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OSCILLATION OF SOLUTIONS TO IMPULSIVE DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this article, we study the oscillation of second order impulsive dynamic equations on time scales. The effect of the moments of impulse are fixed. Using Riccati transformation techniques, we obtain some conditions for the oscillation of all solutions

1. Introduction

This paper concerns the oscillation of second-order impulsive dynamic equations on time scales. We consider the system

$$\begin{gathered}(a(t)x^{\Delta}(t))^{\Delta} + p(t)x(\sigma(t)) = 0, \quad t \in \mathbb{J}_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}, t \neq t_k,\ k = 1, 2, \dots, \\ x(t_k^+) = b_k x(t_k), \quad x^{\Delta}(t_k^+) = c_k x^{\Delta}(t_k), \quad k = 1, 2, \dots, \\ x(t_0^+) = x(t_0), \quad x^{\Delta}(t_0^+) = x^{\Delta}(t_0),\end{gathered} \tag{1.1}$$

where $\mathbb{T}$ is a time scales, unbounded-above, with $t_k \in \mathbb{T}$, $0 \le t_0 < t_1 < t_2 < \dots < t_k < \dots$, $\lim_{k\to\infty} t_k = \infty$ and $y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$, $y^{\Delta}(t_k^+) = \lim_{h\to 0^+} y^{\Delta}(t_k+h)$, which represent right limits of $y(t), y^{\Delta}(t)$ at $t = t_k$ in the sense of time scales. We can define $y(t_k^-)$, $y^{\Delta}(t_k^-)$ similarly.

In this paper, we assume that $a(t) \in C_{\rm rd}(\mathbb{T}, \mathbb{R}^+)$, $p(t) \in C_{\rm rd}(\mathbb{T}, \mathbb{R}^+)$, $b_k > 0$, $c_k > 0$, $d_k = \frac{c_k}{b_k}$, t_k are right dense, where $C_{\rm rd}$ denotes the set of rd-continuous functions, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\mathbb{R}^+ = \{x : x > 0\}$.

Definition 1.1. A function x is a solution of (1.1), if it satisfies $(a(t)x^{\Delta}(t))^{\Delta} + p(t)x(\sigma(t)) = 0$ a.e. on $\mathbb{J}_{\mathbb{T}} \backslash \{t_k\}$, $k = 1, 2, \dots$, and for each $k = 1, 2, \dots, x$ satisfies the impulsive condition $x(t_k^+) = b_k x(t_k)$, $x^{\Delta}(t_k^+) = c_k x^{\Delta}(t_k)$ and the initial condition $x(t_0^+) = x(t_0)$, $x^{\Delta}(t_0^+) = x^{\Delta}(t_0)$.

Definition 1.2. A solution x of (1.1) is oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called non-oscillatory. Equation (1.1) is called oscillatory if all solutions are oscillatory.

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In recently years, there has been an increasing interest in studying the oscillation and non-oscillation of solutions of various equations on time scales, and we refer the reader to papers [5, 10, 11, 12] and references cited therein. The time scales calculus has a tremendous potential for applications in mathematical models of real processes. Impulsive dynamic equations on time scales have been investigated by Agarwal [1], Benchohra [2] and so forth. Benchohra [2] considered the existence of extremal solutions for a class of second order impulsive dynamic equations on time sales.

The oscillation of impulsive differential equations and difference equations have been investigated by many authors and numerous papers have been published on this class of equations and good results were obtained (see [7, 9] and the references therein). But fewer papers are on the oscillation of impulsive dynamic equations on time scales.

For example, Huang [8] considered the equation

$$\begin{gathered} y^{\Delta\Delta}(t)+f(t,y^{\sigma}(t))=0, \quad t\in\mathbb{J}_{\mathbb{T}}:=[0,\infty)\cap\mathbb{T}, \quad t\neq t_k,\ k=1,2,\dots, \\ y(t_k^+)=g_k(y(t_k)), \quad y^{\Delta}(t_k^+)=h_k(y^{\Delta}(t_k)), \quad k=1,2,\dots, \\ y(t_0^+)=y(t_0), \quad y^{\Delta}(t_0^+)=y^{\Delta}(t_0). \end{gathered}$$

Using Riccati transformation techniques, they obtain sufficient conditions for oscillations of all solutions.

2. Results

In the following, we assume the solutions of (1.1) exist in $\mathbb{J}_{\mathbb{T}}$. To the best of our knowledge, the question of the oscillation for second order self-conjugate impulsive dynamic equations has not been yet considered.

Lemma 2.1. *Suppose that $x(t)>0$, $t\geq t_0'\geq t_0$ is a solution of* (1.1). *If*

$$\int_{t_0}^{t_1}\frac{\Delta s}{a(s)}+d_1\int_{t_1}^{t_2}\frac{\Delta s}{a(s)}+d_1d_2\int_{t_2}^{t_3}\frac{\Delta s}{a(s)}+\dots+d_1d_2\dots d_n\int_{t_n}^{t_{n+1}}\frac{\Delta s}{a(s)}+\dots=\infty, \tag{2.1}$$

then $x^{\Delta}(t_k^+)\geq 0$ and $x^{\Delta}(t)\geq 0$ for $t\in(t_k,t_{k+1}]_{\mathbb{T}}$, where $t_k\geq t_0'$

The proof is similar to that in [8, Lemma 2.1]; so we omit it. We remark that when $a(t)\equiv 1$, Lemma 2.1 reduces to [8, Lemma 2.1].

Lemma 2.2. *Assume that $q(t)\in C_{\rm rd}(\mathbb{T},\mathbb{R}^+)$, if*

$$\omega^{\Delta\Delta}(t)+q(t)\omega^{\Delta}(t)\leq 0$$

has a positive solution, then

$$\omega^{\Delta\Delta}(t)+q(t)\omega^{\Delta}(t)=0$$

has a positive solution.

The proof is similar to that in [6, Lemma 4.1.2]; so we omit it.

Theorem 2.3. *Assume that $a(t)\equiv 1$ and* (2.1) *holds.*

$$\Big(\prod_{T\leq t_k<t}d_k^{-1}\omega^{\Delta}\Big)^{\Delta}+\prod_{T\leq t_k<t}d_k^{-1}p(t)\omega(t)=0, \quad a.e.\ t>T\geq t_0 \tag{2.2}$$

is oscillatory, then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that (1.1) has a non-oscillatory solution $x(t)$, we may assume that $x(t)$ is eventually positive solution of (1.1); i.e., $x(t) > 0$, $t \geq T \geq t_0$. From Lemma 2.1, we have $x^{\Delta}(t) \geq 0$, $x^{\Delta}(t_k^+) \geq 0$, $t \geq T$, $t \in \mathbb{T}$. Let $z(t) = \frac{x^{\Delta}(t)}{x(t)} \geq 0$. For $t \neq t_k$, we get

$$z^{\Delta}(t) = \frac{x^{\Delta\Delta}(t)x(t) - (x^{\Delta}(t))^2}{x(t)x(\sigma(t))} = -p(t) - \frac{z^2(t)}{1+\mu(t)z(t)},$$
$$z(t_k^+) = \frac{x^{\Delta}(t_k^+)}{x(t_k^+)} = \frac{c_k x^{\Delta}(t_k)}{b_k x(t_k)} = d_k z(t_k).$$

Thus we arrive at

$$z^{\Delta}(t) + \frac{z^2(t)}{1+\mu(t)z(t)} + p(t) = 0, \quad t \in [T,\infty) \cap \mathbb{T},\ t \neq t_k,$$
$$z(t_k^+) = d_k z(t_k).$$

Now we define $v(t) = (\prod_{T \leq t_k < t} d_k^{-1}) z(t)$, $t > T$, $t \in \mathbb{T}$. Then for $t_n > T$,

$$v(t_n^+) = (\prod_{T \leq t_k \leq t_n} d_k^{-1}) z(t_n^+) = (\prod_{T \leq t_k \leq t_n} d_k^{-1}) d_n z(t_n) = v(t_n),$$

which implies that $v(t)$ is rd-continuous on $(T,\infty) \cap \mathbb{T}$. For $t \neq t_n$, we have

$$\begin{aligned}
v^{\Delta}(t) &= \prod_{T \leq t_k < t} d_k^{-1} z^{\Delta}(t) \\
&= \prod_{T \leq t_k < t} d_k^{-1} [-p(t) - \frac{z^2(t)}{1+\mu(t)z(t)}] \\
&= \prod_{T \leq t_k < t} d_k^{-1} [-p(t) - \frac{(\prod_{T \leq t_k < t} d_k)^2 v^2(t)}{1 + \prod_{T \leq t_k < t} d_k \mu(t) v(t)}] \\
&= -\prod_{T \leq t_k < t} d_k \frac{v^2(t)}{1 + \prod_{T \leq t_k < t} d_k \mu(t) v(t)} - \prod_{T \leq t_k < t} d_k^{-1} p(t).
\end{aligned}$$

For $t = t_n$, the left-hand derivative of $v(t)$ at $t = t_n$ is given by

$$\begin{aligned}
v^{\Delta}(t_n^-) &= \prod_{T \leq t_k < t_n} d_k^{-1} z^{\Delta}(t_n^-) \\
&= \prod_{T \leq t_k < t_n} d_k^{-1} \lim_{t \to t_n^-} [-p(t) - \frac{z^2(t)}{1+\mu(t)z(t)}] \\
&= \prod_{T \leq t_k < t_n} d_k^{-1} [-p(t_n) - \frac{z^2(t_n)}{1+\mu(t_n)z(t_n)}] \\
&= \prod_{T \leq t_k < t_n} d_k^{-1} [-p(t_n) - \frac{\prod_{T \leq t_k < t_n} d_k^2 v^2(t_n)}{1+\mu(t_n)\prod_{T \leq t_k < t_n} d_k v(t_n)}] \\
&= -\prod_{T \leq t_k < t_n} d_k^{-1} p(t_n) - \prod_{T \leq t_k < t_n} d_k \frac{v^2(t_n)}{1+\mu(t_n)\prod_{T \leq t_k < t_n} d_k v(t_n)}].
\end{aligned}$$

Similarly, we obtain

$$v^{\Delta}(t_n^+) = -\prod_{T\le t_k\le t_n} d_k \frac{v^2(t_n)}{1+\prod_{T\le t_k\le t_n} d_k\mu(t_n)v(t_n)} - \prod_{T\le t_k\le t_n} d_k^{-1}p(t_n).$$

So for $t > T$,

$$v^{\Delta}(t) + \prod_{T\le t_k<t} d_k \frac{v^2(t)}{1+\prod_{T\le t_k<t} d_k\mu(t)v(t)} + \prod_{T\le t_k<t} d_k^{-1}p(t) = 0, \quad \text{a.e.} \tag{2.3}$$

Now we define $q(t) = \prod_{T\le t_k<t} d_k v(t)$, $w(t) = e_q(t,\, t_0) > 0$, $t > T$. Then

$$w^{\Delta}(t) = q(t)w(t) = \prod_{T\le t_k<t} d_k v(t)w(t),$$

$$\begin{aligned}\Big(\prod_{T\le t_k<t} d_k^{-1}w^{\Delta}\Big)^{\Delta} &= (v(t)w(t))^{\Delta} = w^{\Delta}(t)v(t) + w(\sigma(t))v^{\Delta}(t)\\ &= \prod_{T\le t_k<t} d_k v^2(t)w(t) + e_q(\sigma(t),\, t_0)v^{\Delta}(t).\end{aligned}$$

Since

$$e_q(\sigma(t),\, t_0) = (1+\mu(t)q(t))e_q(t,\, t_0) = (1+\mu(t)v(t)\prod_{T\le t_k<t} d_k)w,$$

by (2.3) we obtain

$$\begin{aligned}\Big(\prod_{T\le t_k<t} d_k^{-1}w^{\Delta}\Big)^{\Delta} &= w[\prod_{T\le t_k<t} d_k v^2(t) + (1+\mu(t)v(t)\prod_{T\le t_k<t} d_k)v^{\Delta}(t)]\\ &= -w[1+\mu(t)v(t)\prod_{T\le t_k<t} d_k]\prod_{T\le t_k<t} d_k^{-1}p(t)\\ &\le -w(t)\prod_{T\le t_k<t} d_k^{-1}p(t), \quad \text{a.e.}\end{aligned}$$

This implies

$$\Big(\prod_{T\le t_k<t} d_k^{-1}w^{\Delta}\Big)^{\Delta} + \prod_{T\le t_k<t} d_k^{-1}p(t)w \le 0, \quad \text{a.e.}$$

has a positive solution. By Lemma 2.2, we obtain

$$\Big(\prod_{T\le t_k<t} d_k^{-1}\omega^{\Delta}\Big)^{\Delta} + \prod_{T\le t_k<t} d_k^{-1}p(t)\omega = 0, \quad \text{a.e.}$$

has a positive solution, a contradiction, and so, the proof is complete. □

Theorem 2.4. *Assume that $b_k = c_k, t_k$ are right dense for all $k = 1, 2, \dots$. Then the oscillation of all solutions of (1.1) is equivalent to the oscillation of all solutions of the equation*

$$(a(t)y^{\Delta}(t))^{\Delta} + p(t)y(\sigma(t)) = 0. \tag{2.4}$$

Proof. Let $y(t)$ be any solution of (2.4). Set $x(t) = y(t)\prod_{t_0\le t_k<t} b_k$ for $t > t_0$, then

$$x(t_n^+) = y(t_n^+)\prod_{t_0\le t_k\le t_n} b_k = b_n x(t_n).$$

Furthermore, for $t \neq t_n$, we have

$$x^{\Delta}(t) = (\prod_{t_0 \le t_k < t} b_k) y^{\Delta}(t),$$

$$x^{\Delta}(t_n^+) = (\prod_{t_0 \le t_k \le t_n} b_k) y^{\Delta}(t_n^+) = b_n x^{\Delta}(t_n) = c_n x^{\Delta}(t_n).$$

For $t \neq t_n$,

$$\begin{aligned}
(a(t)x^{\Delta}(t))^{\Delta} &= [a(t)(\prod_{t_0 \le t_k < t} b_k) y^{\Delta}(t)]^{\Delta} = \prod_{t_0 \le t_k < t} b_k (a(t) y^{\Delta}(t))^{\Delta} \\
&= -p(t) \prod_{t_0 \le t_k < t} b_k y(\sigma(t)) = -p(t) \prod_{t_0 \le t_k < \sigma(t)} b_k y(\sigma(t)) \\
&= -p(t) x(\sigma(t)).
\end{aligned}$$

Thus $x(t)$ is the solution of (1.1).

Conversely, if $x(t)$ is the solution of (1.1). Set $y(t) = x(t) \prod_{t_0 \le t_k < t} b_k^{-1}$. Thus we have

$$y(t_n^+) = x(t_n^+) \prod_{t_0 \le t_k \le t_n} b_k^{-1} = y(t_n).$$

Furthermore for $t \neq t_n$, $n = 1, 2, \dots$, we have

$$\begin{aligned}
y^{\Delta}(t) &= x^{\Delta}(t) \prod_{t_0 \le t_k < t} b_k^{-1}, \\
y^{\Delta}(t_n^-) &= x^{\Delta}(t_n^-) \prod_{t_0 \le t_k < t_n} b_k^{-1} = x^{\Delta}(t_n) \prod_{t_0 \le t_k < t_n} b_k^{-1}, \\
y^{\Delta}(t_n^+) &= x^{\Delta}(t_n^+) \prod_{t_0 \le t_k \le t_n} b_k^{-1} = x^{\Delta}(t_n) \prod_{t_0 \le t_k < t_n} b_k^{-1}.
\end{aligned}$$

For $t \neq t_n$, $n = 1, 2, \dots$,

$$\begin{aligned}
(a(t)y^{\Delta}(t))^{\Delta} &= [a(t)(\prod_{t_0 \le t_k < t} b_k^{-1}) x^{\Delta}(t)]^{\Delta} = \prod_{t_0 \le t_k < t} b_k^{-1} (a(t) x^{\Delta}(t))^{\Delta} \\
&= -p(t) \prod_{t_0 \le t_k < t} b_k^{-1} x(\sigma(t)) = -p(t) \prod_{t_0 \le t_k < \sigma(t)} b_k^{-1} x(\sigma(t)) \\
&= -p(t) y(\sigma(t)).
\end{aligned}$$

For $t = t_n$,

$$\begin{aligned}
(a(t_n^-)y^{\Delta}(t_n^-))^{\Delta} &= (a(t_n^-) \prod_{t_0 \le t_k < t_n} b_k^{-1} x^{\Delta}(t_n^-))^{\Delta} \\
&= -\prod_{t_0 \le t_k < t_n} b_k^{-1} p(t_n) x(\sigma(t_n)) = -p(t_n) y(\sigma(t_n)), \\
(a(t_n^+)y^{\Delta}(t_n^+))^{\Delta} &= (a(t_n) y^{\Delta}(t_n^+))^{\Delta} = -p(t_n) y(\sigma(t_n)).
\end{aligned}$$

Thus we arrive at

$$(a(t)y^{\Delta}(t))^{\Delta} + p(t) y(\sigma(t)) = 0.$$

This shows that $y(t)$ is a solution of (2.4). The proof is complete. □

Example 2.5. Consider the equation

$$
\begin{gathered}
x^{\Delta\Delta}+p(t)x(\sigma(t))=0, \quad t\in\mathbb{T}=\mathbb{P}_{\frac{1}{2},\frac{1}{2}},\ t\neq k+\frac{1}{5},\ k=0,1,2,\dots,\\
x((k+\frac{1}{5})^{+})=b_k x(k+\frac{1}{5}),\ x^{\Delta}((k+\frac{1}{5})^{+})=b_k x^{\Delta}(k+\frac{1}{5}),\quad b_k>0,\ k=1,2,\dots,\\
x((\frac{1}{5})^{+})=x(\frac{1}{5}),\quad x^{\Delta}((\frac{1}{5})^{+})=x^{\Delta}(\frac{1}{5}),
\end{gathered} \tag{2.5}
$$

where $p(t)\in C_{\rm rd}(\mathbb{T},\mathbb{R}^{+})$, $\mathbb{P}_{\frac{1}{2},\frac{1}{2}}=\bigcup_{k=0}^{\infty}[k,k+\frac{1}{2}]$. Assume that for each $t_0\geq 0$ there exists $k_0\in N$ and $l_0\in N$ such that $k_0\geq t_0$ and

$$
\sum_{j=1}^{l_0}\int_{k_0+j}^{k_0+j+\frac{1}{2}}p(t)dt+\frac{1}{2}\sum_{j=0}^{l_0-1}p(k_0+j+\frac{1}{2})\geq 4.
$$

From [3, Theorem 4.46], we know that

$$
x^{\Delta\Delta}+p(t)x(\sigma(t))=0
$$

is oscillatory on $\mathbb{T}$. By Theorem 2.4, (2.5) is oscillatory.

Example 2.6. Consider the equation

$$
\begin{gathered}
(\frac{\sigma(t)}{t}x^{\Delta})^{\Delta}+tx(\sigma(t))=0, \quad t\geq 1,\ t\neq k,\ k=1,2,\dots,\\
x(t_k^{+})=b_k x(t_k), \quad x^{\Delta}(t_k^{+})=b_k x^{\Delta}(t_k), \quad b_k>0,\ k=1,2,\dots\\
x(1^{+})=x(1), \quad x^{\Delta}(1^{+})=x^{\Delta}(1),
\end{gathered} \tag{2.6}
$$

where $\mu(t)=\sigma(t)-t\leq ct$, c is a positive constant. Since $\mu(t)\leq ct$, we get

$$
\frac{t}{\sigma(t)}=\frac{t}{t+\mu(t)}\geq\frac{1}{1+c}.
$$

It is easy to see that

$$
\begin{gathered}
\int_1^{\infty}\frac{t}{\sigma(t)}\Delta t\geq\frac{1}{1+c}\int_1^{\infty}\Delta t=\infty,\\
\int_1^{\infty}t\Delta t=\infty.
\end{gathered}
$$

By [4, Theorem 3.2], we see that

$$
(\frac{\sigma(t)}{t}x^{\Delta})^{\Delta}+tx(\sigma(t))=0,
$$

is oscillatory. So (2.6) is oscillatory.

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