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# POSITIVE SOLUTIONS TO NONLINEAR SECOND-ORDER THREE-POINT BOUNDARY-VALUE PROBLEMS FOR DIFFERENCE EQUATION WITH CHANGE OF SIGN

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Abstract. In this paper we investigate the existence of positive solution to the discrete second-order three-point boundary-value problem

$$\Delta^2 x_{k-1} + h(k)f(x_k) = 0, \quad k \in [1,n],$$
$$x_0 = 0, \quad ax_l = x_{n+1},$$

where $n \in [2,\infty)$, $l \in [1,n]$, $0 < a < 1$, $(1-a)l \geq 2$, $(1+a)l \leq n+1$, $f \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $h(t)$ is a function that may change sign on $[1,n]$. Using the fixed-point index theory, we prove the existence of positive solution for the above boundary-value problem.

## 1. Introduction

Recently, some authors considered the existence of positive solutions to discrete boundary-value problems and obtained some existence results; see for example,[1, 2, 3, 6, 7]. Motivated by the papers [5, 8], we consider the existence of positive solution for the nonlinear discrete three-point boundary-value problem

$$\begin{gathered} \Delta^2 x_{k-1} + h(k)f(x_k) = 0, \quad k \in [1,n], \\ x_0 = 0, \quad ax_l = x_{n+1}, \end{gathered} \tag{1.1}$$

where $n \in \{2,3,\dots\}$, $l \in [1,n] = \{1,2,\dots,n\}$, $0 < a < 1$, $(1-a)l \geq 2$, $(1+a)l \leq n+1$, $f \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $h \in C(\mathbb{R}^+,\mathbb{R})$.

When $h(k) \equiv 1$, Equation (1.1) reduces to the nonlinear discrete three-point boundary-value problem studied by Zhang and Medina [8]. Using the same approach as in [8], we obtain the the existence of positive solutions to (1.1) when $h \in C(\mathbb{R}^+,\mathbb{R}^+)$. To the author's knowledge, no one has studied the existence of positive solution for (1.1) when $h$ is allowed to change sign on $[1,n]$. Hence, the aim of the present paper is to establish simple criteria for the existence of at least one positive solution of the (1.1). Our main tool is the fixed-point index theory [4].

**Theorem 1.1** ([4]). *Suppose $E$ is a real Banach space, $K \subset E$ is a cone, and $\Omega_r = \{u \in K : \|u\| \leq r\}$. Let the operator $T : \Omega_r \to K$ be completely continuous and satisfy $Tx \neq x$, for all $x \in \partial\Omega_r$. Then*

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(i) *If* $\|Tx\| \leq \|x\|$, *for all* $x \in \partial\Omega_r$, *then* $i(T, \Omega_r, K) = 1$;
(ii) *If* $\|Tx\| \geq \|x\|$, *for all* $x \in \partial\Omega_r$, *then* $i(T, \Omega_r, K) = 0$.

In this paper, by a positive solution $x$ of (1.1), we mean a solution of the (1.1) satisfying $x_k > 0$, $k \in [1, n+1]$.

We will use the following notation: $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$; $\mathbb{N} = \{0, 1, 2, \dots\}$; $[m, n] = \{m, m+1, m+2, \dots, n\} \subset \mathbb{Z}$; $[x]$ is the integer value function; $\Delta y_k = y_{k+1} - y_k$, $\Delta^n y_k = \Delta(\Delta^{n-1} y_k)$, $n \geq 2$, $k \in \mathbb{N}$.

Moreover, we shall use the following assumptions:

(H1) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ is continuous and nondecreasing.
(H2) $h : [1, n] \to (-\infty, +\infty)$ such that $h(k) \geq 0$, $k \in [1, l]; h(k) \leq 0$, $k \in [l, n]$. Moreover, $h(k)$ does not vanish identically on any subinterval of $[1, n]$.
((H3) There exist nonnegative constants in the extended reals, $f_0$, $f_\infty$, such that
$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}.$$

This paper is organized as follows: In section 2, preliminary lemmas are given. In section 3, we prove the existence of positive solutions for (1.1). In section an example is provided.

## 2. Preliminaries

In this section, we give some lemmas that will be used to prove our main results.

**Lemma 2.1** ([8]). *Let* $al \neq n+1$. *For* $\{y_k\}_{k=0}^{n+1}$, *the problem*
$$\begin{gathered} \Delta^2 x_{k-1} + y_k = 0, \quad k \in [1, n], \\ x_0 = 0, \quad a x_l = x_{n+1}\,. \end{gathered} \tag{2.1}$$
*has a unique solution*
$$x_k = \frac{k}{n+1-al}\Big(\sum_{i=0}^{n}\sum_{j=0}^{i} y_j - a\sum_{i=0}^{l-1}\sum_{j=0}^{i} y_j\Big) - \sum_{i=0}^{k-1}\sum_{j=0}^{i} y_j, \quad k \in [0, n+1].$$

Using the above lemma, for $0 < a < 1$, it is easy to prove the following result.

**Lemma 2.2.** *The Green's function for* (2.1) *is*
$$G(k, j) = \begin{cases} \dfrac{((a-1)k + n + 1 - al)j}{n+1-al}, & j < k,\ j < l, \\ \dfrac{((a-1)j + n + 1 - al)k}{n+1-al}, & k \leq j < l, \\ \dfrac{al(k-j) + (n+1-k)j}{n+1-al}, & l < j \leq k, \\ \dfrac{(n+1-j)k}{n+1-al}, & j \geq k,\ j \geq l, \end{cases} \tag{2.2}$$

We want to point that this Green's function is new.

**Remark 2.3.** Note that $G(k, j) \geq 0$ for $(k, j) \in [0, n+1] \times [1, n]$.

**Lemma 2.4.** *Let* $(1-a)l \geq 2$. *For all* $j_1 \in [\tau, l]$ *and* $j_2 \in [l, n]$, *we have*
$$G(k, j_1) \geq M G(k, j_2), \quad k \in [0, n+1], \tag{2.3}$$
*where* $\tau \in [[al] + 1, l - 1]$ *and* $M = a^2 l / n$.

*Proof.* It is easy to check that $(1-a)l \geq 2$ implies $[[al]+1, l-1]$. We divide the proof into two cases.
**Case 1:** $k \leq l$. By (2.2),

$$
\begin{aligned}
\frac{G(k,j_1)}{G(k,j_2)} &= \begin{cases} \frac{[(a-1)k+n+1-al]j_1}{(n+1-j_2)k}, & j_1 \leq k, \\ \frac{[(a-1)j_1+n+1-al]k}{(n+1-j_2)k}, & k < j_1 \end{cases} \\
&\geq \begin{cases} \frac{(n+1-l)\tau}{(n+1-l)k}, & j_1 \leq k, \\ \frac{(n+1-l)k}{(n+1-l)k}, & k < j_1 \end{cases} \\
&\geq \min\{\frac{\tau}{l}, 1\} \\
&= \frac{\tau}{l} \\
&\geq a > M.
\end{aligned}
$$

**Case 2:** $k \geq l$. For $j_2 \in [l,n]$ and $(1+a)l \leq n+1$, it is easy to check that

$$
\begin{aligned}
(n+1-k-al)j_2 + alk &= (n+1-al)j_2 + (al-j_2)k \\
&\leq (n+1-al)j_2 + (al-j_2)l \\
&= (n+1-al-l)j_2 + al^2 \\
&\leq (n+1-al-l)n + al^2 \\
&= (n-al)(n+1-l) + al,
\end{aligned} \tag{2.4}
$$

and

$$
(a-1)k+n+1-al \geq (a-1)(n+1)+n+1-al = a(n+1-l), \tag{2.5}
$$

From (2.2), (2.4) and (2.5), we obtain

$$
\begin{aligned}
\frac{G(k,j_1)}{G(k,j_2)} &= \begin{cases} \frac{[(a-1)k+n+1-al]j_1}{(n+1-k-al)j_2+alk}, & j_2 \leq k, \\ \frac{[(a-1)k+n+1-al]j_1}{(n+1-j_2)k}, & k < j_2 \end{cases} \\
&\geq \begin{cases} \frac{a(n+1-l)\tau}{(n-al)(n+1-l)+al}, & j_2 \leq k, \\ \frac{a(n+1-l)\tau}{(n+1-l)n}, & k < j_2 \end{cases} \\
&\geq \min\Big\{\frac{a^2l(n+1-l)}{(n-al)(n+1-l)+al}, \frac{a^2l}{n}\Big\} \\
&= \frac{a^2l}{n} = M.
\end{aligned}
$$

Hence, $G(t,j_1) \geq MG(t,j_2)$ holds. $\square$

Let $C[0,n+1]$ be the Banach space with the norm $\|x\| = \sup_{k\in[0,n+1]}|x_k|$. Denote

$$
C_0^+[0,n+1] = \{x \in C[0,n+1] : \min_{k\in[0,n+1]} x_k \geq 0 \text{ and } x_0 = 0,\ x_{n+1} = ax_l\},
$$

$$P = \{x \in C_0^+[0, n+1] : x_k \text{ is concave on } [0, l] \text{ and convex on } [l, n+1]\}.$$

It is obvious that $P$ is a cone in $C[0, n+1]$.

**Lemma 2.5.** *If $x \in P$, then*

$$x_k \geq A(k)x_l, \quad k \in [0, l]; \quad x_k \leq A(k)x_l, \quad k \in [l, n+1].$$

*where*

$$A(k) = \begin{cases} k/l, & k \in [0, l], \\ \frac{n+1-al+(a-1)k}{n+1-l}, & k \in [l+1, n+1], \end{cases}$$

*Proof.* Since $x \in P$, we have $x_k$ is concave on $[0, l]$, convex on $[l, n+1]$, $x_0 = 0$, and $x_{n+1} = ax_l$. Thus,

$$x_k \geq x_0 + \frac{x_l - x_0}{l} k = \frac{k}{l} x_l, \quad \text{for } k \in [0, l],$$

and

$$x_k \leq x_{n+1} + \frac{x_{n+1} - x_l}{n+1-l}(k-1-n) = \frac{n+1-al+(a-1)k}{n+1-l} x_l, \quad \text{for } k \in [l, n+1].$$

Hence, we have

$$x_k \geq A(k)x_l, \quad k \in [0, l]; \quad x_k \leq A(k)x_l, \quad k \in [l, n+1].$$

□

**Lemma 2.6.** *Assume that $(1-a)l \geq 2$. Let $x \in P$, then*

$$x_k \geq \mu\|x\|, \quad k \in [[al]+1, \tau],$$

*where $\mu = \min\{a, 1 - \frac{\tau}{l}\}$, $\tau \in [[al]+1, l-1]$.*

*Proof.* Let $x \in P$, then $x$ is concave on $[0, l]$, and convex on $[l, n+1]$. Since $0 < a < 1$, $x_{n+1} = ax_l < x_l$, then $\|x\| = \sup_{k\in[0,n+1]} |x_k| = \sup_{k\in[0,l]} |x_k|$. Set $r = \inf\{r \in [0, l] : \sup_{k\in[0,l]} x_k = x_r\}$. We now consider the following two cases:
**Case (i):** $k \in [0, r]$. By the concavity of $x_k$, we have

$$x_k \geq x_0 + \frac{x_r - x_0}{r} k = \frac{k}{r} x_r \geq \frac{k}{l} x_r = \frac{k}{l}\|x\|.$$

**Case(ii):** $k \in [r, l]$. Similarly, we obtain

$$\begin{aligned} x_k &\geq x_r + \frac{x_r - x_l}{r - l}(k - r) \\ &= \frac{l-k}{l-r} x_r + \frac{k-r}{l-r} x_l \\ &\geq \frac{l-k}{l-r} x_r \geq \frac{l-k}{l} x_r \\ &= \big[1 - \frac{k}{l}\big]\|x\|. \end{aligned}$$

Thus, we have

$$x_k \geq \min\big\{\frac{k}{l}, 1 - \frac{k}{l}\big\}\|x\|, \quad k \in [0, l],$$

which yields

$$\min_{k\in[[al]+1,\tau]} x_k \geq \min\{a, 1 - \frac{\tau}{l}\}\|x\| = \mu\|x\|.$$

The proof is completed. □

**Lemma 2.7.** *Assume that* $(1-a)l \geq 2$. *If conditions* (H1), (H2), (H4) *hold* $\forall k \in [0, n-l]$, *then there exists a constant* $\tau \in [[al]+1, l-2]$ *such that*

$$B(k) = h^+(l - [\delta k]) - \frac{1}{M} h^-(l+k) \geq 0, \tag{2.6}$$

*where* $h^+(k) = \max\{h(k), 0\}, h^-(k) = -\min\{h(k), 0\}$, $\delta = \frac{l-\tau-1}{n-l+1}$, *and* $M = a^2 l/n$. *Then for all* $q \in [0, \infty)$, *we have*

$$\sum_{j=\tau+1}^{n} G(k,j)h(j)f(qA(j)) \geq 0. \tag{2.7}$$

*Proof.* By the definition of $A(k)$, it is easy to check that

$$A(l - [\delta r]) = \frac{1}{l}\Big(l - [\frac{l-\tau-1}{n-l+1} r]\Big) = 1 - \frac{1}{l}[\frac{l-\tau-1}{n-l+1} r], \quad r \in [0, n-l+1], \tag{2.8}$$

and

$$A(l+r) = 1 - \frac{r}{n-l+1}(1-a), \quad r \in [0, n-l]. \tag{2.9}$$

Set $j = l - [\delta r]$, $r \in [0, n-l+1]$ ($\delta$ as in (H4)). For all $q \in [0, \infty)$, by view of Lemma 2.4, Remark 2.3, (2.6), (2.8), (2.9), (H4), and that $f$ is nondecreasing, we have

$$\begin{aligned}
&\sum_{j=\tau+1}^{l} G(k,j)h^+(j)f(qA(j)) \\
&= \sum_{r=0}^{n-l+1} G(k, l-[\delta r])h^+(l-[\delta r])f(qA(l-[\delta r])) \\
&= \sum_{r=0}^{n-l+1} G(k, l-[\delta r])h^+(l-[\delta r])f\Big(q\big(1 - \frac{1}{l}[\frac{l-\tau-1}{n-l+1} r]\big)\Big) \\
&\geq \sum_{r=0}^{n-l+1} G(k, l-[\delta r])h^+(l-[\delta r])f\Big(q\big(1 - \frac{r}{n-l+1}\big(1 - \frac{\tau+1}{l}\big)\big)\Big) \\
&\geq M\sum_{r=0}^{n-l} G(k, l+r)h^+(l-[\delta r])f\Big(q\big(1 - \frac{r}{n-l+1}\big(1 - \frac{\tau+1}{l}\big)\big)\Big) \\
&\geq \sum_{r=0}^{n-l} G(k, l+r)h^-(l+r)f\Big(q\big(1 - \frac{r}{n-l+1}(1-a)\big)\Big).
\end{aligned} \tag{2.10}$$

Again, setting $j = l + r$, $r \in [0, n-l]$, for $q \in [0, \infty)$, we obtain

$$\sum_{j=l+1}^{n} G(k,j)h^-(j)f(qA(j)) = \sum_{r=1}^{n-l} G(k, l+r)h^-(l+r)f\Big(q\big(1 - \frac{r}{n-l+1}(1-a)\big)\Big). \tag{2.11}$$

Thus, by (2.10) and (2.11), we get

$$\begin{aligned}
&\sum_{j=\tau+1}^{n} G(k,j)h(j)f(qA(j)) \\
&= \sum_{j=\tau+1}^{l} G(k,j)h^+(j)f(qA(j)) - \sum_{j=l+1}^{n} G(k,j)h^-(j)f(qA(j)) \geq 0.
\end{aligned}$$

The proof is completed. □

We define the operator $T: C[0,n+1] \to C[0,n+1]$ by

$$(Tx)_k = \sum_{j=1}^{n} G(k,j)h(j)f(x_j), \quad (k,j) \in [0,n+1] \times [1,n]. \tag{2.12}$$

where $G(k,j)$ as in (2.2). From Lemma 2.4, we easily know that $x(t)$ is a solution of the (1.1) if and only if $x(t)$ is a fixed point of the operator $T$.

**Lemma 2.8.** *Let* $(1-a)l \geq 2$. *Assume that conditions* (H1), (H2),(H4) *are satisfied. Then $T$ maps $P$ into $P$.*

*Proof.* For $x \in P$, by Lemmas 2.5, 2.7, and $f$ is nondecreasing, we have

$$\begin{aligned}
&\sum_{j=\tau+1}^{n} G(k,j)h(j)f(x_j) \\
&= \sum_{j=\tau}^{l} G(k,j)h^+(j)f(x_j) - \sum_{j=l+1}^{n} G(k,j)h^-(j)f(x_j) \\
&\geq \sum_{j=\tau+1}^{l} G(k,j)h^+(j)f(A(j)x_l) - \sum_{j=l+1}^{n} G(k,j)h^-(j)f(A(j)x_l) \\
&= \sum_{j=\tau+1}^{n} G(k,j)h(j)f(x_l A(j)) \geq 0,
\end{aligned} \tag{2.13}$$

which implies

$$\begin{aligned}
(Tx)_k &= \sum_{j=1}^{n} G(k,j)h(j)f(x_j) \\
&= \sum_{j=1}^{\tau} G(k,j)h^+(j)f(x_j) + \sum_{j=\tau+1}^{n} G(k,j)h(j)f(x_j) \\
&\geq \sum_{j=1}^{\tau} G(k,j)h^+(j)f(x_j) \geq 0,
\end{aligned}$$

again $(Tx)_0 = 0$, $(Tx)_{n+1} = a(Tx)_l$, it follows that $T: P \to C_0^+[0,n+1]$. On the other hand,

$$\begin{gathered}
\Delta^2(Tx)_k = -h^+(j)f(x_j) \leq 0, \quad j \in [0,l], \\
\Delta^2(Tx)_k = h^-(j)f(x_j) \geq 0, \quad j \in [l,n+1].
\end{gathered}$$

Thus, $T$ maps $P$ into $P$. □

**Lemma 2.9.** *Let* $(1-a)l \geq 2$. *Assume that conditions* (H1), (H2), (H4) *are satisfied. If $z \in P$ is a fixed point of $T$ and $\|z\| > 0$, then $z$ is a positive solution of the* (1.1).

*Proof.* At first, we claim that $z_l > 0$. Otherwise, $z_l = 0$ implies $z_{n+1} = az_l = 0$. By the convexity and the nonnegativity of $z$ on $[l,n+1]$, we have

$$z_k \equiv 0, \quad k \in [l,n+1],$$

this implies $\Delta z_l = z_{l+1} - z_l = 0$. Since $z = Tz$, we have $\Delta^2 z_k = -h^+(k)f(z_k) \le 0$, $k \in [0, l]$. Then

$$\Delta z_k \ge \Delta z_l = 0, \quad k \in [0, l-1].$$

Thus, $z_k \le z_l = 0$, $k \in [0, l]$. By the nonnegativity of $z$, we get

$$z_k \equiv 0, \quad k \in [0, l],$$

which yields a contradiction with $\|z\| > 0$. □

Next, in view of Lemma 2.1, for $z \in P$, we have

$$z_k \ge \frac{k}{l} z_l > 0, \quad k \in [1, l]. \tag{2.14}$$

Note that $h(k)$ does not vanish identically on any subinterval of $k \in [1, l]$, for any $k \in [1, n]$. By (2.13) we have

$$\begin{aligned} z_k &= (Tz)_k \\ &= \sum_{j=1}^{n} G(k,j)h(j)f(z_j) \\ &= \sum_{j=1}^{\tau} G(k,j)h^+(j)f(z_j) + \sum_{j=\tau}^{n} G(k,j)h(j)f(z_j) \\ &\ge \sum_{j=1}^{\tau} G(k,j)h^+(j)f(z_j) > 0. \end{aligned}$$

Thus, we assert that $z$ is a positive solution of (1.1).

## 3. Existence of solutions

For convenience, we set

$$M = \Big(\mu \max_{k \in [0,n+1]} \sum_{j=[al]+1}^{\tau} G(k,j)h^+(j)\Big)^{-1},$$

$$m = \Big(\max_{k \in [0,n+1]} \sum_{j=1}^{l} G(k,j)h^+(j)\Big)^{-1}$$

where $\mu$ as in Lemma 2.6.

**Theorem 3.1.** *Let* $(1-a)l \ge 2$. *Assume that conditions* (H1)–(H4) *are satisfied. If* (H5), $0 \le f_0 < m$, *and* $M < f_\infty \le +\infty$ *hold, then* (1.1) *has at least one positive solution.*

*Proof.* By Lemma 2.8, $T : P \to P$. Moreover, it is easy to check by Arzela-Ascoli theorem that $T$ is completely continuous. By (H5), we have $f_0 < m$. There exist $\rho_1 > 0$ and $\varepsilon_1 > 0$ such that

$$f(u) \le (m - \varepsilon_1)u, \quad for \quad 0 < u \le \rho_1. \tag{3.1}$$

Let $\Omega_1 = \{x \in P : \|x\| < \rho_1\}$. For $x \in \partial\Omega_1$, by (3.1), we have

$$(Tx)_k = \sum_{j=1}^{n} G(k,j)h(j)f(x_j)$$

$$
\begin{aligned}
&= \sum_{j=1}^{l} G(k,j)h^{+}(j)f(x_j) - \sum_{k=l+1}^{n} G(k,j)h^{-}(j)f(x_j) \\
&\le \sum_{j=1}^{l} G(k,j)h^{+}(j)f(x_j) \\
&\le \rho_1(m-\varepsilon_1) \max_{k\in[0,n+1]} \sum_{j=1}^{l} G(k,j)h^{+}(j) \\
&= \rho_1(m-\varepsilon_1)\frac{1}{m} \\
&< \rho_1 = \|x\|,
\end{aligned}
$$

which yields $\|Tx\| < \|x\|$ for $x \in \partial\Omega_1$. Then by Theorem 3.1, we have

$$
i(T,\Omega_1,P) = 1. \tag{3.2}
$$

On the other hand, from (H5), we have $f_\infty > M$. There exist $\rho_2 > \rho_1 > 0$ and $\varepsilon_2$ such that

$$
f(u) \ge (M+\varepsilon_2)u, \quad \text{for} \quad u \ge \mu\rho_2. \tag{3.3}
$$

Set $\Omega_2 = \{x \in P : \|x\| < \rho_2\}$. For any $x \in \partial\Omega_2$, from Lemma 2.6, we have $x_k \ge \mu\|x\| = \mu\rho_2$, for $k \in [[al]+1,\tau]$. Then from (2.7) and (3.3), we obtain

$$
\begin{aligned}
\|Tx\| &= \max_{k\in[0,n+1]} \Big[ \sum_{j=1}^{\tau} G(k,j)h^{+}(j)f(x_j) + \sum_{j=\tau+1}^{n} G(k,j)h(j)f(x_j) \Big] \\
&\ge \max_{k\in[0,n+1]} \sum_{j=1}^{\tau} G(k,j)h^{+}(j)f(x_j) \\
&\ge \max_{k\in[0,n+1]} \sum_{j=[al]+1}^{\tau} G(k,j)h^{+}(j)f(x_j) \\
&\ge \mu(M+\varepsilon_2)\rho_2 \max_{k\in[0,n+1]} \sum_{j=[al]+1}^{\tau} G(k,j)h^{+}(j) \\
&= \frac{1}{M}(M+\varepsilon_2)\rho_2 > \rho_2 = \|x\|;
\end{aligned}
$$

this is, $\|Tx\| > \|x\|$, for $x \in \partial\Omega_2$. Then, by Theorem 3.1,

$$
i(T,\Omega_2,P) = 0. \tag{3.4}
$$

Therefore, by (3.2), (3.4), and $\rho_1 < \rho_2$, we have

$$
i(T,\Omega_2\setminus\bar{\Omega}_1,P) = -1.
$$

Then operator $T$ has a fixed point in $\Omega_2 \setminus \bar{\Omega}_1$. So, (1.1) has at least one positive solution. $\square$

**Theorem 3.2.** *Let* $(1-a)l \ge 2$. *Assume that* (H1)–(H4) *are satisfied. If* (H6), $M < f_0 \le +\infty$, *and* $0 \le f_\infty < m$ *hold, then* (1.1) *has at least one positive solution.*

*Proof.* At first, by (H6), we get $f_0 > M$, there exist $\rho_3$ and $\varepsilon_3$ such that

$$
f(u) \ge (M+\varepsilon_3)u, \quad \text{for } 0 < u < \rho_3. \tag{3.5}
$$

Set $\Omega_3 = \{x \in P : \|x\| < \rho_3\}$. For any $x \in \partial\Omega_3$, by Lemma 2.6, we get $x_k \geq \mu\|x\| = \mu\rho_3$, for $k \in [[al]+1, \tau]$, then by (2.7) and (3.5), we have

$$\begin{aligned}
\|Tx\| &= \max_{k\in[0,n+1]} \Big[\sum_{k=1}^{\tau} G(k,j)h^+(j)f(x_j) + \sum_{j=\tau+1}^{n} G(k,j)h(j)f(x_j)\Big] \\
&\geq \max_{k\in[0,n+1]} \sum_{j=1}^{\tau} G(k,j)h^+(j)f(x_j) \\
&\geq \max_{k\in[0,n+1]} \sum_{j=[al]+1}^{\tau} G(k,j)h^+(j)f(x_j) \\
&\geq \mu(M+\varepsilon_3)\rho_3 \max_{k\in[0,n+1]} \sum_{j=[al]+1}^{\tau} G(k,j)h^+(j) \\
&= \frac{1}{M}(M+\varepsilon_3)\rho_3 > \rho_3 = \|x\|;
\end{aligned}$$

this is, $\|Tx\| > \|x\|$, for $x \in \partial\Omega_3$. Thus, by Theorem 1.1,

$$i(T, \Omega_3, P) = 0. \tag{3.6}$$

Next, from (H6), we have $f_\infty < m$. There exist $\rho_4 > 0$ and $0 < \varepsilon_4 < \rho_4$ such that

$$f(u) \leq (m - \varepsilon_4)u, \quad \text{for } u \geq \rho_4.$$

Set $L = \max_{0\leq u\leq\rho_4} f(u)$. Then

$$f(u) \leq L + (m-\varepsilon_4)u, \quad \text{for} \quad u \geq 0. \tag{3.7}$$

Choose $\rho_5 > \max\{\rho_4, L/\varepsilon_4\}$. Let $\Omega_4 = \{x \in P : \|x\| < \rho_5\}$. Then for $x \in \partial\Omega_4$, by (2.7) and (3.7), we have

$$\begin{aligned}
(Tx)_k &= \sum_{j=1}^{n} G(k,j)h(j)f(x_j) \\
&= \sum_{j=1}^{l} G(k,j)h^+(j)f(x_j) - \sum_{j=l+1}^{n} G(k,j)h^-(j)f(x_j) \\
&\leq \sum_{j=1}^{l} G(k,j)h^+(j)f(x_j) \\
&\leq (L + (m-\varepsilon_4)\rho_5)\frac{1}{m} \\
&= \rho_5 - (\varepsilon_4\rho_5 - L)\frac{1}{m} \\
&< \rho_5 = \|x\|.
\end{aligned}$$

Thus, by Theorem 3.1,

$$i(T, \Omega_4, P) = 1. \tag{3.8}$$

Therefore, by (3.6), (3.8), and $\rho_3 < \rho_5$, we have

$$i(T, \Omega_4 \setminus \bar{\Omega}_3, P) = 1.$$

Then operator $T$ has a fixed point in $\Omega_4 \setminus \bar{\Omega}_3$. So, (1.1) has at least one positive solution. $\square$

## 4. Example

In this section, we illustrates our main results. Consider the boundary-value problem

$$
\begin{gathered}
\Delta^2 x_{k-1} + h(k) x_k^\alpha = 0, \quad k \in [1, 11], \\
x_0 = 0, \quad \frac{1}{3} x_6 = x_{12},
\end{gathered}
\tag{4.1}
$$

where $0 < \alpha < 1$, and

$$
h(k) = \begin{cases} 3(k-8)^2, & k \in [1, 8], \\ \frac{8}{99}(8-k)^3, & k \in [8, 11]. \end{cases}
$$

Let $n = 11$, $l = 8$, $a = 1/3$, then we have $M = 8/99$. Now taking $\tau = 3$, then $\tau \in [[al] + 1, l - 2] = [3, 6]$, $\delta = 1$, and for all $k \in [0, n - l] = [0, 3]$, we have

$$
B(k) = h^+(l - [\delta k]) - \frac{1}{M} h^-(l + k) = k^2(3 - k) \geq 0.
$$

Hence, Condition (H2) and (H4) hold. Set $f(u) = u^\alpha$, it is easy to see that

$$
f_0 = \infty, \quad f_\infty = 0,
$$

that is, (H6) holds. Thus, by Theorem 3.2, (4.1) has at least one positive solution.

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