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POSITIVE SOLUTIONS FOR SEMI-LINEAR ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS

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Abstract. We prove the existence of a solution, decaying to zero at infinity, for the second order differential equation

$$\frac{1}{A(t)}(A(t)u'(t))' + \phi(t) + f(t,u(t)) = 0, \quad t \in (a,\infty).$$

Then we give a simple proof, under some sufficient conditions which unify and generalize most of those given in the bibliography, for the existence of a positive solution for the semilinear second order elliptic equation

$$\Delta u + \varphi(x,u) + g(|x|)x.\nabla u = 0,$$

in an exterior domain of the Euclidean space $\mathbb{R}^n, n \geq 3$.

1. Introduction

The semilinear elliptic equation

$$\Delta u + \varphi(x,u) + g(|x|)x.\nabla u = 0, \quad x \in G_\delta = \{x \in \mathbb{R}^n : |x| > \delta > 0\}, \tag{1.1}$$

constitutes the object of numerous investigations in the last few years (see [1, 4, 5, 6, 7, 8, 9, 13, 14]). The function φ is nonnegative and locally Hölder continuous in $G_\delta \times \mathbb{R}$ for which there exist two continuous functions $q : [\delta,\infty) \to [0,\infty)$ and $\omega : [0,\infty) \to [0,\infty)$ such that

$$0 \leq \varphi(x,t) \leq q(|x|)\omega(t), \quad t \in [0,\infty),\ x \in G_\delta\,.$$

So far, the optimal sufficient condition stated to ensure the existence of a positive solution, decaying to zero at infinity, for (1.1) in some G_B with $B > \delta$ is

$$\int_\delta^\infty r\ \big[q(r) + g^-(r)\big]\ dr < \infty\,, \tag{1.2}$$

where $g^-(r) = \max(-g(r),0)$ for $r \geq \delta$.

To apply the method of sub-solutions and super-solutions developed in [13] and other works, the scaling function $|x| = r = \beta(s) = (\frac{s}{n-2})^{1/(n-2)}$ plays a capital role in finding a radial super-solution for (1.1) of the form $u(x) = h(|x|) = h(r)$, where h is chosen so that $y(s) = sh(\beta(s))$ satisfies a nonlinear differential equation

$$y''(s) + G(s,y(s),y'(s)) = 0 \quad s \geq s_0 = (n-2)\delta^{n-2}. \tag{1.3}$$

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As a sub-solution of (1.1) we understand any function $\omega \in C^2(G_B) \cap C(\overline{G_B})$ such that $\Delta\omega(x) + \varphi(x, \omega(x)) + g(|x|)x \cdot \nabla\omega(x) \geq 0$ in G_B. For the super-solution, the sign of the inequality should be reversed.

Our aim in this paper is twofold. Firstly, we study in section 2 the existence of solutions, having a nonnegative limit at infinity, for the problem

$$\frac{1}{A(t)}(A(t)u'(t))' + \phi(t) + f(t, u(t)) = 0, \quad t \in (a, \infty), \tag{1.4}$$

where A and f satisfy some hypothesis stated in the next section. Secondly, in section 3, we omit the scaling function β defined before and we give a simple proof for the existence of positive solutions, decaying to zero at infinity, in some G_B, $B > \delta$ for the semi-linear elliptic equation (1.1). This will be done under sufficient conditions given by the hypotheses (A3)-(A4) below, which improve and generalize (1.2). More precisely we will prove the existence of a positive solution to (1.1) even when $\int_\delta^\infty r\, g^-(r)\, dr = \infty$.

2. Positive solutions of second-order ODEs

In this section, we are concerned with the existence of positive solutions for the problem

$$\begin{gathered} \frac{1}{A(t)}(A(t)u'(t))' + \phi(t) + f(t, u(t)) = 0, \quad \text{for } t \geq a > 1 \\ Au'(a) = -\alpha \leq 0\,, \quad \lim_{t\to\infty} u(t) = \lambda \geq 0\,, \quad \text{with } \alpha + \lambda > 0\,, \end{gathered} \tag{2.1}$$

where A is a positive and differentiable function on $[1, \infty)$, ϕ is a nonnegative continuous function on $[1, \infty)$ and $f : [1, \infty) \times [0, \infty) \to [0, \infty)$ is continuous such that $f(x, 0) = 0$.

In the sequel we suppose that $\int_1^\infty \frac{1}{A(t)}\, dt < \infty$ and we denote by

$$G(t) = A(t)\Big(\int_t^\infty \frac{1}{A(s)}\, ds\Big)$$

for $t \geq 1$. The following hypotheses satisfied by A, ϕ and f throughout this section:

- (A1) $\int_1^\infty G(t)\phi(t)\, dt < \infty$;
- (A2) For each $c > 0$, there exists a continuous function $k : [1, \infty) \to [0, \infty)$ such that
$$|f(t, u) - f(t, v)| \leq k(t)|u - v| \quad \text{for any } (t, u, v) \in [1, \infty) \times [0, c] \times [0, c]$$
and $\int_1^\infty G(t)\, k(t)\, dt < \infty$.

Our first existence result is the following.

Theorem 2.1. *Let $\alpha \geq 0$ and $\lambda \geq 0$ with $\alpha + \lambda > 0$. Under the hypotheses* (A1)-(A2)*, there exists $a > 1$ such that* (2.1) *has a unique positive solution $u \in C^1([a, \infty), \mathbb{R})$.*

Proof. Let

$$c > M := \lambda + \alpha \int_1^\infty \frac{1}{A(t)}\, dt + \int_1^\infty G(t)\phi(t)\, dt.$$

From (A2), there exists a k such that $|f(s, u) - f(s, v)| \leq k(s)|u - v|$ for any $(s, u, v) \in [1, \infty) \times [0, c] \times [0, c]$ and $\int_1^\infty G(t)\, k(t)\, dt < \infty$. Let $a > 1$ such that

$$\int_a^\infty G(t)k(t)\, dt < 1 - \frac{M}{c} := \sigma.$$

We denote by $C_b([a,\infty),\mathbb{R})$ the set of continuous bounded real valued functions on $[a,\infty)$ and by

$$\Gamma := \{u \in C_b([a,\infty),\mathbb{R}) : \lambda \le u \le c\}.$$

Then Γ endowed with the supremum norm is a Banach space. To apply a fixed point argument, we define the operator T on Γ by

$$Tu(r) = \lambda + \alpha \int_r^\infty \frac{1}{A(t)}dt + \int_r^\infty \frac{1}{A(t)}\Big(\int_a^t A(s)[\phi(s) + f(s,u(s))]ds\Big)dt. \quad (2.2)$$

First, we claim that $T(\Gamma) \subset \Gamma$. Indeed, from (A2) and Fubini theorem, we get that for each $u \in \Gamma$ any $r \ge a$,

$$\begin{aligned}\lambda \le Tu(r) &\le \lambda + \alpha \int_a^\infty \frac{1}{A(t)}dt + \int_a^\infty \frac{1}{A(t)}\Big[\int_a^t A(s)\big(\phi(s) + ck(s)\big)\,ds\Big]\,dt \\ &\le \lambda + \alpha \int_a^\infty \frac{1}{A(t)}dt + \int_a^\infty G(s)\phi(s)\,ds + c\int_a^\infty G(s)k(s)\,ds \le c.\end{aligned}$$

Now, we have to show that T is a contraction on $(\Gamma, \|.\|_\infty)$. Indeed, let $u, v \in \Gamma$ and $r \in [a,\infty)$. Then by the assumption (A2) and Fubini theorem we have

$$\begin{aligned}|Tu(r) - Tv(r)| &\le \int_r^\infty \frac{1}{A(t)}\Big(\int_a^t A(s)k(s)|u(s) - v(s)|\,ds\Big)dt \\ &\le \|u - v\|_\infty \int_a^\infty A(s)k(s)\Big(\int_s^\infty \frac{1}{A(t)}\,dt\Big)ds,\end{aligned}$$

which implies that $\|Tu - Tv\|_\infty \le \sigma\,\|u - v\|_\infty$. Thus, by the Banach fixed point theorem, there exists a unique point $u \in (\Gamma, \|.\|_\infty)$ such that $Tu = u$. It is easy to verify that u is the unique solution in $C^1([a,\infty),\mathbb{R})$ for (2.1). This completes the proof. □

It is worth pointing out that for any given $u(a) \ge 0$ and $u'(a) \le 0$, the corresponding solution to the equation is unique and defined for all times (that is, blowup is not possible), see [2, 3, 11]. Also and under more restrictive conditions, the asymptotic behavior of the solutions have been studied, see [12].

Example 2.2. Let $\sigma > 0$ and $\theta : [1,\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{t\to\infty}\theta(t) = 0$. Let $A(t) = t^{\sigma+1}\exp\big(\int_1^t \frac{\theta(s)}{s}\,ds\big)$. Then $\lim_{t\to\infty}\frac{t\,A'(t)}{A(t)} = \sigma+1 > 1$. So $\int_1^\infty \frac{1}{A(s)}ds < \infty$ and we have $\int_t^\infty \frac{1}{A(s)}\,ds \sim \frac{t}{\sigma\,A(t)}$ as $t \to \infty$. Consequently $G(t) \sim \frac{t}{\sigma}$ as $t \to \infty$.

Let q, ρ be respectively two nontrivial nonnegative continuous function on $[1,\infty)$ and $[0,\infty)$ such that $\int_1^\infty t\,q(t)dt < \infty$ and put $f(t,u) = q(t)\int_0^u \rho(s)ds$. Then for each nonnegative continuous function ϕ on $[1,\infty)$ satisfying $\int_1^\infty t\,\phi(t)dt < \infty$, there exists $a > 1$ such that (2.1) has a unique positive solution $u \in C^1([a,\infty),\mathbb{R})$.

3. Applications to elliptic equations

In this section, we are concerned with the nonlinear second order elliptic equation (1.1) in an exterior domain $G_\delta = \{x \in \mathbb{R}^n : |x| > \delta\}$, where $n \ge 3$ and $\delta \ge 0$. We prove, under some assumptions on the functions φ, g, that (1.1) has a positive solution in G_B for $B \ge \delta$ decaying to zero as $|x|$ tends to infinity. More precisely, we omit the function β defined in section 1 and we apply the result in section 2 to give a simple proof for the existence of positive solution, decaying to zero, for (1.1) in G_B with B large enough.

To this aim, we consider two continuous functions φ and g satisfying

(A3) $\varphi \in C(G_\delta \times \mathbb{R}, \mathbb{R}_+)$ and there exists a nonnegative continuous function f on $[\delta, \infty) \times \mathbb{R}$ such that $f(t, 0) = 0$ and a nonnegative continuous function ϕ on $[\delta, \infty)$ such that $0 \le \varphi(x, u) \le f(|x|, u) + \phi(|x|)$. Moreover for each $c > 0$, there exists a nontrivial nonnegative continuous function k defined on $[\delta, \infty)$ such that,
$$|f(t, u) - f(t, v)| \le k(t)|u - v|, \quad \forall u, v \in [0, c],\ \forall t \ge \delta;$$

(A4)
$$\int_\delta^\infty [k(t) + \phi(t)]A(t)\Big(\int_t^\infty \frac{1}{A(r)} dr\Big) dt < \infty,$$
where $A(t) = t^{n-1} \exp\big(-\int_\delta^t \xi\, g^-(\xi) d\xi\big)$ and $g^- = \max(-g, 0)$.

In the particular case when $\int_\delta^\infty r\, g^-(r)\, dr < \infty$, hypothesis (A4) reduces to $\int_\delta^\infty t\, [k(t) + \phi(t)]\, dt < \infty$. So hypothesis (A4) is weaker than the condition (1.2) given in the introduction where $\phi = 0$.

Next, we recall the following two lemmas needed to achieve the proof of our second main result.

Lemma 3.1 ([13]). *If for some $B \ge \delta$, there exists a nonnegative sub-solution w and a nonnegative super-solution v to (1.1) in G_B, such that $w(x) \le v(x)$ for all $x \in \overline{G_B}$, then (1.1) has a solution u in G_B, such that $w \le u \le v$ in $\overline{G_B}$ and $u = v$ on $S_B = \{x \in \mathbb{R}^n / |x| = B\}$.*

Lemma 3.2 ([10, Theorem 3.5]). *Let $\pounds$ be a uniformly elliptic operator on a domain Ω. Let $u \in C^2(\Omega)$ such that $\pounds u \ge 0$ in Ω. If there exists $x_0 \in \Omega$ satisfying $\sup_{x \in \Omega} u(x) = u(x_0)$, then u is constant in all Ω.*

Now, we give our main result in this section.

Theorem 3.3. *Let $\delta > 0$ and assume (A3)-(A4). Then (1.1) has a positive solution u in G_B for some $B \ge \delta$, such that $\lim_{x \to \infty} u(x) = 0$.*

Proof. We will apply Lemma 3.1. Clearly the trivial function $w = 0$ is a sub-solution of (1.1) in G_δ. Next, we try to find a positive radial super-solution $y(r) = y(|x|)$ for (1.1) with $\lim_{r \to \infty} y(r) = 0$. Taking into account (A3), it suffices to find a function y such that
$$y'' + [\frac{n-1}{r} + r g(r)] y' + f(r, y) + \phi(r) \le 0 \quad \text{for } r > B > \delta$$
$$\lim_{r \to \infty} y(r) = 0.$$

Now, taking into account of Theorem 2.1, it suffices to find $B > \delta$ and a solution for the problem
$$y'' + [\frac{n-1}{r} - r g^-(r)] y' + f(r, y) + \phi(r) = 0, \quad r > B$$
$$y'(r) < 0, \quad r > B, \quad \lim_{r \to \infty} y(r) = 0.$$

Or equivalently,

$$\begin{gathered}\frac{1}{A(r)}(A(r)y'(r))' + f(r,y) + \phi(r) = 0, \quad r > B\\ y'(r) < 0, \quad r > B, \quad \lim_{r\to\infty} y(r) = 0\,,\end{gathered}\tag{3.1}$$

where

$$A(r) = r^{n-1}\exp\Big(-\int_\delta^r \xi g^-(\xi)d\xi\Big).$$

So it follows from hypotheses (A3)-(A4) and Theorem 2.1 that there exists $B > \delta$ such that (3.1) has a positive solution $y(r)$ on $[B,\infty)$. Obviously y is a supersolution for (1.1) in G_B. Hence, by Lemma 3.1, problem (1.1) has a solution u in G_B such that $0 \le u(x) \le y(|x|)$ in G_B and $u = y > 0$ on S_B.

Next, we prove that the solution u is positive in G_B. Suppose that there exists $x_0 \in G_B$ such that $u(x_0) = 0$. Then, the uniformly elliptic operator $\pounds u := \Delta u + g(|x|)x.\nabla u$ satisfies $\pounds(-u) \ge \varphi(x,u) \ge 0$ in G_B and $\sup_{x\in G_B}(-u(x)) = -u(x_0) = 0$. Hence by Lemma 3.2 we obtain $u = 0$ in G_B. From the continuity of u in $\overline{G_B}$, this contradicts the fact that $u > 0$ on S_B and shows that $u(x) > 0$, for all $x \in G_B$. □

Example 3.4. In the sequel, we define by $\operatorname{Log}_0 t = t$ and $\operatorname{Log}_m t = \operatorname{Log}(\operatorname{Log}_{m-1} t)$ for $m \in \mathbb{N}^\star$ and t large enough. Let $\delta_m > 0$ such that $\operatorname{Log}_m(\delta_m) = 1$ and let g be a continuous function on $[\delta_m,\infty)$ such that

$$g^-(r) = \max(-g(r),0) = \frac{\gamma}{r\prod_{k=0}^m \operatorname{Log}_k(r)},\tag{3.2}$$

where $\gamma > 0$ if $m \in \mathbb{N}^\star$ and $0 < \gamma < n-2$ if $m = 0$. Then $t\,g^-(t) = \gamma\frac{d}{dt}(\operatorname{Log}_{m+1} t)$ and so

$$\exp\Big(\int_{\delta_m}^t s\,g^-(s)\,ds\Big) = (\operatorname{Log}_m t)^\gamma.$$

Thus, $\int_{\delta_m}^\infty r\,g^-(r)\,dr = \infty$ and (A4) is satisfied if and only if

$$\int_{\delta_m}^\infty t[k(t) + \phi(t)]dt < \infty.$$

Indeed, this follows from Example 2.2 with $\theta(s) = -s^2 g^-(s), \sigma = n-2$ if $m \in \mathbb{N}^\star$ and $\theta = 0, \sigma = n-2-\gamma$ if $m = 0$.

Now, using this fact we deduce that if g is a function where g^- is given by (3.2), if ϕ and k are two nonnegative continuous functions on $[\delta_m,\infty)$ satisfying $\int_{\delta_m}^\infty t\,[k(t) + \phi(t)]\,dt < \infty$ and if $0 \le \varphi(x,u) \le k(|x|)u^\alpha + \phi(|x|)$ for $\alpha \ge 1$, then there exists $B > \delta_m$ such that (1.1) has a positive solution u on G_B decaying to zero at infinity.

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References

[1] R. P. Agarwal, O. G. Mustafa: *Riccatian approach to the decay of solutions of certain semi-linear PDE's*, Appl. Math. Lett. 20 (2007), 1206–1210.

[2] S. R. Bernfeld: *The extendability of solutions of perturbed scalar differential equations*, Pacific J. Math. 42 (1972), 277–288.

[3] A. Constantin: *Global existence of solutions for perturbed differential equations*, Ann. Mat. Pura Appl. 168 (1995), 237–299.
[4] A. Constantin: *Existence of positive solutions of quasilinear elliptic equations*, Bull. Austral. Math. Soc. 54 (1996) 147–154.
[5] A. Constantin: *Positive solutions of quasilinear elliptic equations*, J. Math. Anal. Appl. 213 (1997) 334–339.
[6] A. Constantin: *On the existence of positive solutions of second order differential equations*, Ann. Mat. Pura App. 184 (2005) 131–138.
[7] M. Ehrnström: *On radial solutions of certain semi-linear elliptic equations*, Nonlinear. Anal. TMA .64 (2006) 1578–1586.
[8] M. Ehrnström: *Positive solutions for second-order nonlinear differential equations*, Nonlinear. Anal. TMA .64 (2006) 1608–1620.
[9] M. Ehrnström, O. G. Mustafa: *On positive solutions of a class of nonlinear elliptic equations*, Nonlinear Anal. 67 (2007) 1147–1154.
[10] D. Gilbarg, N. S. Trudinger: *Elliptic partial differential equations of second order*, Springer (1983).
[11] T. Hara, T. Yoneyama and J. Sugie: *Continuability of solutions of perturbed differential equations*, Nonlinear Anal. 8 (1984), 963–975.
[12] O. Lipovan: *On the asymptotic behavior of the solutions to a class of second order nonlinear differential equations*, Glasgow Math. J. 45 (2003), 179–187.
[13] E. S. Noussair, C. A. Swanson: *Positive solutions of quasilinear elliptic equations in exterior domains*, J. Math. Anal. Appl. 75 (1980), 121–133.
[14] F. H. Wong, S. P. Wang, and C. C. Yeh: *Positive solutions of nonlinear elliptic equations*, Appl. Math. Lett. 21 (2008), 298–302.

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