Electronic Journal of Differential Equations, Vol. 2009(2009), No. 123, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

THE RIEMANN PROBLEM IN GASDYNAMICS

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Abstract. In this note we give a proof of the existence of a solution to the Riemann problem in one-dimensional gasdynamics. Lax's 1957 paper on conservation laws leaves no doubt that such a solution exists, but it seems to us that there may be interest in a brief and explicit proof favorable to numerical computations. Our procedure also allows us to give a simple characterization of those problems in which a given wave is a shock or a rarefaction wave. In the final section we prove a result of Von Neumann's concerning the overtaking of two shocks. This paper was written in 1969 and is being published now at the suggestion of Jerry Goldstein, whose editorial note is included.

Editorial note by Jerry Goldstein: This work was done in 1969 and presented in the PDE Seminar at Tulane, where the two authors and I were colleagues. I was struck by the beauty and simplicity of the result, although Ed and Steve felt that the experts understood that the result was implicitly contained in Peter Lax's earlier paper [4]. Later these results were included in Joel Smoller's book [7], but I still felt that the Conway-Rosencrans global in time analysis of the Riemann problem in gasdynamics deserved a wider audience. Ed Conway died in 1985. In March 2009, I met Steve Rosencrans in New Orleans and urged him again to publish this lovely short note. I am very pleased that he finally agreed to do so.

1. Introduction

In this note we give a proof of the existence of a solution to the Riemann problem in gasdynamics. Lax's paper [4] leaves no doubt that such a solution exists, but it seems to us that there may be interest in a brief and explicit proof favorable to numerical computations. Our procedure also allows us to give a simple characterization of those problems in which a given wave is a shock or a rarefaction wave. In the final section we prove a result of Von Neumann's concerning the overtaking of two shocks.

The proof we give follows the general method given by Lax although a familiarity with that work will not be needed. In the general case treated by Lax existence is proved only for data close to a constant state (see below) but in the special case of gasdynamics, it is possible to prove a global result.

2000 *Mathematics Subject Classification.* 35L03, 35L65, 35L67, 76L05.
Key words and phrases. Riemann problem; shock wave; rarefaction wave.

Submitted August 2, 2009. Published September 29, 2009.
Supported in part by grant GP9019 from the National Science Foundation.

The Riemann problem is of current interest because of several recent papers (e.g., [3], [6]) considering weak solutions of conservation laws, in which existence is proved by approximating the given Cauchy problem by a sequence of Riemann problems. The Riemann problem is the Cauchy problem for one-dimensional gasdynamics with the initial data

$$\mathbf{v}(0,x) = \begin{cases} \mathbf{v}_r & x \geq 0, \\ \mathbf{v}_l & x < 0, \end{cases}$$

where $\mathbf{v}_r$ and $\mathbf{v}_l$ are any 3D vectors and generally

$$\mathbf{v}(t,x) = \begin{pmatrix} \rho(t,x) \\ p(t,x) \\ u(t,x) \end{pmatrix}$$

($t \geq 0$ and $-\infty < x < \infty$) and ρ, p, u are the density, pressure, and velocity. The function v must satisfy the equations of gasdynamics and the Rankine-Hugoniot conditions [2, eqns. 34.06 and 54]. We assume that the gas is polytropic, i.e.,

$$p = \text{Const.} \times e^{S/c_v} \rho^{\gamma},$$

where c_v is the specific heat at constant volume, S is the entropy per unit mass, and $1 < \gamma < 2$. The sound speed is then $c = \sqrt{\gamma p/\rho}$.

2. Method of solution of the Riemann problem

Following Lax we restate the results of [2] as follows: given any state $\mathbf{v}_l$ there exist three one-parameter families of states

$$\{B(y)\mathbf{v}_l\}, \quad \{C(y)\mathbf{v}_l\}, \quad \{F(y)\mathbf{v}_l\}$$

$-\infty < y < \infty$ such that

- $\mathbf{v}_l$ can be joined to $B(y)\mathbf{v}_l$ on the right by a backward-facing shock (if $y < 0$) or a centered backward-facing rarefaction wave (if $y > 0$)
- $\mathbf{v}_l$ can be joined to $C(y)\mathbf{v}_l$ on the right by a contact discontinuity.
- $\mathbf{v}_l$ can be joined to $F(y)\mathbf{v}_l$ on the right by a forward-facing shock (if $y < 0$) or a centered forward-facing rarefaction wave (if $y > 0$).

We solve the Riemann problem by finding y_1, y_2, y_3 such that

$$F(y_3)C(y_2)B(y_1)\mathbf{v}_l = \mathbf{v}_r.$$

The solution then consists of constant states separated by rarefaction waves, shocks, or contact discontinuities (see [2]).

In the following sections we determine the maps B, C, F. Actually, they are given almost explicitly in [2], and we use these results. The choice of the logarithm of the pressure ratio as a parameter makes the formulas explicit except for the inversion of one scalar function.

3. One-parameter family of forward-facing waves

First we shall derive the set of all states reached from $\mathbf{v}_l$ by a centered forward-facing rarefaction wave. In this case $p_r > p_l$, see [2, Section 81] so we can write

$$\frac{p_r}{p_l} = e^y, \quad y \geq 0.$$

Then

$$\frac{p_r}{p_l} = \left(\frac{\rho_r}{\rho_l}\right)^{1/\gamma} = e^{y/\gamma}$$

(see [2, Eqn. 40.10]). Finally [2, Eqn. 40.09] implies

$$\frac{u_r - u_l}{c_l} = \frac{2}{\gamma - 1}(e^{\tau y} - 1),$$

where

$$\tau = \frac{\gamma - 1}{2\gamma}$$

and c_l is the left-hand value of the sound speed. Now we compute the set of states reached from $\mathbf{v}_l$ by a forward-facing shock. We need the following two equations, [2, Eqn. 67.02]

$$\frac{\rho_1}{\rho_0} = \frac{p_1 + \mu^2 p_0}{p_0 + \mu^2 p_1}, \quad \mu^2 = \frac{\gamma - 1}{\gamma + 1} \tag{3.1}$$

and [2, Eqn. 71.05]

$$\frac{|u_1 - u_0|}{c_0} = \frac{1 - \mu^2}{\sqrt{1 + \mu^2}} \frac{p_1 - p_0}{p_0} \sqrt{\frac{p_0}{p_1 + \mu^2 p_0}} \tag{3.2}$$

where the subscript 0 stands for the state in front of (i.e., "before") the shock, and 1 stands for the state in back of (i.e., "after") the shock. In a forward-facing wave, particles cross the shock from right to left, so $1 = l$ and $0 = r$. Furthermore in a forward-facing shock $p_r \le p_l$, so we can write

$$\frac{p_r}{p_l} = e^x \quad (x \le 0).$$

Then (3.1) implies

$$\frac{\rho_r}{\rho_l} = \frac{e^x + \mu^2}{1 + \mu^2 e^x}\,.$$

In a forward-facing shock $u_1 \ge u_0$ so (3.2) implies

$$\frac{u_r - u_l}{c_l} = -\frac{1 - \mu^2}{\sqrt{1 + \mu^2}} \frac{1 - e^x}{\sqrt{\mu^2 + e^x}}.$$

The above calculations show that

$$F(y)\mathbf{v} = \begin{pmatrix} f_3(y) v_1 \\ e^y v_2 \\ v_3 + \sqrt{\frac{\gamma v_2}{v_1}} h_3(y) \end{pmatrix},$$

where

$$h_3(y) = \begin{cases} \frac{2}{\gamma - 1}(e^{\tau y} - 1) & y \ge 0 \\ \sqrt{\frac{2}{\gamma(\gamma+1)}} \frac{e^y - 1}{\sqrt{\mu^2 + e^y}} & y < 0 \end{cases}$$

and

$$f_3(y) = \begin{cases} e^{y/\gamma} & y \ge 0 \\ \frac{\mu^2 + e^y}{1 + \mu^2 e^y} & y < 0. \end{cases}$$

4. One-parameter family of backward-facing waves

The procedure is as in the previous section. The results are

$$B(y)\mathbf{v} = \begin{pmatrix} f_1(y)v_1 \\ e^{-y}v_2 \\ v_3 + \sqrt{\frac{\gamma v_2}{v_1}}h_1(y) \end{pmatrix},$$

where

$$f_1(y) = 1/f_3(y) \quad -\infty < y < \infty$$

and

$$h_1(y) = \begin{cases} \frac{2}{\gamma-1}(1-e^{-\tau y}) & y \geq 0 \\ \sqrt{\frac{2}{\gamma(\gamma+1)}}\frac{1-e^{-y}}{\sqrt{\mu^2+e^{-y}}} & y < 0. \end{cases}$$

5. One-parameter family of contact discontinuities

In this case the density suffers an arbitrary change while p and u are constant. Thus we may take

$$C(y)\mathbf{v} = \begin{pmatrix} e^y v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

6. Solution of the Riemann problem

We need the following observation:

$$\frac{h_3(x)}{\sqrt{f_1(x)}}\frac{e^{-x/2}}{h_1(x)} = 1 \quad -\infty < x < \infty. \tag{6.1}$$

We define

$$A_\rho = \frac{\rho_r}{\rho_l}, \quad A_p = \frac{p_r}{p_l}, \quad A_u = \frac{u_r - u_l}{c_l}.$$

It is easy to see that h_1 is monotonic increasing and that

$$h_1([-\infty,\infty]) = [-\infty, \frac{2}{\gamma-1}].$$

Therefore, the equation

$$h_1(y_1) + \sqrt{\frac{A_p}{A_\rho}}h_1\left(y_1 + \log A_p\right) = A_u \tag{6.2}$$

has a unique solution y_1 if and only if

$$A_u < \frac{2}{\gamma-1}\Big(1 + \sqrt{A_p/A_\rho}\Big). \tag{6.3}$$

Let

$$y_3 = y_1 + \log A_p$$
$$y_2 = \log\Big(\frac{A_\rho f_1(y_3)}{f_1(y_1)}\Big).$$

It is then a straightforward calculation to show that

$$F(y_3)C(y_2)B(y_1)\mathbf{v}_l = \mathbf{v}_r$$

so that we have a solution to the Riemann problem for arbitrary $\mathbf{v}_r$ and $\mathbf{v}_l$ satisfying (6.2). [1]

Note that this solution is explicit except for the inversion of only one function, i.e. (6.2). Thus the method is well-suited to computations.

We also have some information about the solution. The backward wave is a rarefaction wave if and only if $y_1 > 0$, which by (6.2) is equivalent to

$$\sqrt{\frac{A_p}{A_\rho}}h_1(\log A_p) < A_u < \frac{2}{\gamma-1}\Big(1+\sqrt{A_p/A_\rho}\Big)$$

and is a shock otherwise. Similarly, we find that the forward-facing wave is a rarefaction wave if and only if $y_3 > 0$, which by (6.2) is equivalent to

$$h_1(-\log A_p) < A_u < \frac{2}{\gamma-1}\Big(1+\sqrt{A_p/A_\rho}\Big)$$

and is a shock otherwise. These explicit criteria seem to be new.

7. Overtaking of two weak forward-facing shock waves

We now apply the above results to the interaction of two forward-facing shock waves. This problem has been treated by von Neumann (unpublished). (However, see related material in [5].) He showed that if $\gamma \le 5/3$, then the resulting configuration after the waves meet consists of a backward-facing rarefaction wave, a contact discontinuity, and a forward-facing transmitted shock. The proof is included in the report [1].

The criteria derived at the end of Section 6 immediately enable us to formulate the problem as an inequality. As a simple application, we prove von Neumann's result for sufficiently **weak** interacting shocks: the resulting backward wave **is** a rarefaction wave if $\gamma < 5/3$ and **is not** a rarefaction wave if $\gamma > 5/3$.

The problem is formulated as follows. We are given three states, $\mathbf{v}_l$, $\mathbf{v}_m$, $\mathbf{v}_r$ such that

$$\mathbf{v}_m = F_s(\mathbf{v}_l) \quad s < 0$$
$$\mathbf{v}_r = F_t(\mathbf{v}_m) \quad t < 0$$

and we wish to solve the Riemann problem with initial states $\mathbf{v}_l$, $\mathbf{v}_r$. From these equations we see

$$\mathbf{v}_r = F_t(F_s(\mathbf{v}_l))$$

[1]Physical interpretation: if (6.3) is violated, the two sections of the gas are moving away from one another so fast that a vacuum is formed

which implies

$$\begin{gathered} A_\rho = f_3(t)f_3(s) \\ A_p = e^{t+s} \\ A_u = h_3(s) + h_3(t)\sqrt{e^s/f_3(s)}. \end{gathered}$$

From Section 6 we know that (assuming (6.3)) the backward-facing wave is a rarefaction wave if and only if

$$A_u > \sqrt{\frac{A_p}{A_\rho}}\, h_1(\log A_p).$$

Using (6.1) we see that this is equivalent to

$$h_1(t+s) < h_1(t) + \frac{h_1(s)h_1(t)}{h_3(t)}. \tag{7.1}$$

Let

$$\begin{gathered} \beta := \frac{\gamma+1}{\gamma-1} \\ \psi(x) := \frac{1-e^{-x}}{\sqrt{1+\beta e^{-x}}} \\ G(t,s) := \psi(t+s)\psi(-t) - \psi(t)\psi(-t) + \psi(t)\psi(s) \\ \phi(x) := \psi(x)/x \end{gathered}$$

Then (7.1) is equivalent to

$$G(t,s) < 0 \quad \text{for } t<0,\ s<0.$$

The following rearrangement:

$$\begin{aligned} \frac{G(t,s)}{t^2(s+t)} &= 2\Big\{\frac{\phi(s)-\phi(-t)}{s+t}\Big\}\Big\{\frac{\phi(t)-\phi(-t)}{s+t}\Big\} \\ &\quad - \phi(-t)\Big\{\frac{\frac{\phi(s+t)-\phi(t)}{s} - \frac{\phi(s)-\phi(-t)}{s+t}}{t}\Big\} \end{aligned}$$

allows us to calculate the limit

$$\begin{aligned} \lim_{s,t\uparrow 0} \frac{G(t,s)}{t^2 s(s+t)} &= 2\phi'(0)^2 - \frac{3}{2}\phi(0)\phi''(0) \\ &= \frac{3}{64}\,\frac{\gamma^2-1}{\gamma^3}\big(\gamma - \frac{5}{3}\big), \end{aligned}$$

which proves the stated result.

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