POSITIVE SOLUTIONS TO NONLINEAR SECOND-ORDER THREE-POINT BOUNDARY-VALUE PROBLEMS FOR DIFFERENCE EQUATION WITH CHANGE OF SIGN

CHUNLI WANG, XIAOSHUANG HAN, CHUNHONG LI

Abstract. In this paper we investigate the existence of positive solution to the discrete second-order three-point boundary-value problem

\[ \Delta^2 x_{k-1} + h(k)f(x_k) = 0, \quad k \in [1, n], \]
\[ x_0 = 0, \quad ax_l = x_{n+1}, \]

where \( n \in [2, \infty), \ l \in [1, n], \ 0 < a < 1, \ (1-a)l \geq 2, \ (1+a)l \leq n + 1, \ f \in C(\mathbb{R}^+, \mathbb{R}^+) \) and \( h(t) \) is a function that may change sign on \([1, n]\). Using the fixed-point index theory, we prove the existence of positive solution for the above boundary-value problem.

1. Introduction

Recently, some authors considered the existence of positive solutions to discrete boundary-value problems and obtained some existence results; see for example, [1, 2, 3, 6, 7]. Motivated by the papers [5, 8], we consider the existence of positive solution for the nonlinear discrete three-point boundary-value problem

\[ \Delta^2 x_{k-1} + h(k)f(x_k) = 0, \quad k \in [1, n], \]
\[ x_0 = 0, \quad ax_l = x_{n+1}, \]

\[ (1.1) \]

where \( n \in \{2, 3, \ldots \}, \ l \in [1, n] = \{1, 2, \ldots, n\}, \ 0 < a < 1, \ (1-a)l \geq 2, \ (1+a)l \leq n + 1, \ f \in C(\mathbb{R}^+, \mathbb{R}^+) \) and \( h(t) \) is \( C(\mathbb{R}, \mathbb{R}) \) and \( h(t) \) is a function that may change sign on \([1, n]\). Using the same approach as in [8], we obtain the existence of positive solutions to (1.1) when \( h \in C(\mathbb{R}^+, \mathbb{R}) \). To the author's knowledge, no one has studied the existence of positive solution for (1.1) when \( h \) is allowed to change sign on \([1, n]\). Hence, the aim of the present paper is to establish simple criteria for the existence of at least one positive solution of the (1.1). Our main tool is the fixed-point index theory [4].

Theorem 1.1 ([4]). Suppose \( E \) is a real Banach space, \( K \subset E \) is a cone, and \( \Omega_r = \{u \in K : \|u\| \leq r\} \). Let the operator \( T : \Omega_r \to K \) be completely continuous and satisfy \( Tx \neq x \), for all \( x \in \partial \Omega_r \). Then

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Lemma 2.1. Example is provided.

In section 3, we prove the existence of positive solutions for (1.1). In section 2 C. WANG, X. HAN, C. LI EJDE-2008/87

Moreover, we shall use the following assumptions:

(H1) \( f \in C(\mathbb{R}^+, \mathbb{R}^+) \) is continuous and nondecreasing.

(H2) \( h : [1, n] \to (-\infty, +\infty) \) such that \( h(k) \geq 0, \ k \in [1, l]; h(k) \leq 0, \ k \in [l, n]. \)

Moreover, \( h(k) \) does not vanish identically on any subinterval of \([1, n]\).

(H3) There exist nonnegative constants in the extended reals, \( f_0, f_\infty \), such that

\[
 f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}. 
\]

This paper is organized as follows: In section 2, preliminary lemmas are given. In section 3, we prove the existence of positive solutions for (1.1). In section an example is provided.

2. Preliminaries

In this section, we give some lemmas that will be used to prove our main results.

Lemma 2.1 ([S]). Let \( n \neq n + 1 \). For \( \{y_k\}_{k=0}^{n+1} \), the problem

\[
 \Delta^2 x_{k-1} + y_k = 0, \quad k \in [1, n],
\]

\[
 x_0 = 0, \quad a x_l = x_{n+1}. \tag{2.1}
\]

has a unique solution

\[
 x_k = \frac{k}{n + 1 - al} \left( \sum_{i=0}^{n} \sum_{j=0}^{i} y_j - a \sum_{i=0}^{l-1} \sum_{j=0}^{i} y_j \right) - \sum_{i=0}^{k-1} \sum_{j=0}^{i} y_j, \quad k \in [0, n+1].
\]

Using the above lemma, for \( 0 < a < 1 \), it is easy to prove the following result.

Lemma 2.2. The Green’s function for (2.1) is

\[
 G(k, j) = \begin{cases} 
 (a - 1)k + n + 1 - al \frac{1}{j}, & j < k, j < l, \\
 n + 1 - al & k \leq j < l, \\
 al(k - j) + (n + 1 - k)j & l < j \leq k, \\
 n + 1 - al & j \geq k, j \geq l.
\end{cases} \tag{2.2}
\]

We want to point that this Green’s function is new.

Remark 2.3. Note that \( G(k, j) \geq 0 \) for \((k, j) \in [0, n+1] \times [1, n]\).

Lemma 2.4. Let \((1 - a)l \geq 2\). For all \( j_1 \in [\tau, l] \) and \( j_2 \in [l, n] \), we have

\[
 G(k, j_1) \geq MG(k, j_2), \quad k \in [0, n+1], \tag{2.3}
\]

where \( \tau \in [al] + 1, l - 1 \) and \( M = a^2l/n \).
Proof. It is easy to check that \((1 - a)l \geq 2\) implies \([[al] + 1, l - 1]\). We divide the proof into two cases.

Case 1: \(k \leq l\). By (2.2),

\[
G(k, j_1) \quad \frac{[(a - 1)k + n + 1 - al]j_1}{(n + 1 - j_2)k}, \quad j_1 \leq k,
\]

\[
\geq \frac{(n + 1 - l)\tau}{(n + 1 - l)k}, \quad j_1 \leq k,
\]

\[
\geq \frac{(n + 1 - l)k}{(n + 1 - l)k}, \quad k < j_1
\]

\[
\geq \min \{\frac{\tau}{l}, 1\}
\]

\[
= \frac{\tau}{l}
\]

\[
\geq a > M.
\]

Case 2: \(k \geq l\). For \(j_2 \in [l, n]\) and \((1 + a)l \leq n + 1\), it is easy to check that

\[
(n + 1 - k - al)j_2 + alk = (n + 1 - al)j_2 + (al - j_2)k
\]

\[
\leq (n + 1 - al)j_2 + (al - j_2)l
\]

\[
= (n + 1 - al - l)j_2 + al^2
\]

\[
\leq (n + 1 - al - l)n + al^2
\]

\[
= (n - al)(n + 1 - l) + al,
\]

and

\[
(a - 1)k + n + 1 - al \geq (a - 1)(n + 1) + n + 1 - al = a(n + 1 - l),
\]

(2.5)

From (2.2), (2.4) and (2.5), we obtain

\[
G(k, j_1) \quad \frac{[(a - 1)k + n + 1 - al]j_1}{(n + 1 - j_2)k}, \quad j_2 \leq k,
\]

\[
\geq \frac{a(n + 1 - l)\tau}{(n - al)(n + 1 - l) + al^2}, \quad j_2 \leq k,
\]

\[
\geq \frac{a(n + 1 - l)\tau}{(n + 1 - l)k}, \quad k < j_2
\]

\[
\geq \min \{\frac{a^2l(n + 1 - l)}{(n - al)(n + 1 - l) + al^2}, \frac{a^2l}{n}\}
\]

\[
= a^2l
\]

\[
= M.
\]

Hence, \(G(t, j_1) \geq MG(t, j_2)\) holds. \(\Box\)

Let \(C[0, n + 1]\) be the Banach space with the norm \(\|x\| = \sup_{k \in [0, n + 1]} |x_k|\).

Denote

\[
C_0^+ [0, n + 1] = \{x \in C[0, n + 1]: \min_{k \in [0, n + 1]} x_k \geq 0 \text{ and } x_0 = x_{n+1} = ax_1\},
\]

\[
C_0^+ [0, n + 1] = \{x \in C[0, n + 1]: \min_{k \in [0, n + 1]} x_k \geq 0 \text{ and } x_0 = 0, x_{n+1} = ax_1\},
\]
\[ P = \{ x \in C^+_0[0, n + 1] : x_k \text{ is concave on } [0, l] \text{ and convex on } [l, n + 1] \} . \]

It is obvious that \( P \) is a cone in \( C[0, n + 1] \).

**Lemma 2.5.** If \( x \in P \), then
\[ x_k \geq A(k)x_l, \quad k \in [0, l]; \quad x_k \leq A(k)x_l, \quad k \in [l, n + 1]. \]

where
\[ A(k) = \begin{cases} \frac{k}{l}, & k \in [0, l], \\ \frac{n+1-al+(a-1)k}{n+1-l}, & k \in [l+1, n+1]. \end{cases} \]

**Proof.** Since \( x \in P \), we have \( x_k \) is concave on \([0, l]\), convex on \([l, n+1]\), \( x_0 = 0 \), and \( x_{n+1} = a x_l \). Thus,
\[ x_k \geq x_0 + \frac{x_l - x_0}{l} k = \frac{k}{l} x_l \quad \text{for } k \in [0, l], \]
and
\[ x_k \leq x_{n+1} + \frac{x_{n+1} - x_l}{n+1-l} (k - 1 - n) = \frac{n+1-al+(a-1)k}{n+1-l} x_l \quad \text{for } k \in [l, n+1]. \]

Hence, we have
\[ x_k \geq A(k)x_l, \quad k \in [0, l]; \quad x_k \leq A(k)x_l, \quad k \in [l, n+1]. \]

\[ \square \]

**Lemma 2.6.** Assume that \((1-a)l \geq 2\). Let \( x \in P \), then
\[ x_k \geq \mu ||x||, \quad k \in \left[ [al] + 1, \tau \right], \]
where \( \mu = \min\{a, 1 - \frac{\tau}{l}\}, \tau \in \left[ [al] + 1, l - 1 \right]. \)

**Proof.** Let \( x \in P \), then \( x \) is concave on \([0, l]\), and convex on \([l, n+1]\). Since \( 0 < a < 1 \), \( x_{n+1} = ax_l < x_l \), then \( ||x|| = \sup_{k \in [0, n+1]} |x_k| = \sup_{k \in [0, l]} |x_k| \). Set \( r = \inf \{ r \in [0, l] : \sup_{k \in [0, l]} x_k = x_r \} \). We now consider the following two cases:

**Case (i):** \( k \in [0, r] \). By the concavity of \( x_k \), we have
\[ x_k \geq x_0 + \frac{x_r - x_0}{r} k = \frac{k}{r} x_r \geq \frac{k}{l} x_r = \frac{k}{l} ||x||. \]

**Case (ii):** \( k \in [r, l] \). Similarly, we obtain
\[ x_k \geq x_r + \frac{x_r - x_l}{r-l} (k-r) \]
\[ = \frac{l-k}{l-r} x_r + \frac{k-r}{l-r} x_l \]
\[ \geq \frac{l-k}{l-r} x_r \geq \frac{l-k}{l} x_r \]
\[ = \left[ 1 - \frac{k}{l} \right] ||x||. \]

Thus, we have
\[ x_k \geq \min \left\{ \frac{k}{l}, 1 - \frac{k}{l} \right\} ||x||, \quad k \in [0, l], \]
which yields
\[ \min_{k \in [l+1, \tau]} x_k \geq \min \{a, 1 - \frac{\tau}{l} \} ||x|| = \mu ||x||. \]

The proof is completed. \[ \square \]
Lemma 2.7. Assume that \((1-a)l \geq 2\). If conditions (H1), (H2), (H4) hold \(\forall k \in [0, n-l]\), then there exists a constant \(\tau \in [[a]] + 1, l-2\) such that

\[ B(k) = h^+(l - \lceil \delta k \rceil) - \frac{1}{M} h^-(l + k) \geq 0, \]

(2.6)

where \(h^+(k) = \max\{h(k), 0\}\), \(h^-(k) = -\min\{h(k), 0\}\), \(\delta = \frac{t - r - 3}{n - r + 1}\), and \(M = a^2l/n\). Then for all \(q \in [0, \infty)\), we have

\[ \sum_{j=l+1}^{n} G(k, j)h(j) f(qA(j)) \geq 0. \]

(2.7)

Proof. By the definition of \(A(k)\), it is easy to check that

\[ A(l - \lceil \delta r \rceil) = \frac{1}{l} \left( l - \left[ \frac{l - \tau - 1}{n - l + 1} r \right] \right) = 1 - \frac{1}{l} \left[ \frac{l - \tau - 1}{n - l + 1} r \right], \quad r \in [0, n - l + 1], \]

(2.8)

and

\[ A(l + r) = 1 - \frac{r}{n - l + 1} (1 - a), \quad r \in [0, n - l]. \]

(2.9)

Set \(j = l - \lceil \delta r \rceil, r \in [0, n - l + 1]\) \((\delta \text{ as in (H4)})\). For all \(q \in [0, \infty)\), by view of Lemma \(2.4\), Remark \(2.3\), (2.6), (2.8), (2.9), (H4), and that \(f\) is nondecreasing, we have

\[
\begin{align*}
\sum_{j=l+1}^{n} G(k, j)h^+(j) f(qA(j)) \\
= & \sum_{r=0}^{n-l-1} G(k, l - \lceil \delta r \rceil) h^+(l - \lceil \delta r \rceil) f(qA(l - \lceil \delta r \rceil)) \\
= & \sum_{r=0}^{n-l-1} G(k, l - \lceil \delta r \rceil) h^+(l - \lceil \delta r \rceil) f\left(q\left(1 - \frac{1}{l}\left[\frac{l - \tau - 1}{n - l + 1} r\right]\right)\right) \\
\geq & \sum_{r=0}^{n-l-1} G(k, l - \lceil \delta r \rceil) h^+(l - \lceil \delta r \rceil) f\left(q\left(1 - \frac{r}{n - l + 1}(1 - \frac{\tau + 1}{l})\right)\right) \\
\geq & M \sum_{r=0}^{n-l} G(k, l + r) h^+(l + r) f\left(q\left(1 - \frac{r}{n - l + 1}(1 - \frac{\tau + 1}{l})\right)\right) \\
\geq & \sum_{r=0}^{n-l} G(k, l + r) h^-(l + r) f\left(q\left(1 - \frac{r}{n - l + 1}(1 - a)\right)\right).
\end{align*}
\]

(2.10)

Again, setting \(j = l + r, \quad r \in [0, n - l]\), for \(q \in [0, \infty)\), we obtain

\[
\sum_{j=l+1}^{n} G(k, j)h^-(j) f(qA(j)) = \sum_{r=1}^{n-l} G(k, l + r) h^-(l + r) f\left(q\left(1 - \frac{r}{n - l + 1}(1 - a)\right)\right).
\]

(2.11)

Thus, by (2.10) and (2.11), we get

\[
\begin{align*}
\sum_{j=l+1}^{n} G(k, j)h(j) f(qA(j)) \\
= & \sum_{j=l+1}^{n} G(k, j)h^+(j) f(qA(j)) - \sum_{j=l+1}^{n} G(k, j)h^-(j) f(qA(j)) \geq 0.
\end{align*}
\]
The proof is completed.

We define the operator $T : C[0, n + 1] \to C[0, n + 1]$ by
\[
(Tx)_k = \sum_{j=1}^{n} G(k, j) h(j) f(x_j), \quad (k, j) \in [0, n + 1] \times [1, n].
\] (2.12)
where $G(k, j)$ as in (2.2). From Lemma 2.4, we easily know that $x(t)$ is a solution of the (1.1) if and only if $x(t)$ is a fixed point of the operator $T$.

**Lemma 2.8.** Let $(1-a)l \geq 2$. Assume that conditions (H1), (H2), (H4) are satisfied. Then $T$ maps $P$ into $P$.

**Proof.** For $x \in P$, by Lemmas 2.5, 2.7, and $f$ is nondecreasing, we have
\[
\sum_{j=\tau+1}^{l} G(k, j) h(j) f(x_j) = \sum_{j=\tau}^{l} G(k, j) h^+(j) f(x_j) - \sum_{j=\tau+1}^{n} G(k, j) h^-(j) f(x_j)
\geq \sum_{j=\tau+1}^{l} G(k, j) h^+(j) f(A(j)x_l) - \sum_{j=\tau+1}^{n} G(k, j) h^-(j) f(A(j)x_l)
= \sum_{j=\tau+1}^{n} G(k, j) h(j) f(x_l A(j)) \geq 0,
\] (2.13)
which implies
\[
(Tx)_k = \sum_{j=1}^{n} G(k, j) h(j) f(x_j)
= \sum_{j=1}^{\tau} G(k, j) h^+(j) f(x_j) + \sum_{j=\tau+1}^{n} G(k, j) h(j) f(x_j)
\geq \sum_{j=1}^{\tau} G(k, j) h^+(j) f(x_j) \geq 0,
\]
again $(Tx)_0 = 0, (Tx)_{n+1} = a(Tx)_l$, it follows that $T : P \to C_0^+(0, n + 1]$. On the other hand,
\[
\Delta^2(Tx)_k = -h^+(j) f(x_j) \leq 0, \quad j \in [0, l],
\Delta^2(Tx)_k = h^-(j) f(x_j) \geq 0, \quad j \in [l, n + 1].
\]
Thus, $T$ maps $P$ into $P$. □

**Lemma 2.9.** Let $(1-a)l \geq 2$. Assume that conditions (H1), (H2), (H4) are satisfied. If $z \in P$ is a fixed point of $T$ and $\|z\| > 0$, then $z$ is a positive solution of the (1.1).

**Proof.** At first, we claim that $z_l > 0$. Otherwise, $z_l = 0$ implies $z_{n+1} = az_l = 0$. By the convexity and the nonnegativity of $z$ on $[l, n + 1]$, we have
\[
z_k \equiv 0, \quad k \in [l, n + 1],
\]
this implies \( \Delta z_l = z_{l+1} - z_l = 0 \). Since \( z = Tz \), we have \( \Delta^2 z_k = -h^+(k) f(z_k) \leq 0 \), \( k \in [0, l] \). Then

\[
\Delta z_k \geq \Delta z_l = 0, \quad k \in [0, l-1].
\]

Thus, \( z_k \leq z_l = 0 \), \( k \in [0, l] \). By the nonnegativity of \( z \), we get \( z_k \equiv 0 \), \( k \in [0, l] \), which yields a contradiction with \( \|z\| > 0 \).

\( \square \)

Next, in view of Lemma 2.1, for \( z \in P \), we have

\[
z_k \geq k l z_l > 0, \quad k \in [1, l].
\]

(2.14)

Note that \( h(k) \) does not vanish identically on any subinterval of \( k \in [1, l] \), for any \( k \in [1, n] \). By (2.13) we have

\[
z_k = (Tz)_k
\]

\[
= \sum_{j=1}^{n} G(k, j) h(j) f(z_j)
\]

\[
= \sum_{j=1}^{\tau} G(k, j) h^+(j) f(z_j) + \sum_{j=\tau}^{n} G(k, j) h(j) f(z_j)
\]

\[
\geq \sum_{j=1}^{\tau} G(k, j) h^+(j) f(z_j) > 0.
\]

Thus, we assert that \( z \) is a positive solution of (1.1).

3. Existence of solutions

For convenience, we set

\[
M = \left( \mu \max_{k \in [0, n+1]} \sum_{j=|a|+1}^{\tau} G(k, j) h^+(j) \right)^{-1},
\]

\[
m = \left( \max_{k \in [0, n+1]} \sum_{j=1}^{l} G(k, j) h^+(j) \right)^{-1}
\]

where \( \mu \) as in Lemma 2.6.

**Theorem 3.1.** Let \((1 - a)l \geq 2\). Assume that conditions (H1)–(H4) are satisfied. If (H5), \( 0 \leq f_0 < m \), and \( M < f_\infty \leq +\infty \) hold, then (1.1) has at least one positive solution.

**Proof.** By Lemma 2.8 \( T : P \to P \). Moreover, it is easy to check by Arzela-Ascoli theorem that \( T \) is completely continuous. By (H5), we have \( f_0 < m \). There exist \( \rho_1 > 0 \) and \( \varepsilon_1 > 0 \) such that

\[
f(u) \leq (m - \varepsilon_1) u, \quad \text{for} \quad 0 < u \leq \rho_1.
\]

(3.1)

Let \( \Omega_1 = \{ x \in P : \|x\| < \rho_1 \} \). For \( x \in \partial \Omega_1 \), by (3.1), we have

\[
(Tx)_k = \sum_{j=1}^{n} G(k, j) h(j) f(x_j)
\]
\[
\sum_{j=1}^{l} G(k, j)h^+(j)f(x_j) - \sum_{k=i+1}^{n} G(k, j)h^-(j)f(x_j)
\leq \sum_{j=1}^{l} G(k, j)h^+(j)f(x_j)
\leq \rho_1(m - \varepsilon_1) \max_{k \in [0, n+1]} \sum_{j=1}^{l} G(k, j)h^+(j)
= \rho_1(m - \varepsilon_1) \frac{1}{m}
< \rho_1 = \|x\|,
\]
which yields \(\|Tx\| < \|x\|\) for \(x \in \partial\Omega_1\). Then by Theorem 3.1, we have
\[
i(T, \Omega_1, P) = 1. \tag{3.2}
\]
On the other hand, from (H5), we have \(f_{\infty} > M\). There exist \(\rho_2 > \rho_1 > 0\) and \(\varepsilon_2\) such that
\[
f(u) \geq (M + \varepsilon_2)u, \quad \text{for } u \geq \mu \rho_2. \tag{3.3}
\]
Set \(\Omega_2 = \{x \in P : \|x\| < \rho_2\}\). For any \(x \in \partial\Omega_2\), from Lemma 2.6 we have \(x_k \geq \mu\|x\| = \mu \rho_2\), for \(k \in [[a\ell] + 1, \tau]\). Then from (2.7) and (3.3), we obtain
\[
\|Tx\| = \max_{k \in [0, n+1]} \left[ \sum_{j=1}^{\tau} G(k, j)h^+(j)f(x_j) + \sum_{j=\tau+1}^{n} G(k, j)h(j)f(x_j) \right]
\geq \max_{k \in [0, n+1]} \sum_{j=1}^{\tau} G(k, j)h^+(j)f(x_j)
\geq \max_{k \in [0, n+1]} \sum_{j=\lfloor a\ell \rfloor + 1}^{\tau} G(k, j)h^+(j)f(x_j)
\geq \mu(M + \varepsilon_2)\rho_2 \max_{k \in [0, n+1]} \sum_{j=\lfloor a\ell \rfloor + 1}^{\tau} G(k, j)h^+(j)
= \frac{1}{M}(M + \varepsilon_2)\rho_2 > \rho_2 = \|x\|,
\]
this is, \(\|Tx\| > \|x\|\), for \(x \in \partial\Omega_2\). Then, by Theorem 3.1
\[
i(T, \Omega_2, P) = 0. \tag{3.4}
\]
Therefore, by (3.2), (3.4), and \(\rho_1 < \rho_2\), we have
\[
i(T, \Omega_2 \setminus \bar{\Omega}_1, P) = -1.
\]
Then operator \(T\) has a fixed point in \(\Omega_2 \setminus \bar{\Omega}_1\). So, (1.1) has at least one positive solution. \(\square\)

**Theorem 3.2.** Let \((1 - a)l \geq 2\). Assume that (H1)–(H4) are satisfied. If (H6), \(M < f_0 \leq +\infty\), and \(0 \leq f_{\infty} < m\) hold, then (1.1) has at least one positive solution.

**Proof.** At first, by (H6), we get \(f_0 > M\), there exist \(\rho_3\) and \(\varepsilon_3\) such that
\[
f(u) \geq (M + \varepsilon_3)u, \quad \text{for } 0 < u < \rho_3. \tag{3.5}
\]
Set $\Omega_3 = \{ x \in P : \| x \| < \rho_3 \}$. For any $x \in \partial \Omega_3$, by Lemma 2.6 we get $x_k \geq \mu \| x \| = \mu \rho_3$, for $k \in [a[l] + 1, \tau]$, then by (2.7) and (3.5), we have

$$
\| Tx \| = \max_{k \in [0,n+1]} \left[ \sum_{k=1}^{\tau} G(k,j)h^+(j)f(x_j) + \sum_{j=\tau+1}^{n} G(k,j)h(j)f(x_j) \right]
$$

$$
\geq \max_{k \in [0,n+1]} \sum_{j=1}^{\tau} G(k,j)h^+(j)f(x_j)
$$

$$
\geq \max_{k \in [0,n+1]} \sum_{j=\tau+1}^{\tau} G(k,j)h^+(j)f(x_j)
$$

$$
\geq \mu(M + \varepsilon_3)\rho_3 \max_{k \in [0,n+1]} \sum_{j=\tau+1}^{\tau} G(k,j)h^+(j)
$$

$$
= \frac{1}{M}(M + \varepsilon_3)\rho_3 > \rho_3 = \| x \|
$$

this is, $\| Tx \| > \| x \|$, for $x \in \partial \Omega_3$. Thus, by Theorem 1.1

$$
i(T,\Omega_3, P) = 0. \quad \text{(3.6)}
$$

Next, from (H6), we have $f_\infty < m$. There exist $\rho_4 > 0$ and $0 < \varepsilon_4 < \rho_4$ such that

$$
f(u) \leq (m - \varepsilon_4)u, \quad \text{for } u \geq \rho_4.
$$

Set $L = \max_{0 \leq u \leq \rho_4} f(u)$. Then

$$
f(u) \leq L + (m - \varepsilon_4)u, \quad \text{for } u \geq 0. \quad \text{(3.7)}
$$

Choose $\rho_5 > \max\{\rho_4, L/\varepsilon_4\}$. Let $\Omega_4 = \{ x \in P : \| x \| < \rho_3 \}$. Then for $x \in \partial \Omega_4$, by (2.7) and (3.7), we have

$$
(Tx)_k = \sum_{j=1}^{n} G(k,j)h(j)f(x_j)
$$

$$
= \sum_{j=1}^{l} G(k,j)h^+(j)f(x_j) - \sum_{j=\tau+1}^{n} G(k,j)h^-(j)f(x_j)
$$

$$
\leq \sum_{j=1}^{l} G(k,j)h^+(j)f(x_j)
$$

$$
\leq (L + (m - \varepsilon_4)\rho_5)\frac{1}{m}
$$

$$
= \rho_5 - (\varepsilon_4\rho_5 - L)\frac{1}{m}
$$

$$
< \rho_5 = \| x \|.
$$

Thus, by Theorem 3.1

$$
i(T,\Omega_4, P) = 1. \quad \text{(3.8)}
$$

Therefore, by (3.6), (3.8), and $\rho_3 < \rho_5$, we have

$$
i(T,\Omega_4 \setminus \bar{\Omega}_3, P) = 1.
$$

Then operator $T$ has a fixed point in $\Omega_4 \setminus \bar{\Omega}_3$. So, (1.1) has at least one positive solution. □
4. Example

In this section, we illustrate our main results. Consider the boundary-value problem

\[ \Delta^2 x_{k-1} + h(k)x_k^\alpha = 0, \quad k \in [1, 11], \]
\[ x_0 = 0, \quad \frac{1}{3}x_6 = x_{12}, \]

where \( 0 < \alpha < 1 \), and

\[ h(k) = \begin{cases} 
3(k - 8)^2, & k \in [1, 8], \\
\frac{8}{99}(8 - k)^3, & k \in [8, 11]. 
\end{cases} \]

Let \( n = 11, \ l = 8, \ a = 1/3 \), then we have \( M = 8/99 \). Now taking \( \tau = 3 \), then \( \tau \in ([al] + 1, l - 2) = [3, 6], \delta = 1, \) and for all \( k \in [0, n - \ell] = [0, 3] \), we have

\[ B(k) = h^+(l - [\delta k]) - \frac{1}{M}h^-(l + k) = k^2(3 - k) \geq 0. \]

Hence, Condition (H2) and (H4) hold. Set \( f(u) = u^\alpha \), it is easy to see that

\[ f_0 = \infty, \quad f_\infty = 0, \]

that is, (H6) holds. Thus, by Theorem 3.2, (4.1) has at least one positive solution.

References