \end{aligned} \tag{3.3}$$

For the sake of convenience, for each $j = 1, 2, 3, 4$, define $\Delta_j\colon \mathbb{C} \to \mathbb{C}$ by

$$\Delta_j(\lambda) = \lambda + 1 - k\alpha e^{-\lambda\tau} - 2k\beta e^{-\lambda\tau}\cos\frac{\pi j}{2}.$$

Then, (3.3) can be rewriten

$$\prod_{j=1}^{4}\Delta_j(\lambda) = 0$$

Theorem 3.1. *If the parameters satisfy* $|\beta| < \frac{1}{2}(\frac{1}{k} - |\alpha|)$, *then the equilibrium* (x^*, x^*, x^*, x^*) *is locally asymptotically stable for all* $\tau \geq 0$, *where* $k = f'(x^*)$.

Proof. Let $\lambda = \upsilon + i\omega$, $\upsilon, \omega \in \mathbb{R}$, and $\Delta_j(\lambda) = R_j(\upsilon,\omega) + iI_j(\upsilon,\omega)$, then

$$R_j(\upsilon,\omega) = \upsilon + 1 - k\alpha e^{-\upsilon\tau}\cos(\omega\tau) - 2k\beta e^{-\upsilon\tau}\cos(\omega\tau)\cos\frac{\pi j}{2},$$

$$I_j(\upsilon,\omega) = \omega + k\alpha e^{-\upsilon\tau}\sin(\omega\tau) + 2k\beta e^{-\upsilon\tau}\sin(\omega\tau)\cos\frac{\pi j}{2}.$$

Therefore, one can obtain

$$R_j(\upsilon,\omega) \geq \upsilon + 1 - k|\alpha|e^{-\upsilon\tau} - 2k|\beta|e^{-\upsilon\tau}. \tag{3.4}$$

Denoting by $R(\upsilon)$ the right-hand side of (3.4), it is easy to see that

$$R(0) = 1 - k(|\alpha| + 2|\beta|) > 0.$$

Note that

$$\frac{\mathrm{d}R}{\mathrm{d}\upsilon} = 1 + k(|\alpha| + 2|\beta|)\tau e^{-\upsilon\tau} > 0,$$

then we have $R(\upsilon) > 0$ for all $\upsilon \geq 0$, and $R_j(\upsilon,\omega) > 0$ for all $\upsilon \geq 0$, $\omega \in \mathbb{R}$.

Now let $\lambda = \upsilon + i\omega$ be an arbitrary root of the characteristic equation, then $R_j(\upsilon,\omega) = I_j(\upsilon,\omega) = 0$ for some $j \in \{1, 2, 3, 4\}$. It follows from the previous discussion that $\upsilon < 0$. Therefore all roots of the characteristic equation have negative real parts, which means that the equilibrium (x^*, x^*, x^*, x^*) is locally asymptotically stable for any $\tau \geq 0$. □

Remark 3.2. Theorem 3.1 can be applied to uncoupled oscillators. Namely, the equilibrium (x^*, x^*, x^*, x^*) is locally asymptotically stable when $\beta = 0$, $|\alpha| < \frac{1}{k}$ and $\tau \geq 0$.

Theorem 3.3. *The equilibrium* (x^*, x^*, x^*, x^*) *is unstable for any* $\tau \geq 0$, *if either* $k(\alpha + 2\beta) > 1$, *or* $k(\alpha - 2\beta) > 1$, *or* $k\alpha > 1$, *where* $k = f'(x^*)$.

Proof. At the equilibrium point (x^*, x^*, x^*, x^*), we have

$$\begin{gathered}\Delta_1(0) = \Delta_3(0) = 1 - k\alpha < 0 \quad \text{when } k\alpha > 1,\\ \Delta_2(0) = 1 - k(\alpha + 2\beta) < 0 \quad \text{when } k(\alpha + 2\beta) > 1,\\ \Delta_4(0) = 1 - k(\alpha - 2\beta) < 0 \quad \text{when } k(\alpha - 2\beta) > 1.\end{gathered}$$

On the other hand, Δ_j, $j = 1, 2, 3, 4$ are continuous functions satisfying

$$\lim_{\lambda\to\infty} \Delta_j = +\infty \quad (j = 1, 2, 3, 4).$$

Thus, under the assumptions of Theorem 3.3, there exists some $j \in \{1, 2, 3, 4\}$ such that the function $\Delta_j(\cdot)$ has at least one positive real zero. Namely, (3.3) has at least one positive real root, and hence the equilibrium (x^*, x^*, x^*, x^*) is unstable. $\square$

For the equilibrium $(0, x, 0, -x)$, we have $k_1 = k_3 = 1$, $k_2 = k_4 = k = f'(x)$. Then the associated characteristic equation becomes

$$\begin{aligned}&(1 + \lambda - \alpha e^{-\lambda\tau})(1 + \lambda - k\alpha e^{-\lambda\tau})\\ &\times\big(k(\alpha^2 - 4\beta^2)e^{-2\lambda\tau} - (1 + k)\alpha e^{-\lambda\tau}(1 + \lambda) + (1 + \lambda)^2\big) = 0.\end{aligned} \tag{3.5}$$

Theorem 3.4. *The equilibrium $(0, x, 0, -x)$ is unstable if it exists.*

Proof. It follows from Theorem 2.4 that if system (1.1) has a nonzero equilibrium $(0, x, 0, x)$ then $\alpha > 1$. However, the condition that $\alpha > 1$ implies that the factor $1 + \lambda - \alpha e^{-\lambda\tau}$ in equation (3.5) has a root with a positive real part, and hence that the characteristic equation of the equilibrium $(0, x, 0, x)$ has a root with a positive real part. Therefore, this kind of equilibrium is always unstable if it exists. $\square$

It follows from Theorem 2.6 that equilibrium $(u, -u, u, -u)$ exists if $\alpha - 2\beta > 1$. Note that $f'(u) = f'(-u)$, the characteristic equation evaluated at this equilibrium $(u, -u, u, -u)$ is same to that of the synchronous equilibria. Hence, similar arguments to the proof of Theorems 3.1 and 3.3 leads to the following result.

Theorem 3.5. *Assume that 1.1 has an equilibrium of the type $(u, -u, u, -u)$, let $k = f'(u)$, then for all $\tau \geq 0$, (i) the equilibrium $(u, -u, u, -u)$ is locally asymptotically stable if $k(|\alpha| + 2|\beta|) < 1$; (ii) the equilibrium $(u, -u, u, -u)$ is unstable if either $k(\alpha + 2\beta) > 1$, or $k(\alpha - 2\beta) > 1$, or $k\alpha > 1$.*

For the equilibrium (x_1, x_2, x_1, x_2), we have $k_3 = k_1 = f'(x_1)$, $k_4 = k_2 = f'(x_2)$. Then the characteristic equation becomes:

$$\begin{aligned}&(1 + \lambda - k_1\alpha e^{-\lambda\tau})(1 + \lambda - k_2\alpha e^{-\lambda\tau})\\ &\times (k_1k_2(\alpha^2 - 4\beta^2)e^{-2\lambda\tau} - (k_1 + k_2)\alpha e^{-\lambda\tau}(1 + \lambda) + (1 + \lambda)^2) = 0.\end{aligned} \tag{3.6}$$

Using similar arguments as the proof of Theorem 3.3, we have the following result.

Theorem 3.6. *The equilibrium (x_1, x_2, x_1, x_2) is unstable for any $\tau \geq 0$ if either $k_1\alpha > 1$ or $k_2\alpha > 1$, where $k_i = f'(x_i)$, $i = 1, 2$.*

For the equilibria (x_1, x_2, x_2, x_1) and $(x_1, x_2, -x_2, -x_1)$, we have $k_4 = k_1 = f'(\pm x_1)$, $k_3 = k_2 = f'(\pm x_2)$. Then the associated characteristic equation becomes

$$\begin{aligned}&k_1^2k_2^2\alpha^2(\alpha^2 - 4\beta^2)e^{-4\lambda\tau} - 2k_1k_2(k_1 + k_2)\alpha(\alpha^2 - 2\beta^2)e^{-3\lambda\tau}(1 + \lambda)\\ &+ ((k_1^2 + 4k_1k_2 + k_2^2)\alpha^2e^{-2\lambda\tau} - (k_1 + k_2)^2\beta^2e^{-2\lambda\tau})(1 + \lambda)^2\\ &- 2(k_1 + k_2)\alpha e^{-\lambda\tau}(1 + \lambda)^3 + (1 + \lambda)^4 = 0\end{aligned} \tag{3.7}$$

However, it is not easy to discuss the distribution of roots of the transcendental equation (3.7). Some relevant results will be reported later.

4. Discussion

In this paper, we investigate the behavior of a neural network model (1.1) consisting of four neurons with delayed self and nearest-neighbor connections. We give analytical results on the existence, patterns, and stability of equilibria of system (1.1). In fact, we may analyze the spatio-temporal patterns of all the bifurcated periodic solutions from the equilibria presented in this paper.

It turns out that we may encounter the following bifurcations of periodic solutions (see, for example, [9, 10]).

(i) Mirror-reflecting waves of the form $x_j(t) = x_{2-j}(t)$, $t \in \mathbb{R}$, $j \,(\mathrm{mod}\, 4)$;
(ii) Standing waves of the form $x_j(t) = x_{2-j}(t - \frac{p}{2})$, $t \in \mathbb{R}$, $j(\mathrm{mod}\ 4)$, where $p > 0$ is a period of x;
(iii) Discrete waves of the form $x_j(t) = x_{j+1}(t \pm \frac{kp}{4})$, $t \in \mathbb{R}$, $j(\mod 4)$, where $p > 0$ is a period of x.

Especially, the discrete waves are also called *synchronous oscillations* (if $k = 0\,(\mod 4)$) or *phase-locked oscillations* (if $k \neq 0\,(\mod 4)$) as each neuron oscillates just like others except not necessarily in phase with each other. Moreover, by using a similar arguments as that in [9], we may expect that the nontrivial synchronous equilibria will have the same types of bifurcations as the trivial equilibrium. Namely, the nontrivial synchronous equilibria given in 2.1 may undergo either a standard Hopf bifurcation leading to synchronous oscillations about both of the nontrivial synchronous equilibria, or an equivariant Hopf bifurcation leading to 10 branches of oscillations: 4 phase-locked, 4 standing wave and 2 mirror reflecting. However, it is much more interesting to discuss the spatio-temporal patterns of periodic solutions bifurcated from the asynchronous equilibria obtained in Theorem 2.2–2.7. Some results about this will be reported later.

Acknowledgements. This work was partially supported by grants: 10601016 from the National Natural Science Foundation of China, [2007]70 from the Program for New Century Excellent Talents in University of Education Ministry of China, and 06JJ3001 from the Hunan Provincial Natural Science Foundation.

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