LOW REGULARITY SOLUTIONS OF THE
CHERN-SIMONS-HIGGS EQUATIONS IN THE LORENTZ
GAUGE

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Abstract. We prove local well-posedness for the 2 + 1-dimensional Chern-
Simons-Higgs equations in the Lorentz gauge with initial data of low regularity.
Our result improves earlier results by Huh [10, 11].

1. Introduction

The Chern-Simon-Higgs model was proposed by Jackiw and Weinberg [12] and
Hong, Pac and Kim [9] in the context of their studies of vortex solutions in the
abelian Chern-Simons theory.

Local well-posedness of low regularity solutions was recently studied in Huh
[10, 11] using a null-form estimate for solutions of the linear wave equation due to
Foschi and Klainerman [8] as well as Strichartz estimates. Our aim in this paper
is to improve the results of [10, 11] in the Lorentz gauge. For this purpose we
use estimates in the restriction spaces $X^{s,b}$ introduced by Bourgain, Klainerman
and Machedon. A key ingredient in our proof is a modified version of a null-form
estimate of Zhou [19] and product rules in $X^{s,b}$ spaces due to D’Ancona, Foschi
and Selberg [6, 7] and Klainerman and Selberg [13]. The Higgs field has fractional
dimension (see below for details), a common feature of systems involving the Dirac
equation, see for example Bournaveas [1, 2], D’Ancona, Foschi and Selberg [6, 7],
Machihara [14, 15], Machihara, Nakamura, Nakanishi and Ozawa [16], Selberg and
Tesfahun [17], Tesfahun [18].

The Chern-Simon-Higgs equations are the Euler-Lagrange equations correspond-
ing to the Lagrangian density

$$
\mathcal{L} = \frac{\kappa}{4} \epsilon^{\mu
u\rho} A_\mu F_{\nu\rho} + D_\mu \phi \bar{D}^\mu \phi - V(|\phi|^2).
$$

Here $A_\mu$ is the gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature, $D_\mu = \partial_\mu - i A_\mu$ is
the covariant derivative, $\phi$ is the Higgs field, $V$ is a given positive function and $\kappa$
is a positive coupling constant. Greek indices run through $\{0, 1, 2\}$, Latin indices run
through $\{1, 2\}$ and repeated indices are summed. The Minkowski metric is defined

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by \((g^{\mu\nu}) = \text{diag}(1, -1, -1)\). We define \(\epsilon^{\mu\nu\rho} = 0\) if two of the indices coincide and \(\epsilon^{\mu\nu\rho} = \pm 1\) according to whether \((\mu, \nu, \rho)\) is an even or odd permutation of \((0, 1, 2)\).

We define Klainerman’s null forms by

\[
Q_{\mu\nu}(u, v) = \partial_\mu u \partial_\nu v - \partial_\nu u \partial_\mu v, \quad (1.1a)
\]

\[
Q_0(u, v) = g^{\mu\nu} \partial_\mu u \partial_\nu v. \quad (1.1b)
\]

Let \(I^n = 2Im(\phi D^n \phi)\). Then the Euler-Lagrange equations are (we set \(\kappa = 2\) for simplicity)

\[
F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha} I^\alpha, \quad (1.2a)
\]

\[
D_\mu D^n \phi = -\phi V'(|\phi|^2). \quad (1.2b)
\]

The system has the positive conserved energy given by

\[
E = \int_{\mathbb{R}^2} \sum_{\mu=0}^2 |D_\mu \phi|^2 + V(|\phi|^2) \, dx.
\]

We are interested in the so-called ‘non-topological’ case in which \(|\phi| \to 0\) as \(|x| \to +\infty\). For the sake of simplicity we follow [10, 11] and set \(V = 0\). It will be clear from our proof that for various classes of \(V\)’s the term \(\phi V'(|\phi|^2)\) can easily be handled.

Under the Lorentz gauge condition \(\partial^\mu A_\mu = 0\) the Euler-Lagrange equations (1.2) become

\[
\partial_0 A_j = \partial_j A_0 + \frac{1}{2} \epsilon_{ij} I_i, \quad (1.3a)
\]

\[
\partial_1 A_2 = \partial_2 A_1 + \frac{1}{2} J_0, \quad (1.3b)
\]

\[
\partial_0 A_0 = \partial_1 A_1 + \partial_2 A_2, \quad (1.3c)
\]

\[
D_\mu D^n \phi = 0. \quad (1.3d)
\]

Alternatively, they can be written as a system of two nonlinear wave equations:

\[
\square A^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} \text{Im}(\overline{D_\gamma \phi} D_\beta \phi - D_\beta \phi D_\gamma \phi) + \frac{1}{2} \epsilon^{\alpha\beta\gamma}(\partial_\beta A_\gamma - \partial_\gamma A_\beta)|\phi|^2, \quad (1.4a)
\]

\[
\square \phi = 2i A^\alpha \partial_\alpha \phi + A^\alpha A_\alpha \phi. \quad (1.4b)
\]

We prescribe initial data in the classical Sobolev spaces \(A^\alpha(0, x) = a^\alpha_0(x) \in H^\alpha\), \(\partial_\beta A^\mu(0, x) = a^\mu_0(x) \in H^{\alpha-1}\), \(\phi(0, x) = \phi_0(x) \in H^b\), \(\partial_\beta \phi(0, x) = \phi_1(x) \in H^{b-1}\).

Dimensional analysis shows that the critical values of \(a\) and \(b\) are \(a_{cr} = 0\) and \(b_{cr} = \frac{1}{2}\). It is well known that in low space dimensions the Cauchy problem may not be locally well posed for \(a\) and \(b\) close to the critical values due to lack of decay at infinity. Observe also that \(\phi\) has fractional dimension.

From the point of view of scaling it is natural to take \(b = a + \frac{1}{2}\). With this choice it was shown in Huh [10] that the Cauchy problem is locally well posed for \(a = \frac{3}{4} + \epsilon\) and \(b = \frac{5}{4} + \epsilon\). This was improved in Huh [11] to

\[
a = \frac{3}{4} + \epsilon, \quad b = \frac{9}{8} + \epsilon \quad (1.5)
\]

(slightly violating \(b = a + \frac{1}{2}\)). The proof relies on the null structure of the right hand side of (1.4a). Indeed,

\[
D_\gamma \phi D_\beta \phi - D_\beta \phi D_\gamma \phi = Q_{\gamma\beta}(\phi^2) + i (A_\gamma \partial_\beta(|\phi|^2) - A_\beta \partial_\gamma(|\phi|^2)).
\]

On the other hand, since in (1.3) the \(A_\mu\) satisfy first order equations and \(\phi\) satisfies a second order equation it is natural to investigate the case \(b = a + 1\). It turns out
that this choice allows us to improve on \( a \) at the expense of \( b \). It is shown in Huh [11] that we have local well posedness for
\[
a = \frac{1}{2}, \quad b = \frac{3}{2}.
\]
To prove this result Huh uncovered the null structure in the right hand side of equation (1.4b). Indeed, if we introduce \( B_\mu \) by \( \partial_\mu B_\mu = 0 \) and \( \partial_\mu B_\nu - \partial_\nu B_\mu = \epsilon_{\mu\nu\lambda} A^\lambda \), then the equations take the form:
\[
\Box B^\gamma = - \text{Im} (\tilde{\phi} D^\gamma \phi) = - \text{Im} (\tilde{\phi} \partial^\gamma \phi) + i \epsilon^{\mu\nu\gamma} \partial_\mu B_\nu |\phi|^2, \quad (1.7a)
\]
\[
\Box \phi = i \epsilon^{\rho\mu\nu} Q_{\rho\mu}(B_\nu, \phi) + Q_0(B_\mu, B^\mu) \phi + Q_{\mu\nu}(B^\mu, B^\nu) \phi. \quad (1.7b)
\]
In this article we shall prove the Theorem stated below which corresponds to equation (1.5), it improves (1.6) by \( \frac{1}{2} - \epsilon \) derivatives in both \( a \) and \( b \). Compared to (1.5), it improves \( a \) by \( \frac{3}{2} \) derivatives at the expense of \( \frac{8}{3} \) derivatives in \( b \).

**Theorem 1.1.** Let \( n = 2 \) and \( \frac{1}{4} < s < \frac{1}{2} \). Consider the Cauchy problem for the system (1.7) with initial data in the following Sobolev spaces:
\[
B^\gamma(0) = b_0^\gamma \in H^{s+1}(\mathbb{R}^2), \quad \partial_\tau B^\gamma(0) = b_1^\gamma \in H^s(\mathbb{R}^2), \quad (1.8a)
\]
\[
\phi(0) = \phi_0 \in H^{s+1}(\mathbb{R}^2), \quad \partial_\tau \phi(0) = \phi_1 \in H^s(\mathbb{R}^2). \quad (1.8b)
\]
Then there exists a \( T > 0 \) and a solution \((B, \phi)\) of (1.7) in \([0, T] \times \mathbb{R}^2\) with \( B, \phi \in C^0([0, T]; H^{s+1}(\mathbb{R}^2)) \cap C^1([0, T]; H^s(\mathbb{R}^2))\).

The solution is unique in a subspace of \( C^0([0, T]; H^{s+1}(\mathbb{R}^2)) \cap C^1([0, T]; H^s(\mathbb{R}^2))\), namely in \( H^{s+1, \theta}\), where \( \frac{3}{4} < \theta < s + \frac{1}{2} \) (the definition of \( H^{s+1, \theta} \) is given in the next section).

Finally, we remark that the problem of global existence is much more difficult. We refer the reader to Chae and Chae [4], Chae and Choe [5] and Huh [10, 11].

2. Bilinear Estimates

In this Section we collect the bilinear estimates we need for the proof of Theorem 1.1. We shall work with the spaces \( H^{s, \theta} \) and \( \mathcal{H}^{s, \theta} \) defined by
\[
H^{s, \theta} = \{ u \in \mathcal{S}' : \Lambda^s \Lambda^\theta u \in L^2(\mathbb{R}^{2+1}) \},
\]
\[
\mathcal{H}^{s, \theta} = \{ u \in H^{s, \theta} : \partial_\tau u \in H^{s-1, \theta} \}
\]
where \( \Lambda \) and \( \Lambda_- \) are defined by
\[
\Lambda^s u(\tau, \xi) = (1 + |\xi|^2)^{s/2} \bar{u}(\tau, \xi),
\]
\[
\Lambda_-^\theta u(\tau, \xi) = \left(1 + \frac{(\tau^2 - |\xi|^2)^2}{1 + \tau^2 + |\xi|^2}\right)^{\theta/2} \bar{u}(\tau, \xi).
\]
Notice that the weight \( (1 + \frac{(\tau^2 - |\xi|^2)^2}{1 + \tau^2 + |\xi|^2})^{\theta/2} \) is equivalent to the weight \( w_- (\tau, \xi)^\theta \), where we define
\[
w_- (\tau, \xi) = 1 + ||\tau| - |\xi||.
\]
We define the norms
\[
\|u\|_{H^{s, \theta}} = \| (\xi)^s w_- (\tau, \xi)^\theta \bar{u}(\tau, \xi) \|_{L^2(\mathbb{R}^{2+1})},
\]
\[
\|u\|_{\mathcal{H}^{s, \theta}} = \|u\|_{H^{s, \theta}} + \|\partial_\tau u\|_{H^{s, \theta}}.
\]
Proposition 2.1. Let \( n = 2, s > 0 \) and let \( \theta \) and \( \delta \) satisfy
\[
\frac{1}{2} < \theta \leq \min\{1, s + \frac{1}{2}\}, \\
0 \leq \delta \leq \min\{1 - \theta, s + \frac{1}{2} - \theta\}.
\]
Then
\[
\|Q_0(\phi, \psi)\|_{H^s \cdot \theta + \delta} \lesssim \|\phi\|_{H^{s+1} \cdot \theta} \|\psi\|_{H^{s+1} \cdot \theta}.
\]

If \( Q = Q_0 \) there is a better estimate.

Proposition 2.2. Let \( n = 2, s > 0 \) and let \( \theta \) and \( \delta \) satisfy
\[
\frac{1}{2} < \theta \leq \min\{1, s + \frac{1}{2}\}, \\
0 \leq \delta \leq \min\{1 - \theta, s + \frac{1}{2} - \theta\}.
\]
Then
\[
\|Q(\phi, \psi)\|_{H^s \cdot \theta + \delta} \lesssim \|\phi\|_{H^{s+1} \cdot \theta} \|\psi\|_{H^{s+1} \cdot \theta}.
\]

The spaces in estimate (2.1) are different, with \( \delta \) defined by (1.1). We only sketch the proof for \( n = 2, s > 0 \) and let \( \theta \) and \( \delta \) satisfy
\[
\frac{1}{2} < \theta \leq \min\{1, s + \frac{1}{2}\}, \\
0 \leq \delta \leq \min\{1 - \theta, s + \frac{1}{2} - \theta\}.
\]
Then
\[
\|Q_0(\phi, \psi)\|_{H^s \cdot \theta + \delta} \lesssim \|\phi\|_{H^{s+1} \cdot \theta} \|\psi\|_{H^{s+1} \cdot \theta}.
\]

Proof of Proposition 2.1. We only sketch the proof for \( Q = Q_{ij} \). The proof for \( Q = Q_{ij} \) is similar. Let
\[
F(\tau, \xi) = \langle \xi \rangle^s w_+^\theta(\tau, \xi)w_-(\tau, \xi)^{\theta}\hat{\phi}(\tau, \xi), \\
G(\tau, \xi) = \langle \xi \rangle^s w_+^\theta(\tau, \xi)w_-(\tau, \xi)^{\theta}\hat{\psi}(\tau, \xi).
\]
Let \( H(\tau, \xi) \) be a test function. We may assume \( F, G, H \geq 0 \). We need to show:
\[
\int \frac{\langle \xi \rangle^s w_+^\theta(\tau, \xi)w_-(\tau, \xi)^{\theta}\hat{\phi}(\tau, \xi)}{\langle \xi \rangle^s w_+^\theta(\tau, \xi)w_-(\tau, \xi)^{\theta}w_+(\tau, \xi)^{\theta}w_-(\tau, \xi)^{\theta}} d\tau d\lambda d\xi d\eta \lesssim \|F\|_{L^2} \|G\|_{L^2} \|H\|_{L^2}.
\]

Using
\[
\langle \xi \rangle^s \leq \langle \xi \rangle^s + \langle \eta \rangle^s
\]
we see that we need to estimate the following integral (and a symmetric one):
\[
\int w_+^\theta(\tau, \xi)w_-(\tau, \xi)^{\theta}w_+(\tau, \xi)^{\theta}w_-(\tau, \xi)^{\theta} d\tau d\lambda d\xi d\eta
\]
We restrict our attention to the region where \( \tau \geq 0, \lambda \geq 0 \). The proof for all other regions is similar. We use

\[
\tau \eta - \lambda \xi = (|\xi|\eta - |\eta|\xi) + (\tau - |\xi|)\eta - (\lambda - |\eta|)\xi
= (|\xi|\eta - |\eta|\xi) + (|\tau| - |\xi|)\eta - (|\lambda| - |\eta|)\xi
\]
to see that, we need to estimate the following three integrals:

\[
R^+ = \int \frac{|||\xi|\eta - |\eta|\xi|F(\tau, \xi)G(\lambda, \eta)H(\tau + \lambda, \xi + \eta)d\tau d\lambda d\xi d\eta}{w_1^{-\theta - \delta}(\tau + \lambda, \xi + \eta)w_+^0(\tau, \xi)w_-(\tau, \xi)(\eta)^s w_+^0(\lambda, \eta)w_0^0(\lambda, \eta)}.
\]

\[
T^+ = \int \frac{|||\tau| - |\xi|F(\tau, \xi)G(\lambda, \eta)H(\tau + \lambda, \xi + \eta)d\tau d\lambda d\xi d\eta}{w_1^{-\theta - \delta}(\tau + \lambda, \xi + \eta)w_+^0(\tau, \xi)w_-(\tau, \xi)(\eta)^s w_+^0(\lambda, \eta)w_0^0(\lambda, \eta)}.
\]

\[
L^+ = \int \frac{|||\lambda| - |\eta|F(\tau, \xi)G(\lambda, \eta)H(\tau + \lambda, \xi + \eta)d\tau d\lambda d\xi d\eta}{w_1^{-\theta - \delta}(\tau + \lambda, \xi + \eta)w_+^0(\tau, \xi)w_-(\tau, \xi)(\eta)^s w_+^0(\lambda, \eta)w_0^0(\lambda, \eta)}.
\]

We start with \( R^+ \). We have

\[
|||\eta|\xi - |\eta|\xi||^2 \lesssim |\xi|^{1/2} |\eta|^{1/2} (|\xi| + |\eta|)^{1/2} (|||\tau| + \lambda| - |\xi + \eta|| + |||\tau| - |\xi|| + |||\lambda| - |\eta||)^{1/2}. \tag{2.7}
\]

Indeed,

\[
|||\eta|\xi - |\eta|\xi|| = 2|||\eta|\xi|| (|||\eta|\xi| - |\xi|\eta)
= |||\eta|\xi|| (|||\eta|\xi| + |\xi + \eta|) (|||\xi|\eta - |\xi + \eta|| + |||\xi|\eta + |\xi + \eta||).
\]

We have \( |||\xi|| + |\eta| + |\xi + \eta| \leq 2 \langle |||\xi|| + |\eta| \rangle \) and

\[
|||\xi|| + |\eta| - |\xi + \eta| = \tau + \lambda - |\xi + \eta| - (\lambda - |\eta|) - (\tau - |\xi|) \leq |\tau + \lambda - \xi| + |\lambda - |\eta|| + |\tau - |\xi||,
\]

therefore (2.7) follows. Following Zhou [19] we use (2.7) to obtain

\[
|||\eta|\xi - |\eta|\xi||^2 \lesssim |||\eta|\xi - |\eta|\xi||^2s|||\xi|| - |\xi|\eta|^{1 - 2s}
\lesssim |||\eta|\xi - |\eta|\xi||^2s |||\xi||^{1 - 2s}|||\eta||^{1/2 - s} (|||\xi|| + |\eta|)^{1/2 - s} |||\tau + \lambda - |\xi + \eta||^{1/2 - s}
+ |||\eta|\xi - |\eta|\xi||^2s |||\xi||^{1 - 2s}|||\eta||^{1/2 - s} (|||\xi|| + |\eta|)^{1/2 - s} |||\tau - |\xi||^{1/2 - s}
+ |||\eta|\xi - |\eta|\xi||^2s |||\xi||^{1 - 2s}|||\eta||^{1/2 - s} (|||\xi|| + |\eta|)^{1/2 - s} |||\lambda - |\eta||^{1/2 - s}.
\]

Therefore,

\[
R^+ \lesssim R_1^+ + R_2^+ + R_3^+,
\]

where

\[
R_1^+ = \int \frac{|||\eta|\xi - |\eta|\xi||^2s |||\xi||^{1/2 - s}|||\eta||^{1/2 - s} (|||\xi|| + |\eta|)^{1/2 - s} |||\tau + \lambda - |\xi + \eta||^{1/2 - s}}{w_1^{-\theta - \delta}(\tau + \lambda, \xi + \eta)w_+^0(\tau, \xi)w_-(\tau, \xi)(\eta)^s w_+^0(\lambda, \eta)w_0^0(\lambda, \eta)}
\times F(\tau, \xi)G(\lambda, \eta)H(\tau + \lambda, \xi + \eta) d\tau d\lambda d\xi d\eta
\]

\[
\leq \int \frac{|||\eta|\xi - |\eta|\xi||^2s (|||\xi|| + |\eta|)^{1/2 - s}}{w_0^0(\tau, \xi)w_0^0(\lambda, \eta)(\eta)^{s + 1/2}w_{s + 1/2}^0(\eta)^{2s} + 1/2}
\times F(\tau, \xi)G(\lambda, \eta)H(\tau + \lambda, \xi + \eta) d\tau d\lambda d\xi d\eta.
\]
We have used the fact that \( w_{\frac{1}{2} - \theta - \delta}(\tau + \lambda, \xi + \eta) \geq 1 \). Indeed, \( s + \frac{1}{2} - \theta - \delta > 0 \) for small \( \delta \), because \( \theta < s + \frac{1}{2} \).

\[
R_2^+ = \int \int \int \frac{||\eta|| \xi - |\xi||\eta||^{2s} |\xi|^{1/2-s} |\eta|^{1/2-s} (|\xi| + |\eta|)^{1/2-s} ||\eta||^{1/2-s} \theta}{w_{1-\theta-\delta}^\theta(\tau + \lambda, \xi + \eta) w_+^\theta(\tau, \xi) w_+^\theta(\lambda, \eta) w_+^\theta(\lambda, \eta)}
\]

\[
\times F(\tau, \xi) G(\lambda, \eta) H(\tau + \lambda, \xi + \eta) d\tau d\lambda d\xi d\eta
\]

\[
\leq \int \int \int \frac{||\eta|| \xi - |\xi||\eta||^{2s} (|\xi| + |\eta|)^{1/2-s} ||\eta||^{1/2-s} \theta}{w_{1-\theta-\delta}^\theta(\tau, \xi) w_+^\theta(\lambda, \eta) |\xi|^{1/2-s} |\eta|^{1/2-s} \theta}
\]

\[
\times F(\tau, \xi) G(\lambda, \eta) H(\tau + \lambda, \xi + \eta) d\tau d\lambda d\xi d\eta
\]

We present the proof for \( R_2^+ \). The proofs for \( R_1^+ \) and \( R_3^+ \) are similar. We change variables \( \tau \mapsto u := |\tau| - |\xi| = \tau - |\xi| \) and \( \lambda \mapsto v := |\lambda| - |\eta| = \lambda - |\eta| \) and we use the notation

\[
f_u(\xi) = F(u + |\xi|, \xi), \quad g_v(\eta) = G(v + |\eta|, \eta), \quad H_{u,v}(\tau', \xi') = H(u + v + \tau', \xi')
\]

to get

\[
R_2^+ = \int \int \frac{1}{(1 + |u|)^{\theta + s - \frac{1}{2}} (1 + |v|)^{\theta}} \left[ \int \frac{||\eta|| \xi - |\xi||\eta||^{2s} (|\xi| + |\eta|)^{1/2-s} \theta}{|\xi|^{1/2-s} |\eta|^{1/2-s} \theta} f_u(\xi) g_v(\eta) H_{u,v}(|\xi| + |\eta|, \xi + \eta) d\xi d\eta \right] du dv.
\]

We have \( ||\eta|| \xi - |\xi||\eta||^{2s} \frac{(|\xi| + |\eta|)^{1/2-s} \theta}{|\xi|^{1/2-s} |\eta|^{1/2-s} \theta} \) therefore

\[
[\cdots] \leq \left( \int f_u(\xi) g_v(\eta)^2 d\xi d\eta \right)^{1/2} K^{1/2}
\]

\[
= ||f_u||_{L^2(\mathbb{R}^2)} ||g_v||_{L^2(\mathbb{R}^2)} K^{1/2},
\]

where

\[
K = \int \frac{(|\xi| + |\eta|)^{1-2s} \theta}{|\xi|^{1/2-s} |\eta|^{1/2-s} \theta} H_{u,v}(|\xi| + |\eta|, \xi + \eta) d\xi d\eta
\]

\[
= \int \frac{(|\xi' - \eta| + |\xi' - \eta|)^{1-2s} \theta}{|\xi' - \eta|^{1/2-s} |\eta|^{1/2-s} \theta} H_{u,v}(|\xi' - \eta| + |\eta|, \xi' + \eta) d\xi' d\eta.
\]
We use polar coordinates $\eta = \rho \omega$ to get
\[
K \lesssim \iint \frac{(\xi' - \rho \omega) + \rho - \xi' \cdot \omega)^{2s}}{\xi' - \rho \omega} \times H_u, v(|\xi' - \rho \omega| + \rho, \xi' \cdot \omega^2 d\xi' d\rho d\omega.
\]
For fixed $\xi'$ and $\omega$, we change variables $\rho \mapsto \tau' := |\xi' - \rho \omega| + \rho$ to get
\[
K \lesssim \int \left[ \tau'^{1-2s} \int_{S^1} (\tau' - \xi' \cdot \omega)^{1-2s} d\omega \right] H(\tau', \xi')^2 d\tau'.
\]
From [19] estimate (3.22) we know that
\[
\tau'^{1-2s} \int_{S^1} (\tau' - \xi' \cdot \omega)^{1-2s} d\omega \lesssim 1;
\]
therefore $K \lesssim \|H\|^2_{\tilde{A}}$. Putting everything together we get:
\[
R^+_2 \lesssim \left( \int \|f_u\|_{L^2(\mathbb{R}^2)} \right) \left( \int \|g_v\|_{L^2(\mathbb{R}^2)} \right) \|H\|.
\]
Since $2\theta + 2s - 1 > 2 \cdot \frac{3}{2} + 2 \cdot \frac{1}{4} - 1 = 1$ and $2\theta > 2 \cdot \frac{3}{2} > 1$ we can use the Cauchy-Schwarz inequality to conclude:
\[
R^+_2 \lesssim \||f_u\|_{L^2(\mathbb{R}^2)} \|g_v\|_{L^2(\mathbb{R}^2)} \|H\| = \|F\|\|G\|_{\tilde{A}}\|H\|_{\tilde{A}}.
\]
This completes the estimates for $R^+_2$.

Next we estimate $T^+$. We use $|\tau| - |\xi| \leq w_+(\tau, \xi)^{1-\theta} w_-(\tau, \xi)^\theta$ to get
\[
T^+ = \int \frac{\|\tau| - |\xi|\|\eta\| F(\tau, \xi) G(\lambda, \eta) H(\tau + \lambda, \xi + \eta) d\tau d\lambda d\xi d\eta
\]
\[
\leq \int \frac{F(\tau, \xi) G(\lambda, \eta) H(\tau + \lambda, \xi + \eta) \langle \xi \rangle^\theta \langle \eta \rangle^\theta w_-(\lambda, \eta) d\tau d\lambda d\xi d\eta.
\]
Changing variables $\tau \mapsto u := |\tau| - |\xi|$ and $\lambda \mapsto v := |\lambda| - |\eta| = \lambda - |\eta|$ we have
\[
T^+ \lesssim \int \int \frac{1}{\langle \xi \rangle^\theta \langle \eta \rangle^\theta} \left[ \int \frac{F(u + |\xi|, \xi) G(v + |\eta|, \eta) H(u + v + |\xi| + |\eta|, \xi + \eta)}{(1 + |v|)^\theta} du dv \right] d\xi d\eta.
\]
For fixed $\xi$ and $\eta$ we apply [19] Lemma A in the $(u, v)$-variables to get
\[
T^+ \lesssim \int \frac{1}{\langle \xi \rangle^\theta \langle \eta \rangle^\theta} \|F(u + |\xi|, \xi)\|_{L^2} \|G(v + |\eta|, \eta)\|_{L^2} \times \|H(w + |\xi| + |\eta|, \xi + \eta)\|_{L^2} d\xi d\eta
\]
\[
= \int \frac{1}{\langle \xi \rangle^\theta \langle \eta \rangle^\theta} \|F(\cdot, \xi)\|_{L^2(\mathbb{R}^2)} \|G(\cdot, \eta)\|_{L^2(\mathbb{R}^2)} \|H(\cdot, \xi)\|_{L^2(\mathbb{R}^2)} d\xi d\eta.
\]
Now we do the same in the $(\xi, \eta)$-variables to get
\[
T^+ \lesssim \|F(\cdot, \xi)\|_{L^2(\mathbb{R}^2)} \|G(\cdot, \eta)\|_{L^2(\mathbb{R}^2)} \|H(\cdot, \xi)\|_{L^2(\mathbb{R}^2)} = ||F||_{\tilde{A}} ||G||_{\tilde{A}} ||H||_{\tilde{A}}.
\]
The proof for $L^+$ is similar.

We are also going to need the following ‘product rules’ in $H^{s, \theta}$ spaces.
Proposition 2.3. Let $n = 2$. Then
\[ \| uv \|_{H^{-c,-\gamma}} \lesssim \| u \|_{H^{a,\alpha}} \| v \|_{H^{b,\beta}}, \]  
provided that
\[ a + b + c > 1 \]  
\[ a + b \geq 0, \quad b + c \geq 0, \quad a + c \geq 0 \]  
\[ a + \beta + \gamma > 1/2 \]  
\[ \alpha, \beta, \gamma \geq 0. \]

Proof. If $a, b, c \geq 0$, the result is contained in [13 Proposition A1]. If not, observe that, due to (2.10), at most one of the $a, b, c$ is negative. We deal with the case $c < 0, a, b \geq 0$. All other cases are similar. Observe that
\[ \langle \xi \rangle^{-c} |\tau| - |\xi| \rangle^{-\gamma} \langle \tau, \xi \rangle \]
\[ \lesssim \langle |\tau| - |\xi| \rangle^{-\gamma} \int \langle \xi - \eta \rangle^{-c} |\tilde{u}(\tau - \lambda, \xi - \eta)| |\tilde{v}(\lambda, \eta)| d\lambda d\eta \]
\[ + \langle |\tau| - |\xi| \rangle^{-\gamma} \int \langle \eta \rangle^{-c} |\tilde{u}(\tau - \lambda, \xi - \eta)| |\tilde{v}(\lambda, \eta)| d\lambda d\eta, \]
therefore
\[ \| uv \|_{H^{-c,-\gamma}} \lesssim \| Uv' \|_{H^{0,-\gamma}} + \| u'V \|_{H^{0,-\gamma}}, \]
where
\[ \tilde{U}(\tau, \xi) = \langle \xi \rangle^{-c} |\tilde{u}(\tau, \xi)|, \]
\[ \tilde{u}'(\tau, \xi) = |\tilde{u}(\tau, \xi)|, \]
\[ \tilde{V}(\tau, \xi) = \langle \xi \rangle^{-c} |\tilde{v}(\tau, \xi)|, \]
\[ \tilde{v}'(\tau, \xi) = |\tilde{v}(\tau, \xi)|. \]

Since $a + c \geq 0$, we have
\[ \| Uv' \|_{H^{0,-\gamma}} \lesssim \| U \|_{H^{a+c,\alpha}} \| v' \|_{H^{b,\beta}} \lesssim \| u \|_{H^{a,\alpha}} \| v \|_{H^{b,\beta}}. \]
Since $b + c \geq 0$, we have
\[ \| u'V \|_{H^{0,-\gamma}} \lesssim \| u' \|_{H^{a,\alpha}} \| V \|_{H^{b+c,\beta}} \lesssim \| u \|_{H^{a,\alpha}} \| v \|_{H^{b,\beta}}. \]
The result follows. □

Proposition 2.4. Let $n = 2$. If $s > 1$ and $\frac{1}{2} < \theta \leq s - \frac{1}{2}$, then $H^{s,\theta}$ is an algebra.

For the proof of the above proposition, see [13 Theorem 7.3].

Proposition 2.5. Let $n = 2$, $s > 1$, $\frac{1}{2} < \theta \leq s - \frac{1}{2}$. Assume that
\[ -\theta \leq \alpha \leq 0 \quad -s \leq a < s + \alpha. \]
Then
\[ H^{a,\alpha} \cdot H^{s,\theta} \hookrightarrow H^{a,\alpha}. \]

The proof can be found in [13 Theorem 7.2].
Proof of Theorem 1.1:} Theorem 1.1 follows by well known methods from the following a-priori estimates (together with the corresponding estimates for differences): For any space-time functions $B, B', \phi, \phi' \in \mathcal{H}^{s+1, \theta}$ and any $\gamma, \mu \in \{0, 1, 2\}$ we have:

$$
\|\phi' \partial^\gamma \phi\|_{H^{s, \theta-1+\delta}} \lesssim \|\phi'\|_{\mathcal{H}^{s+1, \theta}} \|\phi\|_{\mathcal{H}^{s+1, \theta}},
\tag{2.15}
$$

$$
\|(\partial^\mu B) \phi \phi'\|_{H^{s, \theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1, \theta}} \|\phi\|_{\mathcal{H}^{s+1, \theta}} \|\phi'\|_{\mathcal{H}^{s+1, \theta}},
\tag{2.16}
$$

$$
\|Q_{\mu \alpha}(B, \phi)\|_{H^{s, \theta-1+\delta}} \lesssim \|B\|^2_{\mathcal{H}^{s+1, \theta}} \|\phi\|_{\mathcal{H}^{s+1, \theta}},
\tag{2.17}
$$

$$
\|Q_0(B, B') \phi\|_{H^{s, \theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1, \theta}} \|B'\|_{\mathcal{H}^{s+1, \theta}} \|\phi\|_{\mathcal{H}^{s+1, \theta}},
\tag{2.18}
$$

$$
\|Q_{\mu \nu}(B, B') \phi\|_{H^{s, \theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1, \theta}} \|B'\|_{\mathcal{H}^{s+1, \theta}} \|\phi\|_{\mathcal{H}^{s+1, \theta}}.
\tag{2.19}
$$

Here $\frac{1}{4} < \theta < \frac{1}{2}$ and $\delta$ is a sufficiently small positive number.

To prove (2.15) we use Proposition 2.3 to get:

$$
\|\phi' \partial^\gamma \phi\|_{H^{s, \theta-1+\delta}} \lesssim \|\phi'\|_{\mathcal{H}^{s+1, \theta}} \|\partial^\gamma \phi\|_{H^{s, \theta}} \lesssim \|\phi'\|_{\mathcal{H}^{s+1, \theta}} \|\phi\|_{\mathcal{H}^{s+1, \theta}}.
\tag{2.20}
$$

Similarly, for (2.16) we have

$$
\|(\partial^\mu B) \phi \phi'\|_{H^{s, \theta-1+\delta}} \lesssim \|\partial^\mu B\|_{H^{s, \theta}} \|\phi\|_{H^{s+1, \theta}} \|\phi'\|_{H^{s+1, \theta}}.
\tag{2.21}
$$

By Proposition 2.4 and our assumptions on $s$ and $\theta$ it follows that the space $H^{s+1, \theta}$ is an algebra. Therefore,

$$
\|\phi' \phi\|_{H^{s+1, \theta}} \lesssim \|\phi\|_{H^{s+1, \theta}} \|\phi'\|_{H^{s+1, \theta}},
$$

and estimate (2.16) follows.

Estimate (2.17) follows from Proposition (2.1). Finally, we consider estimates (2.18) and (2.19). We use the letter $Q$ to denote any of the null forms $Q_0, Q_{\mu \nu}$.

We have

$$
\|Q(B, B') \phi\|_{H^{s, \theta-1+\delta}} \lesssim \|Q(B, B')\|_{H^{s, \theta-1+\delta}} \|\phi\|_{H^{s+1, \theta}}.
\tag{2.22}
$$

This follows from Proposition 2.5 with $s$ replaced by $s+1$ and $\alpha$ replaced by $\theta-1+\delta$.

Next, by (2.1),

$$
\|Q(B, B')\|_{H^{s, \theta-1+\delta}} \lesssim \|B\|_{\mathcal{H}^{s+1, \theta}} \|B'\|_{\mathcal{H}^{s+1, \theta}},
\tag{2.23}
$$

therefore (2.18) and (2.19) follow.

REFERENCES


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