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THE CAUCHY PROBLEM FOR A SHORT-WAVE EQUATION

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Abstract. We prove existence and uniqueness of solutions for the Cauchy problem of the simplest nonlinear short-wave equation, $u_{tx} = u - 3u^2$, with periodic boundary condition.

1. Introduction

In this paper we consider the Cauchy problem for the short-wave equation

$$u_{tx} = u - 3u^2, \tag{1.1}$$

with the boundary condition ($L > 0$)

$$u(0,t) = u(L,t), \quad t \geq 0, \tag{1.2}$$

and the L-periodic initial condition

$$u(x,0) = \phi(x), \quad \forall x \in \mathbb{R}. \tag{1.3}$$

Here, $u(x,t)$ represents a small amplitude depending on one-dimensional (fast) space variable x and (slow) time t.

Nonlinear evolution of long waves in dispersive media with small amplitude in shallow water is a well known subject. It has been described by many mathematical models such as the Boussinesq equation [3, 8], the KdV equation [5], or the Benjamin-Bona-Mahony-Peregrine equation (BBMP) [1, 7]. In contrast, for short-waves, commonly called ripples, only a few results exist [6, 4, 2]. When we speak of long or short-waves, we are referring to an underlying spacescale, X, to which all space variables have been compared. Thus, for instance, for the surface-wave motion of a fluid, the unperturbed depth serves as a natural parameter. The shortness of the waves is referred to this underlying parameter.

The short-wave equation (1.1) is derived in [6] via multiple-scale perturbation theory from BBMP and governs the leading order term of the asymptotic dynamics of short-waves sustained by BBMP. A first study of equation (1.1) was done in [4]. We sketch here its derivation. Start from BBMP

$$U_T + U_X - U_{XXT} = 3(U^2)_X, \tag{1.4}$$

which is the model equation for the unperturbed equation to which we will find the short-wave limit. Here, $U(X,T)$ represents a small amplitude depending on

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one-dimensional space variable X and time T. Its linear dispersion relation, $\omega(k)$, is real (this means that we are not dealing with dissipative effects) and is given by

$$\omega(k) = \frac{k}{1+k^2}, \tag{1.5}$$

having zero limit when $k \to \infty$. The phase and group velocity are all bounded in the short-wave limit $k \to \infty$. This property allows BBMP to sustain short-waves. In fact, let us consider a short-wave with characteristic length $\ell = \varepsilon \sim k^{-1}$, with $k \gg 1$. Define the scaled (fast) space variable $x = \varepsilon^{-1}X$ ($\varepsilon \ll 1$). The characteristic time associated with short-waves is given by looking at the dispersive relation of the linear part for the time variable. In our case, $\omega(\varepsilon^{-1}) = \varepsilon - \varepsilon^3 + \varepsilon^5 - \dots$. In this way, we obtain the scaled (slow) time variable $t = \varepsilon T$. We are lead thus to the scaled variables $x = \varepsilon^{-1}X$ and $t = \varepsilon T$, which transforms the X and T derivatives into $\partial_X = \varepsilon^{-1}\partial_x$ and $\partial_T = \varepsilon\partial_t$. Assume now the expansion $U = u_0 + \varepsilon u_1 + \dots$. Passing to the x and t variables and integrating in x, we have the lowest order in (1.4) in the form

$$u_{0tx} = u_0 - 3(u_0)^2. \tag{1.6}$$

For simplicity, writing u_0 as u, we obtain (1.1).

In the next section, under certain conditions, we prove the existence and uniqueness of solutions for (1.1)-(1.3).

2. Main result

Let $u = u(x,t)$ be a classical solution to the Cauchy problem, that is, a twice continuously differentiable function satisfying (1.1)-(1.3). Integrating the left-hand side of (1.1) in x, from 0 to L, and using (1.2), we get

$$\frac{d}{dt}\int_0^L u_x(x,t)dx = \frac{d}{dt}\left(u(L,t) - u(0,t)\right) = 0.$$

Therefore, from (1.1), we have

$$0 = \frac{d}{dt}\int_0^L u_x(x,t)dx = \int_0^L \big(u(x,t) - 3u^2(x,t)\big)\,dx. \tag{2.1}$$

Thus, it is natural to consider only initial conditions satisfying (2.1).

Note also that the L_2-norm of $u_x(\cdot,t)$ is a constant. Indeed, multiplying both sides of (1.1) by u_x and integrating in x, from 0 to L, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_x(\cdot,t)|_2^2 &= \frac{d}{dt}\int_0^L \frac{u_x^2(x,t)}{2}dx \\
&= \int_0^L \big(u(x,t) - 3u^2(x,t)\big)u_x(x,t)dx \\
&= \int_0^L \frac{\partial}{\partial x}\big(\frac{u^2(x,t)}{2} - u^3(x,t)\big)dx \\
&= \big(\frac{u^2(L,t)}{2} - u^3(L,t)\big) - \big(\frac{u^2(0,t)}{2} - u^3(0,t)\big) = 0.
\end{aligned} \tag{2.2}$$

This observation is of importance in the proof of a global existence.

We will seek for solutions to problem (1.1)-(1.3) in a generalized sense. Namely, consider a formal Fourier series

$$u(x,t)=\sum_{n=-\infty}^{\infty} u_n(t)e^{i2\pi nx/L}, \quad u_{-n}=\overline{u_n}, \tag{2.3}$$

with coefficients depending on t. Assume that

$$u(x,0)=\phi(x), \quad x\in\mathbb{R},$$

where ϕ is an L-periodic function. It is assumed that $u_{-n}=\overline{u_n}$ or, equivalently, $u(x,t)\in\mathbb{R}$. Formally substituting Fourier series (2.3) in the differential equation we obtain a system of ordinary differential equations

$$\frac{du_n(t)}{dt}=-\frac{iL}{2\pi n}\Big(u_n(t)-3\sum_{\alpha+\beta=n,\, n\in\mathbb{Z}} u_\alpha(t)u_\beta(t)\Big), \quad n\neq 0. \tag{2.4}$$

(Denote $u_n(t)$ simply by u_n.) Note that, for $n=0$, we do not obain a differential equation for u_0, but a constraint relating u_0 to all the others Fourier modes. Since u_0 is the real function u average value over the domain of periodicity, we obtain the equation

$$u_0-3u_0^2=3\sum_{n\in\mathbb{Z},\, n\neq 0}|u_n|^2. \tag{2.5}$$

This equation admits real solutions

$$u_0=\frac{1}{6}\Big(1\pm\sqrt{1-36\sum_{n\in\mathbb{Z},\, n\neq 0}|u_n|^2}\Big), \tag{2.6}$$

only if $\sum_{n\in\mathbb{Z},\, n\neq 0}|u_n|^2\leq 1/36$. For definiteness assume from now on that the sign in formula (2.6) is plus, for example. The other choice is essentially the same, the major difference being the fact that it results in waves travelling in the opposite direction [4].

Rewrite (2.4), in the integral form

$$u_n(t)=\phi_n-\frac{iL}{2\pi n}\int_0^t\Big(u_n(s)-3\sum_{\alpha+\beta=n,\, n\in\mathbb{Z}} u_\alpha(s)u_\beta(s)\Big)ds, \quad n\neq 0, \tag{2.7}$$

Denote by H the space of complex sequences $v=\{v_n\}_{n\in\mathbb{Z}}$ with the norm

$$|v|=\Big(|v_0|^2+\sum_{n\in\mathbb{Z},\, n\neq 0} n^2|v_n|^2\Big)^{1/2}.$$

The space of L-periodic functions u with Fourier coefficients $\{u_n\}_{n=-\infty}^{\infty}\in H$, we shall also denote by H. Let

$$\phi(x)=\sum_{n=-\infty}^{\infty}\phi_n e^{i2\pi nx/L}\in H,$$

with $\phi_{-n}=\overline{\phi_n}$. We say that a function $u\in C\big([0,\infty),H\big)$,

$$t\to u(t)=\sum_{n=-\infty}^{\infty} u_n(t)e^{i2\pi nx/L}, \quad u_{-n}=\overline{u_n},$$

is a solution to problem (1.1)-(1.3), if $\dot u\in L_\infty\big([0,\infty),H\big)$, and the Fourier coefficients u_n satisfy (2.6), (2.7), and $u_n(0)=\phi_n$, for all n.

Now we are in a position to formulate the main result of this paper.

Theorem 2.1. *If $\phi \in H$ satisfies*

$$\sum_{n\in\mathbb{Z},\, n\neq 0} n^2|\phi_n|^2 < 1/72 \quad \textit{and} \quad \int_0^L \big(\phi(x) - 3\phi^2(x)\big)dx = 0,$$

then problem (1.1)-(1.3) has one and only one solution. For all $t \geq 0$, Fourier series (2.3) converges uniformly in x. Its sum is differentiable in x for almost all $x \in [0, L]$. The derivative satisfies the conditions $u_x(\cdot, t) \in L_2([0, L], R)$ and $u_x(x, \cdot) \in C([0, \infty[, R)$. Moreover, u_x is differentiable in t and (1.1) holds for almost all $x \in [0, L]$.

Remark. The uniform convergence of Fourier series (2.3) implies that $u(\cdot, t)$ is a continuous L-periodic function.

The proof of Theorem 2.1 is divided in several steps. First note that the condition

$$\int_0^L (\phi(x) - 3\phi^2(x))dx = 0,$$

implies

$$\phi_0 = 3|\phi_0|^2 + 3 \sum_{n\in\mathbb{Z},\, n\neq 0} |\phi_n|^2.$$

From this, we get

$$\phi_0 = \frac{1}{6}\Big(1 \pm \sqrt{1 - 36 \sum_{n\in\mathbb{Z},\, n\neq 0} |\phi_n|^2}\Big). \tag{2.8}$$

Since

$$\sum_{n\in\mathbb{Z},\, n\neq 0} |\phi_n|^2 \leq \sum_{n\in\mathbb{Z},\, n\neq 0} n^2|\phi_n|^2 < 1/72,$$

it follows that ϕ_0 is well defined. Let $v(\cdot) \in L_\infty([0, T], H)$. The norm in this space we shall denote by $\|v\|$. Define an operator $f : L_\infty([0, T], H) \to L_\infty([0, T], H)$ as follows:

$$f_n(v(\cdot))(t) = \phi_n - \frac{iL}{2\pi n} \int_0^t \Big(v_n(s) - 3 \sum_{k=-\infty}^{\infty} v_k(s)v_{n-k}(s)\Big)ds, \quad n \neq 0, \tag{2.9}$$

$$f_0(v(\cdot))(t) = \frac{1}{6}\Big(1 + \sqrt{1 - 36 \sum_{n\in\mathbb{Z},\, n\neq 0} |f_n(v(\cdot))(t)|^2}\Big). \tag{2.10}$$

Let $M > 0$. Denote by $\Phi \in L_\infty([0, T], H)$ the constant function $\Phi(t) \equiv \phi$ and consider a complete metric space

$$V_{TM} = \{v(\cdot) \in L_\infty([0, T], H) : \|v - \Phi\| \leq M\}$$

with the metric induced by $L_\infty([0, T], H)$. We need the following auxiliary results.

Proposition 2.2. *If $\sum_{n\neq 0} n^2|\phi_n|^2 < 1/72$ and T is sufficiently small, then f is well defined and is a contractive map from V_{TM} into V_{TM}.*

Proof. Since

$$\begin{aligned} f_n(v)(t) - f_n(w)(t) = -\frac{iL}{2\pi n} \int_0^t \Big[&(v_n(s) - w_n(s)) + 3 \sum_{k=-\infty}^{\infty} ((v_k(s) - w_k(s))v_{n-k}(s) \\ &+ w_k(s)(v_{n-k}(s) - w_{n-k}(s))\Big]ds, \quad n \neq 0, \end{aligned}$$

we have

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |f_n(v)(t)-f_n(w)(t)|^2 \\
&\le (\text{const}) \sum_{n\in\mathbb{Z},\, n\neq 0} \Big[\int_0^t \Big[|v_n(s)-w_n(s)| \\
&\quad + \sum_{k=-\infty}^{\infty} (|v_k(s)-w_k(s)||v_{n-k}(s)| + |w_k(s)||v_{n-k}(s)-w_{n-k}(s)|\Big] ds\Big]^2 \\
&\le (\text{const}) t \sum_{n\in\mathbb{Z},\, n\neq 0} \int_0^t \Big[|v_n(s)-w_n(s)| \\
&\quad + \sum_{k=-\infty}^{\infty} |v_k(s)-w_k(s)|(|v_{n-k}(s)|+|w_{n-k}(s)|)\Big]^2 ds \\
&\le (\text{const}) t \sum_{n\in\mathbb{Z},\, n\neq 0} \int_0^t \Big[|v_n(s)-w_n(s)|^2 \\
&\quad + \Big(\sum_{k=-\infty}^{\infty} |v_k(s)-w_k(s)|(|v_{n-k}(s)|+|w_{n-k}(s)|)\Big)^2\Big] ds \\
&\le (\text{const}) t \sum_{n\in\mathbb{Z},\, n\neq 0} \int_0^t \Big[|v_n(s)-w_n(s)|^2 + |v_0(s)-w_0(s)|^2(|v_n(s)|^2+|w_n(s)|^2) \\
&\quad + \Big(\sum_{k\neq 0}\frac{1}{k^2}\Big) \sum_{k=-\infty}^{\infty} k^2|v_k(s)-w_k(s)|^2(|v_{n-k}(s)|^2+|w_{n-k}(s)|^2)\Big] ds \\
&\le (\text{const}) t \int_0^t \Big[\sum_{n\in\mathbb{Z},\, n\neq 0} |v_n(s)-w_n(s)|^2 + |v_0(s)-w_0(s)|^2 \sum_{n\in\mathbb{Z},\, n\neq 0} (|v_n(s)|^2 \\
&\quad + |w_n(s)|^2) \sum_{k=-\infty}^{\infty} k^2 |v_k(s)-w_k(s)|^2 \sum_{n\in\mathbb{Z},\, n\neq 0} (|v_n(s)|^2+|w_n(s)|^2)\Big] ds \\
&\le (\text{const}) t \int_0^t \Big[1+\sum_{n\in\mathbb{Z},\, n\neq 0}(|v_n(s)|^2+|w_n(s)|^2)\Big] ds \|v-w\|^2 \\
&\le (\text{const}) T^2 (1+\|v\|^2+\|w\|^2)\|v-w\|^2.
\end{aligned}
$$

We have thus proved the inequality

$$
\sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |f_n(v)(t)-f_n(w)(t)|^2 \le (\text{const}) T^2 (1+\|v\|^2+\|w\|^2)\|v-w\|^2. \tag{2.11}
$$

We also have

$$
\begin{aligned}
&|f_0(v)(t)-f_0(w)(t)|^2 \\
&= \frac{1}{36}\Big|\sqrt{1-36\sum_{n\in\mathbb{Z},\, n\neq 0}|f_n(v)(t)|^2} - \sqrt{1-36\sum_{n\in\mathbb{Z},\, n\neq 0}|f_n(w)(t)|^2}\Big|^2 \qquad (2.12) \\
&\le \frac{(\text{const})\sum_{n\in\mathbb{Z},\, n\neq 0}(|f_n(v)(t)|^2+|f_n(w)(t)|^2)}{\big|\sqrt{1-36\sum_{n\in\mathbb{Z},\, n\neq 0}|f_n(v)(t)|^2}+\sqrt{1-36\sum_{n\in\mathbb{Z},\, n\neq 0}|f_n(w)(t)|^2}\big|^2}
\end{aligned}
$$

$$\times \sum_{n\in\mathbb{Z},\, n\neq 0} |f_n(v)(t) - f_n(w)(t)|^2.$$

The inclusion $v \in V_{TM}$ implies $\|v\|^2 \le (\|\Phi\| + \|\Phi - v\|)^2 \le (\|\Phi\| + M)^2$. Since $\Phi = f(0)$, from (2.11) we get

$$\sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |f_n(v)(t) - \phi_n|^2 \le (\text{const}) T^2 (1 + (\|\Phi\| + M)^2)^2.$$

Therefore

$$\begin{aligned} \sum_{n\in\mathbb{Z},\, n\neq 0} |f_n(v)(t)|^2 &\le 2 \sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |\phi_n|^2 + 2 \sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |f_n(v)(t) - \phi_n|^2 \\ &\le 2 \sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |\phi_n|^2 + (\text{const}) T^2 (1 + (\|\Phi\| + M)^2)^2 \\ &\le \sigma < \frac{1}{36}, \end{aligned}$$

whenever $T > 0$ is small enough. Thus the map f is well defined (see (2.9) and (2.10)). From (2.11) and (2.12) we obtain

$$\begin{aligned} |f_0(v)(t) - f_0(w)(t)|^2 &\le (\text{const}) \sum_{n\in\mathbb{Z},\, n\neq 0} |f_n(v)(t) - f_n(w)(t)|^2 \\ &\le (\text{const}) \sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |f_n(v)(t) - f_n(w)(t)|^2 \\ &\le (\text{const}) T^2 (1 + \|v\|^2 + \|w\|^2) \|v - w\|^2. \end{aligned}$$

Invoking again (2.11), we get

$$\begin{aligned} \|f(v) - f(w)\|^2 &\le (\text{const}) T^2 (1 + \|v\|^2 + \|w\|^2) \|v - w\|^2 \\ &\le (\text{const}) T^2 (1 + (\|\Phi\| + M)^2) \|v - w\|^2. \end{aligned} \tag{2.13}$$

In particular, we have

$$\|f(v) - \Phi\|^2 \le (\text{const}) T^2 (1 + (\|\phi\| + M)^2)^2 \le M^2,$$

for small $T > 0$. Thus we see that $f : V_{TM} \to V_{TM}$ and from (2.13) it follows that f is a contraction, whenever $T > 0$ is small enough. □

Proposition 2.3. *Let $u \in L_\infty([0, T], H)$ be a solution to the equation $u = f(u)$. Assume that*

$$\sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |u_n(t)|^2 \le \delta < 1/36.$$

Then $u \in C([0, T], H)$ and $\dot{u} \in L_\infty([0, T], H)$.

Proof. Similarly to inequality (2.12) we have

$$\begin{aligned} |u(t_2) - u(t_1)|^2 &= |u_0(t_2) - u_0(t_1)|^2 + \sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |u_n(t_2) - u_n(t_1)|^2 \\ &\le (\text{const}) \sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |u_n(t_2) - u_n(t_1)|^2. \end{aligned}$$

From (2.9) we see that the right side of the inequality is less than or equal to

$$(\text{const}) |t_2 - t_1| \sum_{n\in\mathbb{Z},\, n\neq 0} \Big| \int_{t_1}^{t_2} \Big(|u_n(s) + 3 \sum_{k=-\infty}^{\infty} |u_k(s)| |u_{n-k}(s)| \Big)^2 ds \Big|$$

$$
\begin{aligned}
&\le (\text{const})|t_2-t_1| \sum_{n\in\mathbb{Z},\, n\neq 0} \int_{t_1}^{t_2} \Big(1+|u_0(s)|^2 \\
&\quad + \Big(\sum_{k\in\mathbb{Z},\, k\neq 0} \frac{1}{k^2}\Big) \sum_{k\in\mathbb{Z},\, k\neq 0} k^2|u_k|^2\Big) \sum_{n\in\mathbb{Z},\, n\neq 0} |u_n(s)|^2 ds \\
&\le (\text{const})|t_2-t_1|^2.
\end{aligned}
$$

This proves the continuity of $u(t)$. Since

$$
|\dot u_0|^2 = \frac{9\Big|\sum_{n\in\mathbb{Z},\, n\neq 0}(\dot u_n u_{-n} + u_n \dot u_{-n})\Big|^2}{1-36\sum_{n\in\mathbb{Z},\, n\neq 0}|u_n|^2} \le (\text{const}) \sum_{n\in\mathbb{Z},\, n\neq 0} |\dot u_n|^2 \sum_{n\in\mathbb{Z},\, n\neq 0} |u_n|^2
$$

and

$$
\sum_{n\in\mathbb{Z},\, n\neq 0} n^2|\dot u_n|^2 = \sum_{n\in\mathbb{Z},\, n\neq 0} \big(\frac{L}{2\pi}\big)^2 \Big|u_n - 3\sum_{n\in\mathbb{Z},\, n\neq 0} u_k u_{n-k}\Big|^2,
$$

we have

$$
\begin{aligned}
&|\dot u_0|^2 + \sum_{n\in\mathbb{Z},\, n\neq 0} n^2|\dot u_n|^2 \\
&\le (\text{const}) \sum_{n\in\mathbb{Z},\, n\neq 0} \Big|u_n - 3\sum_{n\in\mathbb{Z},\, n\neq 0} u_k u_{n-k}\Big|^2 \\
&\le (\text{const}) \sum_{n\in\mathbb{Z},\, n\neq 0} \Big(|u_n|^2 + |u_0|^2|u_n|^2 + \Big(\sum_{k\in\mathbb{Z},\, k\neq 0}\frac{1}{k^2}\Big) \sum_{k\in\mathbb{Z},\, k\neq 0} k^2|u_k|^2|u_{n-k}|^2\Big) \\
&\le (\text{const})\Big(1+|u_0|^2 + \Big(\sum_{k\in\mathbb{Z},\, k\neq 0}\frac{1}{k^2}\Big) \sum_{k\in\mathbb{Z},\, k\neq 0} k^2|u_k|^2\Big) \sum_{n\in\mathbb{Z},\, n\neq 0} |u_n|^2 \le (\text{const}).
\end{aligned}
$$

Thus $\dot u \in L_\infty([0,T],H)$. □

Note that we also proved that the function $u \in C([0,T],H)$ is Lipschitzian. Now show that generalized solutions also satisfy property (2.2).

Proposition 2.4. *Assume that $u \in L_\infty([0,T],H)$ satisfies* (2.6). *Then*

$$
\sum_{n\in\mathbb{Z},\, n\neq 0} n^2|u_n(t)|^2 = (\text{const}).
$$

Proof. Indeed, we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{n=-\infty}^{\infty} \big(\frac{2\pi n}{L}\big)^2 |u_n|^2 \\
&= \sum_{n=-\infty}^{\infty} \big(\frac{2\pi n}{L}\big)^2 (\dot u_n u_{-n} + u_n \dot u_{-n}) \\
&= \frac{2\pi i}{L} \sum_{n=-\infty}^{\infty} n\Big[u_n\Big(u_{-n} - 3\sum_{k=-\infty}^{\infty} u_k u_{-n-k}\Big) - u_{-n}\Big(u_n - 3\sum_{k=-\infty}^{\infty} u_k u_{n-k}\Big)\Big] \\
&= -\frac{6\pi i}{L} S,
\end{aligned}
$$

where

$$S = \sum_{n=-\infty}^{\infty} n\Big[u_n \sum_{k=-\infty}^{\infty} u_k u_{-n-k} - u_{-n} \sum_{k=-\infty}^{\infty} u_k u_{n-k}\Big].$$

Observe that

$$S = \sum_{n,k=-\infty}^{\infty} n u_n u_k u_{-n-k} - \sum_{n,k=-\infty}^{\infty} n u_{-n} u_k u_{n-k} = 2 \sum_{n,k=-\infty}^{\infty} n u_n u_k u_{-n-k}.$$

On the other hand, introducing a new summation index $m = n - k$, we can rewrite S in the form

$$S = \sum_{n,k=-\infty}^{\infty} n u_n u_k u_{-n-k} - \sum_{m,k=-\infty}^{\infty} (m+k) u_{-m-k} u_k u_m = - \sum_{m,k=-\infty}^{\infty} k u_{-m-k} u_k u_m.$$

Combining this with the previous equality, we get $S = -S/2$. Thus $S = 0$. □

Proof of Theorem 2.1. From Proposition 2.2 we see that the problem under consideration has one and only one solution $u \in L_\infty([0,T],H)$, whenever $T > 0$ is small enough. By Proposition 2.3 $u \in C([0,T],H)$ and $\dot{u} \in L_\infty([0,T],H)$. Finally, Proposition 2.4 implies the existence of the solution for all $t \geq 0$.

Show that, $u(x,t)$, the sum of Fourier series (2.3) satisfies (1.1). From the inequality

$$\sum_{n\in\mathbb{Z},\, n\neq 0} |u_n(t)| \leq \sqrt{\Big(\sum_{n\in\mathbb{Z},\, n\neq 0} \frac{1}{n^2}\Big) \sum_{n\in\mathbb{Z},\, n\neq 0} n^2 |u_n(t)|^2} = (\text{const})$$

we see that Fourier series (2.3) converges uniformly in x for all $t \geq 0$. The inequality

$$\sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{\infty} u_k(t) u_{n-k}(t) \Big| \leq \sum_{k=-\infty}^{\infty} |u_k(t)| \sum_{n=-\infty}^{\infty} |u_n(t)|$$

implies that the series

$$\sum_{n=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} u_k(t) u_{n-k}(t) \Big) e^{i2\pi nx/L}$$

converges for all $t \geq 0$. Multiplying (2.7) by $e^{i2\pi nx/L}$ and adding the obtained equalities, we get

$$\begin{aligned}\sum_{n=-\infty}^{\infty} i\frac{2\pi}{L} n u_n(t) e^{i2\pi nx/L} &= \sum_{n=-\infty}^{\infty} i\frac{2\pi}{L} n\phi_n e^{i2\pi nx/L} + \sum_{n\in\mathbb{Z},\, n\neq 0} \int_0^t \Big(u_n(s) \\ &\quad - 3 \sum_{\alpha+\beta=n,\, n\in\mathbb{Z}} u_\alpha(s) u_\beta(s)\Big) e^{i2\pi nx/L} ds\end{aligned}$$

From the Lebesgue dominated convergence theorem and the above estimates we have

$$u_x(x,t) = \phi_x(x) + \int_0^t \sum_{n\in\mathbb{Z},\, n\neq 0} \Big(u_n(s) - 3 \sum_{\alpha+\beta=n,\, n\in\mathbb{Z}} u_\alpha(s) u_\beta(s)\Big) e^{i2\pi nx/L} ds.$$

Combining this with (2.5), we obtain

$$u_x(x,t) = \phi_x(x) + \int_0^t (u(x,s) - 3u^2(x,s)) ds.$$

This completes the proof. □

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