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EXISTENCE OF POSITIVE SOLUTIONS FOR p-LAPLACIAN THREE-POINT BOUNDARY-VALUE PROBLEMS ON TIME SCALES

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Abstract. This article shows the existence of positive solutions for a class of p-Laplacian three-point boundary-value problem on time scales. By using several fixed point theorems in cones, we establish conditions for the existence of at least one, two or three positive solutions for the boundary-value problems. Our results are new even for the corresponding differential ($\mathbb{T} = \mathbb{R}$) and difference equation ($\mathbb{T} = \mathbb{Z}$), and for the general time scales setting. An example is also given to illustrate our results.

1. Introduction

Dynamic equations on time scales not only unify differential and difference equations [13], but also exhibit much more complicated dynamics [1, 8, 9]. The study of dynamic equations on time scales has led to important applications in the study of insect population models, biology, heat transfer, stock market, wound healing, and epidemic models [14, 20, 21].

Before introducing the problems of interest for this paper, we present some basic definitions which can be found in [5, 8, 9, 13]. Another source on dynamic equations on time scales is [17].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ with the topology inherited from $\mathbb{R}$. For notation, we shall use the convention that, for each interval J of $\mathbb{R}$, $J_{\mathbb{T}} = J \cap \mathbb{T}$.

The jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \quad \text{and} \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define

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$\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\upsilon(t) := t - \rho(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the Δ-derivative of f at t, denoted by $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the ∇-derivative [5] of f at t, denoted by $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \le \epsilon|\rho(t) - s|,$$

for all $s \in U$.

A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided it is continuous at left dense points in $\mathbb{T}$ and its right sided limit exists (finite) at right dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous. If $\mathbb{T} = \mathbb{Z}$, then any function is ld-continuous. It is known [5] that if f is ld-continuous, then there is a function $F(t)$ such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(\tau)\nabla\tau = F(b) - F(a).$$

For recent results on positive solutions for second order three point boundary value problems on time scales the reader is referred to [2, 3, 6, 10, 15, 18, 22]. Our results have been motivated by those of Anderson, Avery and Henderson [4], and Sun, Tang and Wang [25].

For convenience, throughout this paper we denote $\varphi_p(u) = |u|^{p-2}u$ for $p > 1$ with $(\varphi_p)^{-1} = \varphi_q$, where $1/p + 1/q = 1$.

Anderson, Avery and Henderson [4] considered the problem

$$\begin{gathered} (\varphi_p(u^\Delta(t)))^\nabla + c(t)f(u(t)) = 0, \quad t \in (a, b)_\mathbb{T}, \\ u(a) - B_0(u^\Delta(\nu)) = 0, \quad u^\Delta(b) = 0, \end{gathered}$$

where $\nu \in (a, b)_\mathbb{T}$, $f \in C_{ld}([0, \infty), [0, \infty))$, $c \in C_{ld}((a, b)_\mathbb{T}, [0, \infty))$ and $K_m x \le B_0(x) \le K_M x$ for some positive constants K_m, K_M. They established the existence result of at least one positive solution by a fixed point theorem of cone expansion and compression of functional type.

In [25], the authors considered the eigenvalue problem for the p-Laplacian three-point boundary value problem

$$\begin{gathered} (\varphi_p(u^\Delta(t)))^\nabla + \lambda h(t)f(u(t)) = 0, \quad t \in (0, T)_\mathbb{T}, \\ u(0) - \beta u^\Delta(0) = \gamma u^\Delta(\eta), u^\Delta(T) = 0. \end{gathered}$$

The main tool used in [24] is Krasnoselskii's fixed point theorem.

In this paper we study the existence of solutions for the one-dimensional p-Laplacian three-point boundary value problem on time scales

$$(\varphi_p(u^\Delta(t)))^\nabla + h(t)f(t, u(t)) = 0, \quad t \in (0, T)_\mathbb{T}, \tag{1.1}$$

$$u(0) - \beta u^\Delta(0) = \gamma u^\Delta(\eta), \quad u^\Delta(T) = 0. \tag{1.2}$$

We establish sufficient conditions for the existence of at least one, two or three positive solutions for the boundary value problem. An example is also given to illustrate the main results. The results are new even for the special cases of difference equations and differential equations.

The rest of the paper is organized as follows. In Section 2, we first give four lemmas which are needed throughout this paper and then state several fixed point results: Krasnosel'skii's fixed point theorem in a cone, a new fixed point theorem due to Avery and Henderson and the Leggett-Williams fixed point theorem. In Section 3 we use Krasnosel'skii's fixed point theorem to obtain the existence of at least one positive solutions of problem (1.1)-(1.2). Section 4 will discuss the existence of twin positive solutions of problem (1.1)-(1.2). Two new results and some corollaries will be presented by using a new fixed point theorem due to Avery and Henderson. In Section 5 we develop criteria for the existence of (at least) three positive and arbitrary odd positive solutions of problem (1.1) and (1.2). In particular, our results in this section are new when $\mathbb{T} = \mathbb{R}$ (the continuous case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Finally, in section 6, we give an example to illustrate our main results.

For the sake of convenience, we have the following hypotheses:

- (i) $\mathbb{T}$ is a time scale, with $0, T \in \mathbb{T}$, β, γ are nonnegative constants, $\eta \in (0, \rho(T))_{\mathbb{T}}$.
- (ii) $h \in C_{ld}((0,T)_{\mathbb{T}}, [0,\infty))$ such that $0 < \int_0^T h(s)\nabla s < \infty$, and f is in the space $C([0,\infty),(0,\infty))$.

2. Preliminaries

Let the Banach space $B = C_{ld}([0,T]_{\mathbb{T}})$ (see [2]) be endowed with the norm $\|u\| = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|$, and choose the cone $P \subset B$ defined by

$$\begin{aligned} P = \{u \in B : u(t) \geq 0 \text{ for } t \in [0,T]_{\mathbb{T}} \text{ and} \\ u^{\Delta\nabla}(t) \leq 0 \text{ for } t \in (0,T)_{\mathbb{T}},\ u^{\Delta}(T) = 0\}. \end{aligned}$$

Clearly, $\|u\| = u(T)$ for $u \in P$. Define the operator $A : P \to B$ by

$$\begin{aligned} Au(t) &= \int_0^t \varphi_q\Big(\int_s^T h(\tau) f(\tau, u(\tau))\nabla\tau\Big)\Delta s \\ &\quad + \beta\varphi_q\Big(\int_0^T h(s) f(s,u(s))\nabla s\Big) + \gamma\varphi_q\Big(\int_\eta^T h(s) f(s,u(s))\nabla s\Big) \end{aligned} \tag{2.1}$$

for $t \in [0,T]_{\mathbb{T}}$.

Lemma 2.1 ([24, Lemma 2.6]). *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}_\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous differentiable. Then there exists c in the interval $[\rho(t), t]$ with*

$$(f \circ g)^{\nabla}(t) = f'(g(c)) g^{\nabla}(t).$$

From the definition of A, the monotonicity of $\varphi_q(x)$ and Lemma 2.1, it is easy to see that for each $u \in P$, $Au \in P$ and satisfies (1.2). In addition, since $(\varphi_p(u^{\Delta}(t)))^{\nabla} = -h(t) f(u(t)) < 0$, and $u^{\Delta}(T) = 0$, then $Au(T)$ is the maximum value of $Au(t)$.

Lemma 2.2 ([25, Lemma 2.2]). *$A : P \to P$ is completely continuous.*

Lemma 2.3 ([25, Lemma 2.3]). *If $u \in P$, then $u(t) \geq \frac{t}{T}\|u\|$ for $t \in [0,T]$.*

From the two lemmas above, we see that each fixed point of the operator A in P is a positive solution of (1.1), (1.2).

Lemma 2.4 ([11, 16]). *Let P be a cone in a Banach space B. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either*

(i) *$\|Ax\| \leq \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_2$, or*
(ii) *$\|Ax\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Ax\| \leq \|x\|$ for all $x \in P \cap \partial\Omega_2$.*

Then A has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

In the rest of this section, we provide some background material from the theory of cones in Banach spaces, and we then state several fixed point theorems which we needed later.

Let B be a Banach space and P be a cone in B. A map $\psi : P \to [0, +\infty)$ is said to be a nonnegative, continuous and increasing functional provided ψ is nonnegative, continuous and satisfies $\psi(x) \leq \psi(y)$ for all $x, y \in P$ and $x \leq y$.

Given a nonnegative continuous functional ψ on a cone P of a real Banach space B, we define, for each $d > 0$, the set

$$P(\psi, d) = \{x \in P : \psi(x) < d\}.$$

Lemma 2.5 ([7]). *Let P be a cone in a real Banach space E. Let α and ψ be increasing, nonnegative continuous functional on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\psi(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq H\psi(x)$$

for all $x \in \overline{P(\psi, c)}$. Suppose there exist a completely continuous operator $A : \overline{P(\psi, c)} \to P$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad \text{for } 0 \leq \lambda \leq 1 and x \in \partial P(\theta, b)$$

and

(i) *$\psi(Ax) > c$ for all $x \in \partial P(\psi, c)$;*
(ii) *$\theta(Ax) < b$ for all $x \in \partial P(\theta, b)$;*
(iii) *$P(\alpha, a) \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in \partial P(\alpha, a)$.*

Then, A has at least two fixed points, x_1 and x_2 belonging to $\overline{P(\psi, c)}$ satisfying $a < \alpha(x_1)$ with $\theta(x_1) < b$ and $b < \theta(x_2)$ with $\psi(x_2) < c$.

Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on the cone P. Define the convex sets $P_a, P(\alpha, a, b)$ by

$$P_a = \{x \in P : \|x\| < a\},$$
$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}.$$

Then we state the Leggett-Williams fixed point theorem [19].

Lemma 2.6. *Let P be a cone in a real Banach space B, $A : \overline{P}_c \to \overline{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P with $\alpha(x) \leq \|x\|$ for all $x \in \overline{P}_c$. Suppose there exists $0 < d < a < b \leq c$ such that*

(i) *$\{x \in P(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in P(\alpha, a, b)$;*
(ii) *$\|Ax\| < d$ for $\|x\| \leq d$;*
(iii) *$\alpha(Ax) > a$ for $x \in P(\alpha, a, c)$ with $\|Ax\| > b$.*

Then A has at least three fixed points x_1, x_2, x_3 satisfying $\|x_1\| < d$, $a < \alpha(x_2)$, $\|x_3\| > d$, and $\alpha(x_3) < a$.

3. Existence of One Positive Solution

For convenience, we define some important constants

$$C_1 = (T + \beta + \gamma)^{-1} \varphi_p\Big(\int_0^T h(s) \nabla s\Big), \tag{3.1}$$

$$C_2 = (\eta + \beta + \gamma)^{-1} \varphi_p\Big(\int_\eta^T h(s) \nabla s\Big). \tag{3.2}$$

Theorem 3.1. *Assume there exist positive numbers $a \neq b$ such that the conditions*

- (H1) *There is $a > 0$ such that $f(t, u) \leq \varphi_p(aC_1)$ for $t \in [0, T]_{\mathbb{T}}$ and $0 \leq u \leq a$;*
- (H2) *There is $b > 0$ such that $f(t, u) \geq \varphi_p(bC_2)$ for $t \in [\eta, T]_{\mathbb{T}}$ and $\frac{\eta}{T} b \leq u \leq b$.*

Then (1.1)–(1.2) has at least one positive solution u such that $\|u\|$ lies between a and b.

Proof. Without loss of generality, we may suppose that $0 < a < b$. Define the bounded open ball centered at the origin by

$$\Omega_a = \{u \in B : \|u\| \leq a\}, \quad \Omega_b = \{u \in B : \|u\| \leq b\}.$$

Then $0 \in \Omega_a \subset \Omega_b$. For $u \in P \bigcap \partial\Omega_a$ so that $\|u\| = a$, by (H1) and (3.1), we have

$$\begin{aligned}
\|Au\| &= \sup_{t \in [0,T]} \Big[\int_0^t \varphi_q\Big(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \Big) \Delta s \\
&\quad + \beta \varphi_q\Big(\int_0^T h(s) f(s, u(s)) \nabla s\Big) + \gamma \varphi_q\Big(\int_\eta^T h(s) f(s, u(s)) \nabla s\Big)\Big] \\
&\leq (T + \beta + \gamma) \varphi_q\Big(\int_0^T h(s) f(s, u(s)) \nabla s\Big) \\
&\leq (T + \beta + \gamma) \varphi_q\Big(\int_0^T h(s) \varphi_p(aC_1) \nabla s\Big) \\
&\leq aC_1 (T + \beta + \gamma) \varphi_q\Big(\int_0^T h(s) \nabla s\Big) \leq a.
\end{aligned}$$

Hence, $\|Au\| \leq \|u\|$ for $u \in P \bigcap \partial\Omega_a$. Similarly, let $u \in P \bigcap \partial\Omega_b$ so that $\|u\| = b$. Then

$$\min_{t \in [\eta, T]} u(t) \geq \frac{\eta}{T} b$$

and

$$\begin{aligned}
\|Au\| &\geq Au(\eta) \\
&= \int_0^\eta \varphi_q\Big(\int_s^T h(\tau)f(\tau,u(\tau))\nabla\tau\Big)\Delta s \\
&\quad + \beta\varphi_q\Big(\int_0^T h(s)f(s,u(s))\nabla s\Big) + \gamma\varphi_q(\int_\eta^T h(s)f(s,u(s))\nabla s) \\
&\geq (\eta+\beta+\gamma)\varphi_q(\int_\eta^T h(s)f(s,u(s))\nabla s) \\
&\geq (\eta+\beta+\gamma)bC_2\varphi_q\Big(\int_\eta^T h(s)\nabla s\Big) \\
&= b = \|u\|
\end{aligned}$$

by (H2) and (3.2). Consequently, $\|Au\| \geq \|u\|$ for $u \in P\bigcap\partial\Omega_b$. By Lemma 2.4, A has a fixed point $u \in P\bigcap(\overline{\Omega}_b \setminus \Omega_a)$, which is a positive solution of (1.1), (1.2), such that $a \leq \|u\| \leq b$. $\square$

For $t \in [0,T]_{\mathbb{T}}$, we define

$$f_0(t) = \liminf_{u\to 0^+}\frac{f(t,u)}{\varphi_p(u)}, \quad f_\infty(t) = \liminf_{u\to\infty}\frac{f(t,u)}{\varphi_p(u)}, \tag{3.3}$$

$$f^0(t) = \limsup_{u\to 0^+}\frac{f(t,u)}{\varphi_p(u)}, \quad f^\infty(t) = \limsup_{u\to\infty}\frac{f(t,u)}{\varphi_p(u)}. \tag{3.4}$$

Corollary 3.2. *The boundary-value problem* (1.1), (1.2) *has at least one positive solution provided either*

- (H3) $f^0(t) < \varphi_p(C_1)$ *for* $t \in [0,T]_{\mathbb{T}}$ *and* $f_\infty(t) > \varphi_p\left(\frac{TC_2}{\eta}\right)$ *for* $t \in [\eta,T]_{\mathbb{T}}$ *or*
- (H4) $f_0(t) > \varphi_p\left(\frac{TC_2}{\eta}\right)$ *for* $t \in [\eta,T]_{\mathbb{T}}$ *and* $f^\infty(t) < \varphi_p(C_1)$ *for* $t \in [0,T]_{\mathbb{T}}$,

where $C_1, C_2, f_0, f_\infty, f^0, f^\infty$ *are as in (3.1), (3.2), (3.3), (3.4), respectively. In particular, if f is superlinear in $\varphi_p(u)$ ($f^0(t) = 0$ and $f_\infty(t) = \infty$) or sublinear in $\varphi_p(u)$ ($f_0(t) = \infty$ and $f^\infty(t) = 0$), then* (1.1), (1.2) *has at least one positive solution.*

Proof. First suppose (H3) holds. Then, there are sufficiently small $a > 0$ and sufficiently large $b > 0$ such that

$$\frac{f(t,u)}{\varphi_p(u)} \leq \varphi_p(C_1) \text{ for } t \in [0,T]_{\mathbb{T}}, u \in (0,a],$$

$$\frac{f(t,u)}{\varphi_p(u)} \geq \varphi_p\big(\frac{TC_2}{\eta}\big) \quad \text{for } t \in [\eta,T]_{\mathbb{T}}, u \in [\frac{\eta b}{T}, +\infty).$$

Then

$$f(t,u) \leq \varphi_p(uC_1) \leq \varphi_p(aC_1), \quad t \in [0,T]_{\mathbb{T}}, u \in [0,a],$$

$$f(t,u) \geq \varphi_p\big(\frac{TC_2 u}{\eta}\big) \geq \varphi_p(C_2 b), \quad t \in [\eta,T]_{\mathbb{T}},\ u \in [\frac{\eta b}{T}, b].$$

In particular, both (H1) and (H2) hold, so that by Theorem 3.1, (1.1), (1.2) has at least one positive solution.

Next assume (H4) holds. Then there exist $0 < a < b$ such that

$$\frac{f(t,u)}{\varphi_p(u)} \ge \varphi_p\big(\frac{TC_2}{\eta}\big) \quad \text{for } t \in [\eta,T]_{\mathbb{T}}, u \in (0,a], \tag{3.5}$$

$$\frac{f(t,u)}{\varphi_p(u)} \le \varphi_p(C_1) \quad \text{for } t \in [0,T]_{\mathbb{T}}, u \in [b,+\infty). \tag{3.6}$$

From (3.5) we have $f(t,u) \ge \varphi_p\big(\frac{TC_2u}{\eta}\big) \ge \varphi_p(C_2a)$ for $t \in [\eta,T]_{\mathbb{T}}, u \in [\frac{\eta a}{T}, a]$ satisfying (H2) with respect to a. Now consider (3.6), we wish to show that (H1) holds. To that end, we consider the two cases: (1) $f(t,u)$ is bounded or (2) $f(t,u)$ is unbounded.

Case 1: Suppose there exists $C > 0$ such that $f(t,u) \le C$ for $t \in [0,T]_{\mathbb{T}}$ and $u \in [0,\infty)$. By (3.6), there is $r \ge \max\{b, \frac{\varphi_q(C)}{C_1}\}$ such that $f(t,u) \le C \le \varphi_p(C_1r)$ for $t \in [0,T]_{\mathbb{T}}, u \in [0,r]$. Thus (H1) is satisfied with respect to r.

Case 2: If f is unbounded, there exist $t_0 \in [0,T]_{\mathbb{T}}$ and $r' \ge b$ such that

$$f(t,u) \le f(t_0,r') \le \varphi_p(C_1r') \quad \text{for } t \in [0,T]_{\mathbb{T}} \text{ and } u \in [0,r'].$$

and (H1) is satisfied with respect to r'. Thus in both cases condition (H1) hold and Theorem 3.1 yields the conclusion. $\square$

4. Twin Solutions

In this section, we fix $c \in \mathbb{T}$ such that $\eta < c < T$, and denote

$$C_3 = (c+\beta+\gamma)^{-1}\varphi_p\Big(\int_c^T h(s)\nabla s\Big).$$

Define the nonnegative, increasing and continuous functionals ψ, θ, and α on P by

$$\psi(u) = \min_{t\in[\eta,c]_{\mathbb{T}}} u(t) = u(\eta), \quad \theta(u) = \max_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta),$$
$$\alpha(u) = \max_{t\in[0,c]_{\mathbb{T}}} u(t) = u(c).$$

We observe that, for each $u \in P$,

$$\psi(u) = \theta(u) \le \alpha(u). \tag{4.1}$$

In addition, for each $u \in P$, $\psi(u) = u(\eta) \ge \frac{\eta}{T}\|u\|$. Thus

$$\|u\| \le \frac{T}{\eta}\psi(u), \quad u \in P. \tag{4.2}$$

Finally, we also note that

$$\theta(\lambda u) = \lambda\theta(u), \quad 0 \le \lambda \le 1 \text{ and } u \in \partial P(\theta, b').$$

We now present the results in this section.

Theorem 4.1. *Assume that there are positive numbers $a' < b' < c'$ such that*

$$0 < a' < \frac{C_1}{C_3}b' < \frac{\eta C_1}{TC_3}c'.$$

Assume further that $f(t,u)$ satisfies the following conditions:

(i) $f(t,u) > \varphi_p(c'C_2)$, $(t,u) \in [\eta,T]_{\mathbb{T}} \times [c', \frac{T}{\eta}c']$,
(ii) $f(t,u) < \varphi_p(b'C_1)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0, \frac{T}{\eta}b']$,
(iii) $f(t,u) > \varphi_p(a'C_3)$, $(t,u) \in [c,T]_{\mathbb{T}} \times [a', \frac{T}{c}a']$.

Then (1.1)-(1.2) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$\begin{aligned} a' < \max_{t\in[0,c]_{\mathbb{T}}} u_1(t) \quad &\textit{with} \max_{t\in[0,\eta]_{\mathbb{T}}} u_1(t) < b', \\ b' < \max_{t\in[0,\eta]_{\mathbb{T}}} u_2(t) \quad &\textit{with} \min_{t\in[\eta,c]_{\mathbb{T}}} u_2(t) < c'. \end{aligned}$$

Proof. By the definition of operator A and its properties, it suffices to show that the conditions of Lemma 2.5 hold with respect to A.

We first show that if $u \in \partial P(\psi, c')$, then $\psi(Au) > c'$. Indeed, if $u \in \partial P(\psi, c')$, then $\psi(u) = \min_{t\in[\eta,c]_{\mathbb{T}}} u(t) = u(\eta) = c'$. Since $u \in P$, $\|u\| \leq \frac{T}{\eta}\psi(u) = \frac{T}{\eta}c'$, we have $c' \leq u(t) \leq \frac{T}{\eta}c', t \in [\eta, T]_{\mathbb{T}}$. As a consequence of (i), $f(t, u(t)) > \varphi_p(c'C_2), \quad t \in [\eta, T]_{\mathbb{T}}$. Also, $Au \in P$ implies

$$\begin{aligned} \psi(Au) &= Au(\eta) \\ &= \int_0^\eta \varphi_q\Big(\int_s^T h(\tau)f(\tau, u(\tau))\nabla\tau\Big)\Delta s \\ &\quad + \beta\varphi_q\Big(\int_0^T h(s)f(s, u(s))\nabla s\Big) + \gamma\varphi_q\Big(\int_\eta^T h(s)f(s, u(s))\nabla s\Big) \\ &\geq (\eta + \beta + \gamma)\varphi_q\Big(\int_\eta^T h(s)f(s, u(s))\nabla s\Big) \\ &> (\eta + \beta + \gamma)\frac{\eta c' C_2}{T}\varphi_q\Big(\int_\eta^T h(s)\nabla s\Big) = c'. \end{aligned}$$

Next, we verify that $\theta(Au) < b'$ for $u \in \partial P(\theta, b')$. Let us choose $u \in \partial P(\theta, b')$, then $\theta(u) = \max_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta) = b'$. This implies $0 \leq u(t) \leq b'$, $t \in [0, \eta]_{\mathbb{T}}$. Since $u \in P$, we also have $b' \leq u(t) \leq \|u\| \leq \frac{T}{\eta}u(l) = \frac{T}{\eta}b'$ for $t \in [\eta, T]_{\mathbb{T}}$. So

$$0 \leq u(t) \leq \frac{T}{\eta}b', \quad t \in [0, T]_{\mathbb{T}}.$$

Using (ii), we get

$$f(t, u(t)) < \varphi_p(b'C_1), \quad t \in [0, T]_{\mathbb{T}}.$$

Also, $Au \in P$ implies that

$$\begin{aligned} \theta(Au) &= Au(\eta) \leq Au(T) \\ &= \int_0^T \varphi_q(\int_s^T h(\tau)f(\tau, u(\tau))\nabla\tau)\Delta s \\ &\quad + \beta\varphi_q\Big(\int_0^T h(s)f(s, u(s))\nabla s\Big) + \gamma\varphi_q(\int_\eta^T h(s)f(s, u(s))\nabla s) \\ &\leq (T + \beta + \gamma)\varphi_q\Big(\int_0^T h(s)f(s, u(s))\nabla s\Big) \\ &< (T + \beta + \gamma)b'C_1\varphi_q\Big(\int_0^T h(s)\nabla s\Big) = b'. \end{aligned}$$

Finally, we prove that $P(\alpha, a') \neq \emptyset$ and $\alpha(Au) > a'$ for all $u \in \partial P(\alpha, a')$. In fact, the constant function $\frac{a'}{2} \in P(\alpha, a')$. Moreover, for $u \in \partial P(\alpha, a')$, we have $\alpha(u) = \max_{t\in[0,c]_{\mathbb{T}}} u(t) = u(c) = a'$. This implies $a' \leq u(t) \leq \frac{T}{c}a', \quad t \in [c, T]_{\mathbb{T}}$.

Using assumption (iii), $f(t,u(t)) > \varphi_p(a'C_3), \quad t \in [c,T]_{\mathbb{T}}$. As before $Au \in P$, we obtain

$$
\begin{aligned}
\alpha(Au) &= (Au)(c) \\
&= \int_0^c \varphi_q\Big(\int_s^T h(\tau)f(\tau,u(\tau))\nabla\tau\Big)\Delta s \\
&\quad + \beta\varphi_q\Big(\int_0^T h(s)f(s,u(s))\nabla s\Big) + \gamma\varphi_q\Big(\int_\eta^T h(s)f(s,u(s))\nabla s\Big) \\
&\ge (c+\beta+\gamma)\varphi_q\Big(\int_c^T h(s)f(s,u(s))\nabla s\Big) \\
&> (c+\beta+\gamma)\frac{a'}{L}\varphi_q\Big(\int_c^T h(s)\nabla s\Big) = a'.
\end{aligned}
$$

Thus, by Lemma 2.5, there exist at least two fixed points of A which are positive solutions u_1 and u_2, belonging to $\overline{P(\psi,c')}$, of (1.1)-(1.2) such that

$$
a' < \alpha(u_1) \quad \text{with } \theta(u_1) < b', \quad b' < \theta(u_2) \quad \text{with } \psi(u_2) < c'.
$$

□

In analogy to Theorem 4.1, we have the following result.

Theorem 4.2. *Assume that there are positive numbers $a' < b' < c'$ such that*

$$
0 < a' < \frac{c}{T}b' < \frac{cC_2}{TC_1}c'.
$$

Assume further that $f(t,u)$ satisfies the following conditions:

(i) *$f(t,u) < \varphi_p(c'C_1)$ for $(t,u) \in [0,T]_{\mathbb{T}} \times [0,\frac{T}{\eta}c']$,*
(ii) *$f(t,u) > \varphi_p(b'C_2)$ for $(t,u) \in [\eta,T]_{\mathbb{T}} \times [b',\frac{T}{\eta}b']$,*
(iii) *$f(t,u) < \varphi_p(a'C_1)$ for $(t,u) \in [c,T]_{\mathbb{T}} \times [0,\frac{T}{c}a']$.*

Then (1.1)-(1.2) has at least two positive solutions u_1 and u_2 such that

$$
\begin{gathered}
a' < \max_{t\in[0,c]_{\mathbb{T}}} u_1(t) \quad \textit{with} \max_{t\in[0,\eta]_{\mathbb{T}}} u_1(t) < b', \\
b' < \max_{t\in[0,\eta]_{\mathbb{T}}} u_2(t) \quad \textit{with} \max_{t\in[\eta,c]_{\mathbb{T}}} u_2(t) < c'.
\end{gathered}
$$

Corollary 4.3. *Assume that f satisfies conditions*

(i) *$f_0(t) > \varphi_p(C_2)$, $t \in [\eta,T]_{\mathbb{T}}$ and $f_\infty(t) = \liminf_{u\to\infty}\frac{f(t,u)}{\varphi_p(u)} > \varphi_p(C_3)$, $t \in [c,T]_{\mathbb{T}}$;*
(ii) *there exists $a' > 0$ such that $f(t,u) < \varphi_p(a'C_1)$ for $(t,u) \in [0,T]_{\mathbb{T}} \times [0,\frac{T}{\eta}a']$.*

Then (1.1)-(1.2) has at least two positive solutions.

Corollary 4.4. *Suppose that f satisfies conditions*

(i) *$f_0(t) < \varphi_p(\frac{\eta}{T}C_1)$, $t \in [0,T]_{\mathbb{T}}$ and $f_\infty(t) < \varphi_p(\frac{c}{T}C_1)$, $t \in [c,T]_{\mathbb{T}}$;*
(ii) *there exists $b' > 0$ such that $f(t,u) > \varphi_p(b'C_2)$, for $(t,u) \in [\eta,T]_{\mathbb{T}} \times [b',\frac{T}{\eta}b']$.*

Then (1.1)-(1.2) has at least two positive solutions.

By applying Theorems 4.1 and 4.2, it is easy to prove that Corollaries 4.3 and 4.4 hold, respectively.

5. Existence of three solutions

Let the nonnegative continuous concave functional $\Psi : P \to [0,\infty)$ be defined by

$$\Psi(u) = \min_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta), \quad u \in P.$$

Note that for $u \in P$, $\Psi(u) \le \|u\|$.

Theorem 5.1. *Suppose that there exist constants $0 < d' < a'$ such that*

(i) $f(t,u) < \varphi_p(d'C_1)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0,d']$;
(ii) $f(t,u) \ge \varphi_p(a'C_2)$, $(t,u) \in [\eta,T]_{\mathbb{T}} \times [a', \frac{T}{\eta}a']$;
(iii) *one of the following conditions holds:*
(D1) $\limsup_{u\to\infty} \max_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{\varphi_p(u)} < \varphi_p(C_1)$;
(D2) there exists a number $c' > \frac{T}{\eta}a'$ *such that* $f(t,u) < \varphi_p(c'C_1)$ *for* $(t,u) \in [0,T]_{\mathbb{T}} \times [0,c']$.

Then (1.1)-(1.2) *has at least three positive solutions.*

Proof. By the definition of operator A and its properties, it suffices to show that the conditions of Lemma 2.6 hold with respect to A.

We first show that if (D1) holds, then there exists a number $l' > \frac{T}{\eta}a'$ such that $A : \overline{P}_{l'} \to P_{l'}$. Suppose that

$$\limsup_{u\to\infty} \max_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{\varphi_p(u)} < \varphi_p(C_1)$$

holds, then there are $\tau > 0$ and $\delta < C_1$ such that if $u > \tau$, then

$$\max_{t\in[0,T]_{\mathbb{T}}} \frac{f(t,u)}{\varphi_p(u)} \le \varphi_p(\delta).$$

That is to say,

$$f(t,u) \le \varphi_p(\delta u), \quad (t,u) \in [0,T]_{\mathbb{T}} \times [\tau,\infty).$$

Set $\lambda = \max\{f(t,u) : (t,u) \in [0,T]_{\mathbb{T}} \times [0,\tau]\}$, then

$$f(t,u) \le \lambda + \varphi_p(\delta u), \quad (t,u) \in [0,T]_{\mathbb{T}} \times [0,\infty). \tag{5.1}$$

Take

$$l' > \max\{\frac{T}{\eta}a', \varphi_q(\frac{\lambda}{\varphi_p(C_1) - \varphi_p(\delta)})\}. \tag{5.2}$$

If $u \in \overline{P}_{l'}$, then by (3.1), (5.1) and (5.2), we obtain

$$
\begin{aligned}
\|Au\| &= Au(T) \\
&= \int_0^T \varphi_q\Big(\int_s^T h(\tau)f(\tau,u(\tau))\nabla\tau\Big)\Delta s + \beta\varphi_q\Big(\int_0^T h(s)f(s,u(s))\nabla s\Big) \\
&\quad + \gamma\varphi_q\Big(\int_\eta^T h(s)f(s,u(s))\nabla s\Big) \\
&\le (T+\beta+\gamma)\varphi_q\Big(\int_0^T h(s)f(s,u(s))\nabla s\Big) \\
&\le (T+\beta+\gamma)\varphi_q\Big(\int_0^T h(s)(\lambda+\varphi_p(\delta u(s))\nabla s\Big) \\
&\le (T+\beta+\gamma)\varphi_q(\lambda+\varphi_p(\delta l'))\varphi_q\Big(\int_0^T h(s)\nabla s\Big) \\
&= \varphi_q(\lambda+\varphi_p(\delta l'))\frac{1}{C_1} < l'.
\end{aligned}
$$

Next we verify that if there is a positive number r' such that if $f(t,u) < \varphi_p(r'/N)$ for $(t,u) \in [0,T]_{\mathbb{T}} \times [0,r']$, then $A : \overline{P}_{r'} \to P_{r'}$.

Indeed, if $u \in \overline{P}_{r'}$, then

$$
\begin{aligned}
\|Au\| &= Au(T) \\
&\le (T+\beta+\gamma)\varphi_q\Big(\int_0^T h(s)f(s,u(s))\nabla s\Big) \\
&< \frac{r'}{N}(T+\beta+\gamma)\varphi_q\Big(\int_0^T h(s)\nabla s\Big) = r',
\end{aligned}
$$

thus, $Au \in P_{r'}$. Hence, we have shown that either (D_1) or (D_2) holds, then there exists a number c' with $c' > \frac{T}{\eta}a'$ such that $A : \overline{P}_{c'} \to P_{c'}$. It is also note from (i) that $A : \overline{P}_{d'} \to P_{d'}$.

Now, we show that $\{u \in P(\Psi, a', \frac{T}{\eta}a') : \Psi(u) > a'\} \neq \emptyset$ and $\Psi(Au) > a'$ for all $u \in P(\Psi, a', \frac{T}{\eta}a')$. In fact,

$$
u = \frac{(\eta+T)a'}{2\eta} \in \{u \in P(\Psi, a', \frac{T}{\eta}a') : \Psi(u) > a'\}.
$$

For $u \in P(\Psi, a', \frac{T}{\eta}a')$, we have

$$
a' \le \min_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta) \le u(t) \le \frac{T}{\eta}a',
$$

for all $t \in [\eta, T]_{\mathbb{T}}$. Then, in view of (ii), we know that

$$\begin{aligned}
\Psi(Au) &= \min_{t\in[\eta,T]_{\mathbb{T}}} Au(t) = Au(\eta) \\
&= \int_0^\eta \varphi_q\Big(\int_\tau^T h(\tau)f(\tau,u(\tau))\nabla\tau\Big)\Delta s + \beta\varphi_q(\int_0^T h(s)f(s,u(s))\nabla s) \\
&\quad + \gamma\varphi_q\Big(\int_\eta^T h(s)f(s,u(s))\nabla s\Big) \\
&\geq (\eta+\beta+\gamma)\varphi_q\Big(\int_\eta^T h(s)f(s,u(s))\nabla s\Big) \\
&\geq (\eta+\beta+\gamma)a'C_2\varphi_q\Big(\int_\eta^T h(s)\nabla s\Big) = a'.
\end{aligned}$$

Finally, we assert that if $u \in P(\Psi, a', c')$ and $\|Au\| > \frac{T}{\eta}a'$, then $\Psi(Au) > a'$. Suppose $u \in P(\Psi, a', c')$ and $\|Au\| > \frac{T}{\eta}a'$. Then

$$\begin{aligned}
\Psi(Au) &= \min_{t\in[\eta,T]_{\mathbb{T}}} Au(t) = Au(\eta) \\
&\geq \frac{\eta}{T}Au(T) = \frac{\eta}{T}\|Au\| > a'.
\end{aligned}$$

To sum up, the hypotheses of Lemma 2.6 are satisfied, hence (1.1)–(1.2) has at least three positive solutions u_1, u_2, u_3 such that

$$\|u_1\| < d', a' < \min_{t\in[\eta,T]_{\mathbb{T}}} u_2(t) \quad \textit{and} \quad \|u_3\| > d' \text{ with } \min_{t\in\eta,T]_{\mathbb{T}}} u_3(t) < a'.$$

□

From Theorem 5.1, we see that, when assumptions such as (i), (ii), (iii) are imposed appropriately on f, we can establish the existence of an arbitrary odd number of positive solutions of (1.1), (1.2).

Theorem 5.2. *If*

$$0 < d_1' < a_1' < \frac{T}{\eta}a_1' < d_2' < a_2' < \frac{T}{\eta}a_2' < d_3' < \dots < d_n', \quad n \in \mathbb{N},$$

(i) $f(t,u) < \varphi_p(d_i'C_1)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0, d_i']$;
(ii) $f(t,u) \geq \varphi_p(a_i'C_2)$, $(t,u) \in [\eta,T]_{\mathbb{T}} \times [a_i', \frac{T}{\eta}a_i']$;

then (1.1)-(1.2) *has at least* $2n-1$ *positive solutions.*

Proof. When $n = 1$, it is immediate from condition (i) that $A : \overline{P}_{d_1'} \to P_{d_1'} \subset \overline{P}_{d_1'}$, which means that A has at least one fixed point $u_1 \in \overline{P}_{d_1'}$ by the Schauder fixed point theorem. When $n = 2$, it is clear that Theorem 4.1 holds (with $c_1 = d_2'$). Then we can obtain at least three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\| < d_1', \min_{t\in[\eta,T]_{\mathbb{T}}} u_2(t) > a_1' \quad \textit{and} \quad \|u_3\| > d_1' \text{ with } \min_{t\in[\eta,T]_{\mathbb{T}}} u_3(t) < a_1'.$$

Following this way, we complete the proof by induction. □

6. Example

Let $\mathbb{T}=\{1-(1/2)^{\mathbb{N}_0}\}\cup\{1\}$, where $\mathbb{N}_0$ denote the set of nonnegative integers. Take $T=1$, $p=\frac{3}{2}$, $\beta=\gamma=\frac{1}{4}$, $\eta=\frac{15}{16}$, $c=\frac{31}{32}$. If we let $h(s)\equiv 1$, then by (3.1) and (3.2) we have

$$\begin{gathered}C_1=(T+\beta+\gamma)^{-1}\varphi_p\Big(\int_0^T 1\nabla s\Big)=\frac{2}{3},\\ C_2=(\eta+\beta+\gamma)^{-1}\varphi_p\Big(\int_\eta^T 1\nabla s\Big)=\frac{4}{23}.\end{gathered}$$

Suppose

$$f(t,u)=f(u):=\frac{6+9u}{20(1+u)}(2+\sin u)\sqrt{u},\quad t\in[0,1]_{\mathbb{T}},\ u\geq 0,$$

then $f_0=f^0=\frac{3}{5}$, $f_\infty=\frac{9}{20}$, $f^\infty=\frac{27}{20}$.

Firstly, by easy calculation, it is easy to get

$$\varphi_p(C_1)=\sqrt{\frac{2}{3}}\approx 1.224,\quad \varphi_p(\frac{T}{\eta}C_2)=\sqrt{\frac{64}{345}}\approx 0.431,$$

So the condition (H3) of Corollary 3.2 holds. Thus by Corollary 3.2, the boundary-value problem

$$\big(|u^\Delta(t)|^{-\frac{1}{2}}u^\Delta(t)\big)^\nabla+\frac{6+9u(t)}{20(1+u(t))}(2+\sin u(t))\sqrt{u(t)}=0, \tag{6.1}$$

$$u(0)-\frac{1}{2}u^\Delta(0)=\frac{1}{2}u^\Delta(\frac{15}{16}),\quad u^\Delta(1)=0, \tag{6.2}$$

has at least one positive solution.

Secondly, since

$$f_0=\frac{3}{5}<\varphi_p(\frac{\eta}{T}C_1)=\sqrt{\frac{5}{8}}\approx 0.791,\quad f_\infty=\frac{9}{20}<\varphi_p(\frac{c}{T}C_1)=\sqrt{\frac{11}{48}}\approx 0.479.$$

If we choose $b'=0.7$, then

$$\min_{t\in[0,T]_{\mathbb{T}},u\in[b',\frac{16}{15}b']}f(t,u)\approx 0.800>\varphi_p(b'C_1)\approx 0.683.$$

So all the assumptions of Corollary 4.4 are satisfied. Therefore by Corollary 4.4 the boundary value problem (5.1)–(5.2) has at lest two solutions u_1 and u_2 with $0<\|u_1\|\leq 0.7<\|u_2\|$.

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