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EXISTENCE OF SOLUTIONS FOR QUASILINEAR DELAY INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

KRISHNAN BALACHANDRAN, FRANCIS PAUL SAMUEL

Abstract. We prove the existence and uniqueness of mild and classical solution to a quasilinear delay integrodifferential equation with nonlocal condition. The results are obtained by using C_0-semigroup and the Banach fixed point theorem.

1. Introduction

The existence of solution to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [6]. In that paper, he established the existence and uniqueness of mild, strong and classical solutions of the nonlocal Cauchy problem

$$u'(t) + Au(t) = f(t, u(t)), t \in (0, a] \tag{1.1}$$

$$u(0) + g(t_1, t_2, \dots, t_p, u(t_1), u(t_2) \dots, u(t_p)) = u_0, \tag{1.2}$$

where $0 < t_1 < \dots < t_p \le a$, $-A$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0 \in X$ and $f : [0, a] \times X \to X$, $g : [0, a]^p \times X^p \to X$ are given functions. The symbol $g(t_1, \dots, t_p, u(\cdot))$ is used in the sense that in the place of "$\cdot$" we can substitute only elements of the set $(t_1, \dots, t_p)$. For example

$$g(t_1, \dots, t_p, u(\cdot)) = C_1 u(t_1) + \dots + C_p u(t_p),$$

where C_i $(i = 1, 2 \dots, p)$ are given constants. Subsequently many authors extended the work to various kind of nonlinear evolution equations [3, 4, 7, 8].

Several authors have studied the existence of solutions of abstract quasilinear evolution equations in Banach space [1, 5, 10, 18]. Bahuguna [2], Oka [15] and Oka and Tanaka [16] discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. Kato [12] studied the nonhomogeneous evolution equations and Chandrasekaran [9] proved the existence of mild solutions of the nonlocal Cauchy problem for a nonlinear integrodifferential equation. Dhakne and Pachpatte [11] established the existence of a unique strong solution of a quasilinear

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abstract functional integrodifferential equation in Banach spaces. An equation of this type occurs in a nonlinear conversation law with memory

$$u(t,x)+\Psi(u(t,x))_x=\int_0^t b(t-s)\Psi(u(t,x))_x\,ds+f(t,x),\quad t\in[0,a], \tag{1.3}$$

$$u(0,x)=\phi(x),\quad x\in\mathbb{R}. \tag{1.4}$$

It is clear that if nonlocal condition (1.2) is introduced to (1.3), then it will also have better effect than the classical condition $u(0,x)=\phi(x)$. Therefore, we would like to extend the results for (1.1)-(1.2) to a class of integrodifferential equations in Banach spaces.

The aim of this paper is to prove the existence and uniqueness of mild and classical solutions of quasilinear delay integrodifferential equation with nonlocal conditions of the form

$$u'(t)+A(t,u)u(t)=f(t,u(t),u(\alpha(t)))+\int_0^t k(t,s,u(s),u(\beta(s)))ds, \tag{1.5}$$

$$u(0)+g(u)=u_0, \tag{1.6}$$

where $t\in[0,a]$, $A(t,u)$ is the infinitesimal generator of a C_0-semigroup in a Banach space X, $u_0\in X$, $f:I\times X\times X\to X$, $k:\Delta\times X\times X\to X$, $g:C(I:X)\to X$, $\alpha,\beta:I\to I$ are given functions. Here $I=[0,a]$ and $\Delta=\{(t,s):0\le s\le t\le a\}$. The results obtained in this paper are generalizations of the results given by Pazy [17], Kato [13, 14] and Balachandran and Uchiyama [5].

2. Preliminaries

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach spaces Z the norm of Z is denoted by $\|\cdot\|$ or $\|\cdot\|_Z$. The space of all bounded linear operators from X to Y is denoted by $B(X,Y)$ and $B(X,X)$ is written as $B(X)$. We recall some definitions and known facts from Pazy [17].

Definition 2.1. Let S be a linear operator in X and let Y be a subspace of X. The operator $\tilde{S}$ defined by $D(\tilde{S})=\{x\in D(S)\cap Y: Sx\in Y\}$ and $\tilde{S}x=Sx$ for $x\in D(\tilde{S})$ is called the part of S in Y.

Definition 2.2. Let B be a subset of X and for every $0\le t\le a$ and $b\in B$, let $A(t,b)$ be the infinitesimal generator of a C_0 semigroup $S_{t,b}(s), s\ge 0$, on X. The family of operators $\{A(t,b)\},(t,b)\in I\times B$, is stable if there are constants $M\ge 1$ and ω such that

$$\rho(A(t,b))\supset(\omega,\infty)\quad\text{for } (t,b)\in I\times B,$$

$$\|\prod_{j=1}^{k}R(\lambda:A(t_j,b_j))\|\le M(\lambda-\omega)^{-k}$$

for $\lambda>\omega$ every finite sequences $0\le t_1\le t_2\le\dots\le t_k\le a$, $b_j\in B$, $1\le j\le k$. The stability of $\{A(t,b)\},(t,b)\in I\times B$ implies (see [17]) that

$$\|\prod_{j=1}^{k}S_{t_j,b_j}(s_j)\|\le M\exp\{\omega\sum_{j=1}^{k}s_j\},\quad s_j\ge 0$$

and any finite sequences $0\le t_1\le t_2\le\dots\le t_k\le a$, $b_j\in B$, $1\le j\le k$. $k=1,2,\dots$

Definition 2.3. Let $S_{t,b}(s), s \geq 0$ be the C_0-semigroup generatated by $A(t,b)$, $(t,b) \in I \times B$. A subspace Y of X is called $A(t,b)$-admissible if Y is invariant subspace of $S_{t,b}(s)$ and the restriction of $S_{t,b}(s)$ to Y is a C_0-semigroup in Y.

Let $B \subset X$ be a subset of X such that for every $(t,b) \in I \times B$, $A(t,b)$ is the infinitesimal generator of a C_0-semigroup $S_{t,b}(s), s \geq 0$ on X. We make the following assumptions:

(E1) The family $\{A(t,b)\}, (t,b) \in I \times B$ is stable.
(E2) Y is $A(t,b)$-admissible for $(t,b) \in I \times B$ and the family $\{\tilde{A}(t,b)\}, (t,b) \in I \times B$ of parts $\tilde{A}(t,b)$ of $A(t,b)$ in Y, is stable in Y.
(E3) For $(t,b) \in I \times B$, $D(A(t,b)) \supset Y$, $A(t,b)$ is a bounded linear operator from Y to X and $t \to A(t,b)$ is continuous in the $B(Y,X)$ norm $\|.\|$ for every $b \in B$.
(E4) There is a constant $L > 0$ such that

$$\|A(t,b_1) - A(t,b_2)\|_{Y \to X} \ \leq L\|b_1 - b_2\|_X$$

holds for every $b_1, b_2 \in B$ and $0 \leq t \leq a$.

Let B be a subset of X and $\{A(t,b)\}, (t,b) \in I \times B$ be a family of operators satisfying the conditions (E1)-(E4). If $u \in C(I : X)$ has values in B then there is a unique evolution system $U(t,s;u), 0 \leq s \leq t \leq a$, in X satisfying, (see [17, Theorem 5.3.1 and Lemma 6.4.2, pp. 135, 201-202]

(i) $\|U(t,s;u)\| \leq Me^{\omega(t-s)}$ for $0 \leq s \leq t \leq a$. where M and ω are stability constants.
(ii) $\frac{\partial^+}{\partial t}U(t,s;u)w = A(s,u(s))U(t,s;u)w$ for $w \in Y$, for $0 \leq s \leq t \leq a$.
(iii) $\frac{\partial}{\partial s}U(t,s;u)w = -U(t,s;u)A(s,u(s))w$ for $w \in Y$, for $0 \leq s \leq t \leq a$.

Further we assume that

(E5) For every $u \in C(I : X)$ satisfying $u(t) \in B$ for $0 \leq t \leq a$, we have

$$U(t,s;u)Y \subset Y, \quad 0 \leq s \leq t \leq a$$

and $U(t,s;u)$ is strongly continuous in Y for $0 \leq s \leq t \leq a$.
(E6) Y is reflexive.
(E7) For every $(t,b_1,b_2) \in I \times B \times B$, $f(t,b_1,b_2) \in Y$.
(E8) $g : C(I : B) \to Y$ is Lipschitz continuous in X and bounded in Y, that is, there exist constants $G > 0$ and $G_1 > 0$ such that

$$\|g(u)\|_Y \leq G,$$
$$\|g(u) - g(v)\|_Y \leq G_1 \max_{t \in I} \|u(t) - v(t)\|_X.$$

For the conditions (E9) and (E10) let Z be taken as both X and Y.

(E9) $k : \Delta \times Z \to Z$ is continuous and there exist constants $K_1 > 0$ and $K_2 > 0$ such that

$$\int_0^t \|k(t,s,u_1,v_1) - k(t,s,u_2,v_2)\|_Z ds \leq K_1(\|u_1(t) - u_2(t) + v_1(t) - v_(t)\|_Z),$$

$$K_2 = \max\{\int_0^t \|k(t,s,0,0)\|_Z \ ds : (t,s) \in \Delta\}.$$

(E10) $f : I \times Z \times Z \to Z$ is continuous and there exist constants $K_3 > 0$ and $K_4 > 0$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|_Z \le K_3(\|u_1 - u_2\|_Z + \|v_1 - v_2\|_Z)$$
$$K_4 = \max_{t \in I} \|f(t, 0, 0)\|_Z.$$

Let us take $M_0 = \max\{\|U(t, s; u)\|_{B(Z)}, 0 \le s \le t \le a, u \in B\}$.

(E11) $\alpha, \beta : I \to I$ is absolutely continuous and there exist constants $b > 0$ and $c > 0$ such that $\alpha'(t) \ge b$ and $\beta'(t) \ge c$ respectively for $t \in I$.

(E12)

$$M_0\Big[\|u_0\|_Y + G + r[K_3 a(1 + 1/b) + K_1 a(1 + 1/c)] + a(K_4 + K_2)\Big] \le r$$
$$q = \Big[Ka\|u_0\|_Y + GKa + M_0 G_1 + M_0[K_3 a(1 + 1/b) + K_1 a(1 + 1/c)]$$
$$+ Ka[r(K_3 a(1 + 1/b) + K_1 a(1 + 1/c))] + a(K_4 + K_2)\Big] < 1.$$

Next we prove the existence of local classical solutions of the quasilinear problem (1.5)–(1.6).

For a mild solution of (1.5)–(1.6) we mean a function $u \in C(I : X)$ with values in B and $u_0 \in X$ satisfying the integral equation

$$\begin{aligned} u(t) = U(t, 0; u)u_0 - U(t, 0; u)g(u) + \int_0^t U(t, s; u)[f(s, u(s), u(\alpha(s))) \\ + \int_0^s k(s, \tau, u(\tau), u(\beta(\tau)))d\tau]ds. \end{aligned} \tag{2.1}$$

A function $u \in C(I : X)$ such that $u(t) \in D(A(t, u(t)))$ for $t \in (0, a]$, $u \in C^1((0, a] : X)$ and satisfies (1.5))–(1.6) in X is called a classical solution of (1.5) -(1.6) on I. Further there exists a constant $K > 0$ such that for every $u, v \in C(I : X)$ with values in B and every $w \in Y$ we have

$$\|U(t, s; u)w - U(t, s; v)w\| \le K\|w\|_Y \int_s^t \|u(\tau) - v(\tau)\|d\tau. \tag{2.2}$$

3. Existence Result

Theorem 3.1. *Let* $u_0 \in Y$ *and let* $B = \{u \in X : \|u\|_Y \le r\}$, $r > 0$. *If the assumptions* (E1)–(E12) *are satisfied, then* (1.5)–(1.6) *has a unique classical solution* $u \in C([0, a] : Y) \cap C^1((0, a] : X)$

Proof. Let S be a nonempty closed subset of $C([0, a] : X)$ defined by $S = \{u : u \in C([0, a] : X), \|u(t)\|_Y \le r \ for\ 0 \le t \le a\}$. Consider a mapping P on S defined by

$$(Pu)(t) = U(t, 0; u)u_0 - U(t, 0; u)g(u) + \int_0^t U(t, s; u)\Big[f(s, u(s), u(\alpha(s))) + \int_0^s k(s, \tau, u(\tau), u(\beta(\tau)))d\tau\Big]ds.$$

We claim that P maps S into S. For $u \in S$, we have

$$\|Pu(t)\|_Y$$
$$= \|U(t, 0; u)u_0 - U(t, 0; u)g(u) + \int_0^t U(t, s; u)\Big[f(s, u(s), u(\alpha(s)))$$

$$
\begin{aligned}
&+ \int_0^s k(s,\tau,u(\tau)u(\beta(\tau)))d\tau\Big]ds\| \\
&\le \|U(t,0;u)u_0\| + \|U(t,0;u)g(u)\| \\
&+ \int_0^t \|U(t,s;u)\|\Big[\|f(s,u(s),u(\alpha(s))) - f(s,0,0)\| + \|f(s,0,0)\| \\
&+ \|\int_0^s [k(s,\tau,u(\tau),u(\beta(\tau))) - k(s,\tau,0,0)]d\tau\| + \|\int_0^s k(s,\tau,0,0)d\tau\|\Big]ds.
\end{aligned}
$$

Using assumptions (E8)-(E11), we get

$$
\begin{aligned}
\|Pu(t)\|_Y &\le M_0\|u_0\|_Y + M_0G + \int_0^t M_0\Big[K_3(\|u(s)\| + \|u(\alpha(s))\|) + K_4 \\
&\quad + \int_0^s K_1(\|u(s)\| + u(\beta(\tau))\|)d\tau + \int_0^s K_2 d\tau\Big]ds \\
&\le M_0\|u_0\|_Y + M_0G + M_0\Big[K_3ar + K_3\int_0^t \|u(\alpha(s))\|(\alpha'(s)/b)ds \\
&\quad + K_4a + K_1ar + K_1\int_0^t (\|u(\beta(s))\|(\beta'(s)/c)ds + K_2a\Big] \\
&\le M_0\|u_0\|_Y + M_0G + M_0\Big[K_3ar + (K_3/b)\int_{\alpha(0)}^{\alpha(t)} \|u(s)\|ds + K_4a \\
&\quad + K_1ar + (K_1/c)\int_{\beta(0)}^{\beta(t)} (\|u(s)\|ds + K_2a\Big] \\
&\le M_0\Big[\|u_0\|_Y + G + r[K_3a(1+1/b) + K_1a(1+1/c)] + a(K_4+K_2)\Big]
\end{aligned}
$$

From assumption (E12), one gets $\|Pu(t)\|_Y \le r$. Therefore P maps S into itself. Moreover, if $u, v \in S$, then

$$
\begin{aligned}
&\|Pu(t) - Pv(t)\| \\
&\le \|U(t,0;u)u_0 - U(t,0;v)u_0\| + \|U(t,0;u)g(u) - U(t,0;v)g(v)\| \\
&\quad + \int_0^t \|U(t,s;u)\Big[f(s,u(s),u(\alpha(s))) + \int_0^s k(s,\tau,u(\tau),u(\beta(\tau)))d\tau\Big] \\
&\quad - U(t,s;v)\Big[f(s,v(s),v(\alpha(s))) + \int_0^s k(s,\tau,v(\tau),v((\beta(\tau)))d\tau\Big]\|ds \\
&\le \|U(t,0;u)u_0 - U(t,0;v)u_0\| + \|U(t,0;u)g(u) - U(t,0;v)g(u)\| \\
&\quad - \|U(t,0;v)g(u) - U(t,0;v)g(v)\| \\
&\quad + \int_0^t \Big\{\Big\|U(t,s;u)\Big[f(s,u(s),u(\alpha(s))) + \int_0^s k(s,\tau,u(\tau),u(\beta(\tau)))d\tau\Big] \\
&\quad - U(t,s;v)\Big[f(s,u(s),u(\alpha(s))) + \int_0^s k(s,\tau,u(\tau),u((\beta(\tau)))d\tau\Big]\| \\
&\quad + \|U(t,s;v)\Big[f(s,u(s),u(\alpha(s))) + \int_0^s k(s,\tau,u(\tau),u((\beta(\tau)))d\tau\Big] \\
&\quad - U(t,s;v)\Big[f(s,v(s),v(\alpha(s))) + \int_0^s k(s,\tau,v(\tau),v((\beta(\tau)))d\tau\Big]\|\Big\}ds
\end{aligned}
$$

Using assumptions (E8)-(E12), one can get

$$
\begin{aligned}
&\|Pu(t) - Pv(t)\| \\
&\le Ka\|u_0\|_Y \max_{\tau\in I}\|u(\tau)-v(\tau)\| + GKa\max_{\tau\in I}\|u(\tau)-v(\tau)\| \\
&\quad + M_0G_1\max_{\tau\in I}\|u(\tau)-v(\tau)\| \\
&\quad + Ka\max_{\tau\in I}\|u(\tau)-v(\tau)\|\Big[K_3\int_0^t\|u(s)\|ds + K_3\int_0^t\|u(\alpha(s))\|(\alpha'(s)/b)ds \\
&\quad + K_4a + K_1ar + K_1\int_0^t\|u(\beta(s))\|(\beta'(s)/c)ds + K_2a\Big] \\
&\quad + M_0\Big[K_3\int_0^t\|u(s)-v(s)\|ds + K_3\int_0^t\|u(\alpha(s))-v(\alpha(s))\|(\alpha'(s)/b)ds \\
&\quad + K_1a\max_{\tau\in I}\|u(\tau)-v(\tau)\| + K_1\int_0^t\|u(\beta(s))-v(\beta(s))\|(\beta'(s)/c)ds \\
&\le \Big[Ka\|u_0\|_Y + GKa + M_0G_1 + M_0[K_3a(1+1/b) + K_1a(1+1/c)] \\
&\quad + Ka[r(K_3a(1+1/b) + K_1a(1+1/c))] + a(K_4+K_2)\Big]\max_{\tau\in I}\|u(\tau)-v(\tau)\| \\
&= q\max_{\tau\in I}\|u(\tau)-v(\tau)\|
\end{aligned}
$$

where $0 < q < 1$. From this inequality it follows that for any $t \in I$,

$$
\|Pu(t) - Pv(t)\| \le q\max_{\tau\in I}\|u(\tau)-v(\tau)\|,
$$

so that P is a contraction on S. From the contraction mapping theorem it follows that P has a unique fixed point $u \in S$ which is the mild solution of (1.5))–(1.6) on $[0, a]$. Note that $u(t)$ is in $C(I : Y)$ by (E6) see [17, pp. 135, 201-202 Lemma 7.4]. In fact , $u(t)$ is weakly continuous as a Y-valued function. This implies that $u(t)$ is separably valued in Y, hence it is strongly measurable. Then $\|u(t)\|_Y$ is bounded and measurable function in t. Therefore, $u(t)$ is Bochner integrable (see e.g. [19, Chap.V]). Using relation $u(t) = Pu(t)$, we conclude that $u(t)$ is in $C(I : Y)$.

Now consider the evolution equation

$$
v'(t) + B(t)v(t) = h(t), \quad t \in [0, a] \tag{3.1}
$$

$$
v(0) = u_0 - g(u) \tag{3.2}
$$

where $B(t) = A(t, u(t))$ and $h(t) = f(t, u(t), u(\alpha(t))) + \int_0^t k(t, s, u(s), u(\beta(s))ds$, $t \in [0, a]$ and u is the unique fixed point of P in S. We note that $B(t)$ satisfies (H1)-(H3) in [17, Sec. 5.5.3] and $h \in C(I : Y)$. Theorem 5.5.2 of [17] implies that there exists a unique function $v \in C(I : Y)$ such that $v \in C^1((0, a], X)$ satisfying (3.1) and (3.2) in X and v is given by

$$
\begin{aligned}
v(t) = U(t,0;u)u_0 - U(t,0;u)g(u) + \int_0^t U(t,s;u)[f(s,u(s),u(\alpha(s))) \\
+ \int_0^s k(s,\tau,u(\tau),u(\beta(\tau)))d\tau]ds,
\end{aligned}
$$

where $U(t, s; u)$ is the evolution system generated by the family $\{A(t, u(t))\}, t \in I$ of the linear operators in X. The uniqueness of v implies that $v = u$ on I and hence u is a unique classical solution of (1.5))–(1.6) and $u \in C([0, a] : Y) \cap C^1((0, a] : X)$. $\square$

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Krishnan Balachandran

Department of Mathematics, Bharathiar University, Coimbatore 641 046, India

E-mail address: balachandran_k@lycos.com

Francis Paul Samuel

Department of Mathematics, Bharathiar University, Coimbatore 641 046, India

E-mail address: paulsamuel_f@yahoo.com