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ASYMPTOTIC FORMULAS FOR SOLUTIONS OF PARAMETER-DEPENDING ELLIPTIC BOUNDARY-VALUE PROBLEMS IN DOMAINS WITH CONICAL POINTS

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Abstract. In this article, we study elliptic boundary-value problems, depending on a real parameter, in domains with conical points. We present asymptotic formulas for solutions near singular points, as linear combinations of special singular functions and regular functions. These functions and the coefficients of the linear combination are regular with respect to the parameter.

1. Introduction

Elliptic boundary-value problems in domains with point singularities were thoroughly investigated (see, e.g, [3] and the extensive bibliography in this book). We are concerned with elliptic boundary-value problems depending on a real parameter in domains with conical points. These problems arise in considering initial-boundary-value problems for non-stationary equations with coefficients depending on time (see, e.g, [5], where the initial-boundary-value problem for strongly hyperbolic systems with Dirichlet boundary conditions was considered). We give here as an example the initial-boundary-value problem for the parabolic equation

$$u_t + L(x,t,\partial_x)u = f \quad \text{in } G_T, \tag{1.1}$$

$$B_j(x,t,\partial_x)u = 0, \quad \text{on } S_T,\ j = 1,\dots,m, \tag{1.2}$$

$$u|_{t=0} = \varphi \quad \text{on } G, \tag{1.3}$$

where the sets G, G_T, S_T, and the operators L, B_j are introduced in Section 2. For this problem we have first dealt with the unique solvability and the regularity of the generalized solution with respect to the time variable t (see [6]). After that, to investigate the regularity and the asymptotic of the solution, (1.1) and (1.2) are rewritten in the form

$$L(x,t,\partial_x)u = f - u_t \quad \text{in } G_T, \tag{1.4}$$

$$B_j(x,t,\partial_x)u = 0, \quad \text{on } S_T,\ j = 1,\dots,m. \tag{1.5}$$

Then (1.4), (1.5) can be regarded as a elliptic boundary-value problem depending on the parameter t. This approach was suggested in [2].

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In the present paper we are concerned with asymptotic behaviour of the solutions near the singular points. Firstly, applying the results of the analytic perturbation theory of linear operators ([1]) and the method of linearization of polynomial operator pencils ([10]), we establish the smoothness with respect to the parameter of the eigenvalues, the eigenvectors of the operator pencils generated by the problems. After that, applying the well-known results for elliptic boundary-value problems (without parameter) in the considered domains, we receive the asymptotic formulas of the solutions as a sum of a linear combination of special singular functions and a regular function in which this functions and the coefficients of the linear combinations are regular with respect to the parameter. The present results will be applied to deal with the asymptotic behaviour of the solutions of initial-boundary-value problems for parabolic equations in cylinders with bases containing conical points in a forthcoming work.

Our paper is organized as follows. In Section 2, we introduce some needed notation and definitions. We study the spectral properties of the operator pencil generated by the problem in Section 3. Section 4 is devoted to establishing the asymptotic behaviour of the solutions in a neighborhood of the conical point.

2. Preliminaries

Let G be a bounded domain in $\mathbb{R}^n (n \geq 2)$ with the boundary ∂G. We suppose that $S = \partial G \setminus \{0\}$ is a smooth manifold and G in a neighborhood of the origin 0 coincides with the cone $K = \{x : x/|x| \in \Omega\}$, where Ω is a smooth domain on the unit sphere S^{n-1} in $\mathbb{R}^n$. Let T be a positive real number or $T = +\infty$. If A is a subset of $\mathbb{R}^n$, we set $A_T = A \times (0, T)$. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $\partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$.

Let us introduce some functional space used in this paper. Let l be a nonnegative integer. We denote by $W_2^l(G)$ the usual Sobolev space of functions defined in G with the norm

$$\|u\|_{W_2^l(G)} = \Big(\int_G \sum_{|\alpha| \leq m} |\partial_x^\alpha u|^2 dx \Big)^{1/2},$$

and by $W_2^{l-\frac{1}{2}}(S)$ the space of traces of functions from $W_2^l(G)$ on S with the norm

$$\|u\|_{W_2^{l-\frac{1}{2}}(S)} = \inf \big\{ \|v\|_{W_2^l(G)} : v \in W_2^l(G), v|_S = u \big\}.$$

We define the weighted Sobolev space $V_{2,\gamma}^l(K)$ ($\gamma \in \mathbb{R}$) as the closure of $C_0^\infty(\overline{K} \setminus \{0\})$ with respect to the norm

$$\|u\|_{V_{2,\gamma}^l(K)} = \Big(\sum_{|\alpha| \leq l} \int_K r^{2(\gamma + |\alpha| - l)} |\partial_x^\alpha u|^2 dx \Big)^{1/2}, \tag{2.1}$$

where $r = |x| = \big(\sum_{k=1}^n x_k^2 \big)^{1/2}$. If $l \geq 1$, then $V_\gamma^{l-\frac{1}{2}}(\partial K)$ denote the space consisting of traces of functions from $V_{2,\gamma}^l(K)$ on the boundary ∂K with the norm

$$\|u\|_{V_\gamma^{l-\frac{1}{2}}(\partial K)} = \inf \big\{ \|v\|_{V_{2,\gamma}^l(K)} : v \in V_{2,\gamma}^l(K), v|_{\partial K} = u \big\}. \tag{2.2}$$

It is obvious from the definition that the space $V_{2,\gamma+k}^{l+k}(K)$ is continuously imbedded into the space $V_{2,\gamma}^l(K)$ for an arbitrary nonnegative integer k. An analogous assertion holds for the space $V_{2,\gamma}^{l-\frac{1}{2}}(\partial K)$. The weighted spaces $V_{2,\gamma}^{l}(G)$, $V_{\gamma}^{l-\frac{1}{2}}(S)$ are defined similarly as in (2.1), (2.2) with $K, \partial K$ replaced by G, S, respectively.

Let h be a nonnegative integer and X be a Banach space. Denote by $\mathcal{B}(X)$ the set of all continuous linear operators from X into itself. By $W_2^h((0,T);X)$ we denote the Sobolev space of X-valued functions defined on $(0,T)$ with

$$\|f\|_{W_2^h((0,T);X)} = \Big(\sum_{k=0}^{h}\int_O^T \|\frac{d^k f(t)}{dt^k}\|_X^2 dt\Big)^{1/2} < +\infty.$$

For short, we set

$$W_2^h((0,T)) = W_2^h((0,T);\mathbb{C}), \quad W_2^{l,h}(\Omega_T) = W_2^h((0,T);W_2^l(\Omega)),$$
$$V_{2,\gamma}^{l,h}(G_T) = W_2^h((0,T);V_{2,\gamma}^l(G)), \quad V_{2,\gamma}^{l-\frac{1}{2},h}(S_T) = W_2^h((0,T);V_{2,\gamma}^{l-\frac{1}{2}}(S)).$$

Recall that a X-valued function $f(t)$ defined on $[0,+\infty)$ is said to be continuous or analytic at $t=+\infty$ if the function $g(t) = f(\frac{1}{t})$ is continuous or analytic, respectively, at $t=0$ with a suitable definition of $g(0) \in X$. In these cases we can regard $f(t)$ as a function defined on $[0,+\infty]$ with $f(+\infty) = g(0)$. Denote by $C^{\mathfrak{a}}([0,T];X)$ the set of all X-valued functions defined and analytic on $[0,T]$ (recall that T is a positive real number or $T=+\infty$). It is clear that if $f \in C^{\mathfrak{a}}([0,T];X)$, then f together with all its derivatives are bounded on $[0,T]$.

Let A be a subset of $\mathbb{R}^n$ and $f(x,t)$ be a complex-valued function defined on $A_T = A \times [0,T]$. We will say that f belongs to the class $C^{\infty,\mathfrak{a}}(A_T)$ if and only if $f \in C^a([0,T];C^l(A))$ for all nonnegative integer l.

Let

$$L = L(x,t,\partial_x) = \sum_{|\alpha|\le 2m} a_\alpha(x,t)\,\partial_x^\alpha$$

be a differential operator of order $2m$ defined in Q with coefficients belonging to $C^{\infty,\mathfrak{a}}(\overline{G}_T)$ ($\overline{G}_T = \overline{G}\times[0,T]$). Suppose that $L(x,t,\partial_x)$ is elliptic on $\overline{G}$ uniformly with respect to t on $[0,T]$, i.e, there is a positive constant c_0 such that

$$|L^\circ(x,t,\xi)| \ge c_0|\xi|^{2m} \tag{2.3}$$

for all $\xi \in \mathbb{R}^n$ and for all $(x,t) \in \overline{G}_T$. Here $L^\circ(x,t,\partial_x)$ is principal part of the operator $L(x,t,\partial_x)$; i.e,

$$L^\circ(x,t,\partial_x) = \sum_{|\alpha|=2m} a_\alpha(x,t)\partial_x^\alpha.$$

Let

$$B_j = B_j(x,t,\partial_x) = \sum_{|\alpha|\le\mu_j} b_{j,\alpha}(x,t)\partial_x^\alpha, \quad j=1,\dots,m,$$

be a system of boundary operators on S with coefficients belonging to $C^{\infty,\mathfrak{a}}(\partial G\times[0,T])$, $\operatorname{ord} B_j = \mu_j \le 2m-1$, $j=1,\dots,m$. Suppose that $\{B_j(x,t,\partial_x)\}_{j=1}^m$ is a normal system on S uniformly with respect to t on $[0,T]$; i.e, the two following conditions are satisfied:

(i) $\mu_j \ne \mu_k$ for $j \ne k$,

(ii) there are positive constants c_j such that

$$|B_j^\circ(x,t,\nu(x))| \ge c_j, j=1,\dots,m, \tag{2.4}$$

for all $(x,t) \in S_T$. Here $B_j^\circ(x,t,\partial_x)$ is the principal part of $B_j(x,t,\partial_x)$ and $\nu(x)$ is the unit outer normal to S at point x.

In this paper, we consider asymptotic behaviour near the conical point of solutions of the elliptic boundary-value problem depending on the parameter t:

$$L(x,t,\partial_x)u = f \quad \text{in } G_T, \tag{2.5}$$

$$B_j(x,t,\partial_x)u = g_j \quad \text{on } S_T, j = 1,\dots,m. \tag{2.6}$$

3. Spectral properties of the pencil operator generated

Let $\mathfrak{L} = \mathfrak{L}(t,\partial_x), \mathfrak{B}_j = \mathfrak{B}_j(t,\partial_x)$ be the principal homogenous parts of $L(x,t,\partial_x)$, $B_j(x,t,\partial_x)$ at $x = 0$; i.e,

$$\mathfrak{L} = \mathfrak{L}(t,\partial_x) = \sum_{|\alpha|=2m} a_\alpha(0,t)\,\partial_x^\alpha,$$

$$\mathfrak{B}_j = \mathfrak{B}_j(t,\partial_x) = \sum_{|\alpha|=\mu_j} b_{j\alpha}(0,t)\,\partial_x^\alpha, j = 1,\dots,m.$$

It can be directly verified that the derivative ∂_x^α has the form

$$\partial_x^\alpha = r^{-|\alpha|}\sum_{p=0}^{|\alpha|} P_{\alpha,p}(\omega,\partial_\omega)(r\partial r)^p, \tag{3.1}$$

where $P_{\alpha,p}(\omega,\partial_\omega)$ are differential operators of order $\le |\alpha| - p$ with smooth coefficients on $\overline{\Omega}$, $r = |x|$, ω is an arbitrary local coordinate system on S^{n-1}. Thus we can write $\mathfrak{L}(t,\partial_x), \mathfrak{B}_j(t,\partial_x)$ in the form

$$\mathfrak{L}(t,\partial_x) = r^{-2m}\mathscr{L}(\omega,t,\partial_\omega,r\partial_r),$$

$$\mathfrak{B}_j(t,\partial_x) = r^{-\mu_j}\mathscr{B}_j(\omega,t,\partial_\omega,r\partial_r).$$

We introduce the operator

$$\mathscr{U}(\lambda,t) = (\mathscr{L}(\omega,t,\partial_\omega,\lambda),\mathscr{B}_j(\omega,t,\partial_\omega,\lambda)), \lambda\in\mathbb{C}, t\in[0,T]$$

of the parameter-depending elliptic boundary-value problem

$$\mathscr{L}(\omega,t,\partial_\omega,\lambda)u = f \quad \text{in } \Omega,$$

$$\mathscr{B}_j(\omega,t,\partial_\omega,\lambda)u = g_j \quad \text{on } \partial\Omega, \quad j = 1,\dots,m$$

(Here the parameters are λ and t). For every fixed $\lambda\in\mathbb{C}, t\in[0,T]$ this operator continuously maps

$$\mathcal{X} \equiv W_2^l(\Omega) \text{ into } \mathcal{Y} \equiv W_2^{l-2m}(\Omega)\times\prod_{j=1}^m W_2^{l-\mu_j-\frac{1}{2}}(\partial\Omega)\ (l\ge 2m).$$

We can write $\mathscr{U}(\lambda,t)$ in the form

$$\mathscr{U}(\lambda,t) = A_{2m}(t)\lambda^{2m} + A_{2m-1}(t)\lambda^{2m-1} + \dots + A_0(t),$$

where $A_k(t), k = 2m, 2m-1,\dots,0$ are differential operators in Ω of order $2m-k$ with coefficients belonging to $C^{\infty,\mathfrak{a}}(\Omega_T)$, especially

$$A_{2m}(t) = (\sum_{|\alpha|=2m} a_\alpha(0,t)\omega^\alpha, 0,\dots,0). \tag{3.2}$$

We mention now some well-known definitions ([3]). Let $t_0\in[0,T]$ fixed. If $\lambda_0\in\mathbb{C}$, $\varphi_0\in\mathcal{X}$ such that $\varphi_0\ne 0, \mathscr{U}(\lambda_0,t_0)\varphi_0 = 0$, then λ_0 is called an eigenvalue of $\mathscr{U}(\lambda,t_0)$ and $\varphi_0\in\mathcal{X}$ is called an eigenvector corresponding to λ_0. $\Lambda = \dim\ker\mathscr{U}(\lambda_0,t_0)$ is called the geometric multiplicity of the eigenvalue λ_0.

If the elements $\varphi_1, \dots, \varphi_s$ of $\mathcal{X}$ satisfy the equations

$$\sum_{q=0}^{\sigma} \frac{1}{q!} \frac{d^q}{d\lambda^q} \mathscr{U}(\lambda, t_0)|_{\lambda=\lambda_0} \varphi_{\sigma-q} = 0 \quad \text{for } \sigma = 1, \dots, s,$$

then the ordered collection $\varphi_0, \varphi_1, \dots, \varphi_s$ is said to be a Jordan chain corresponding to the eigenvalue λ_0 of the length $s+1$. The rank of the eigenvector φ_0 ($\operatorname{rank} \varphi_0$) is the maximal length of the Jordan chains corresponding to the eigenvector φ_0.

A canonical system of eigenvectors of $\mathscr{U}(\lambda_0, t_0)$ corresponding to the eigenvalue λ_0 is a system of eigenvectors $\varphi_{1,0}, \dots, \varphi_{\Lambda,0}$ such that $\operatorname{rank} \varphi_{1,0}$ is maximal among the rank of all eigenvectors corresponding to λ_0 and $\operatorname{rank} \varphi_{j,0}$ is maximal among the rank of all eigenvectors in any direct complement in $\ker \mathscr{U}(\lambda_0, t_0)$ to the linear span of the vectors $\varphi_{1,0}, \dots, \varphi_{j-1,0}$ ($j = 2, \dots, \Lambda$). The number $\kappa_j = \operatorname{rank} \varphi_{j,0}$ ($j = 1, \dots, \Lambda$) are called the partial multiplicities and the sum $\kappa = \kappa_1 + \dots + \kappa_\Lambda$ is called the algebraic multiplicity of the eigenvalue λ_0.

The eigenvalue of λ_0 is called simple if both its geometric multiplicity and the rank of the corresponding eigenvector equal to one.

For each $t \in [0, T]$ fixed the set of all complex number λ such that $\mathscr{U}(\lambda, t)$ is not invertible is called the spectrum of $\mathscr{U}(\lambda, t)$. It is known that the spectrum of $\mathscr{U}(\lambda, t)$ is an enumerable set of its eigenvalues (see [3, Th. 5.2.1]). Moreover, there are constants δ, R such that $\mathscr{U}(\lambda, t)$ is invertible for all $t \in [0, T]$ and all λ in the set

$$\mathcal{D} := \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \le \delta |\operatorname{Im} \lambda|, |\lambda| \ge R\} \tag{3.3}$$

(see [3, Thm. 3.6.1]).

Now we use method of linearization to investigate the smoothness of the eigenvalues and the eigenvectors of $\mathscr{U}(\lambda, t)$ with respect to t. Without loss of generality we can assume that the operator $A_0(t)$ is invertible. Indeed, if λ_0 is an eigenvalue of $\mathscr{U}(\lambda, t)$ for all $t \in [0, T]$, then

$$\mathscr{U}(\lambda_0 + \lambda, t) = \sum_{k=0}^{2m} \widetilde{A}_k(t) \lambda^k,$$

where $\widetilde{A}_0(t) = \mathscr{U}(\lambda_0, t)$ is invertible for all $t \in [0, T]$. Setting

$$\mathscr{V}(\lambda, t) = A_0^{-1}(t) \mathscr{U}(\lambda, t), D_k(t) = A_0^{-1}(t) A_k(t), k = 1, \dots, 2m,$$

we have the pencils of continuous operators $\mathscr{V}(\lambda, t)$, $D_k(t)$, $k = 1, \dots, 2m$, from $\mathcal{X}$ into itself, and

$$\mathscr{V}(\lambda, t) = D_{2m}(t) \lambda^{2m} + D_{2m-1}(t) \lambda^{2m-1} + \dots + D_1(t) \lambda + I, \tag{3.4}$$

where I is the identical operator in $\mathcal{X}$. The eigenvalues and the eigenvectors of $\mathscr{V}(\lambda, t)$ are defined analogously as of $\mathscr{U}(\lambda, t)$.

We can verify directly (or see [10, Le. 12.1]) that

$$\mathcal{I} - \lambda \mathscr{A}(t) = \mathscr{C}(t) \mathscr{E}(\lambda, t) \begin{bmatrix} \mathscr{V}(\lambda, t) & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} \mathscr{F}(\lambda, t), \tag{3.5}$$

where

$$\mathscr{A} = \begin{bmatrix} -D_1 & \dots & -D_{2m-1} & -D_{2m} \\ I & & & \\ & \ddots & & \\ & & I & \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} I & -D_1 & \dots & -D_{2m-1} \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix},$$

$$\mathscr{E} = \begin{bmatrix} I & \sum_{k=1}^{2m} D_k\lambda^{k-1} & \sum_{k=2}^{2m} D_k\lambda^{k-2} & \dots & D_{2m-1} + D_{2m}\lambda \\ & I & & & \\ & & \ddots & & \\ & & & & I \end{bmatrix},$$

$$\mathscr{F} = \begin{bmatrix} I & & & \\ -\lambda I & I & & \\ & \ddots & \ddots & \\ & & -\lambda I & I \end{bmatrix},$$

(in operator matrices all the elements not indicated are assumed to be zero, and the argument t has been omitted for the sake of brevity) and $\mathcal{I}$ is the identical operator in $\mathcal{X}^{2m}$. Verifying directly we see that $\mathscr{C}(t)$, $\mathscr{E}(\lambda, t)$, $\mathscr{F}(\lambda, t)$ are invertible elements of $\mathcal{B}(\mathcal{X}^{2m})$ with

$$\mathscr{C}^{-1} = \begin{bmatrix} I & D_1 & \dots & D_{2m-1} \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}, \quad \mathscr{F}^{-1} = \begin{bmatrix} I & & & \\ -\lambda I & I & & \\ & \ddots & \ddots & \\ & & -\lambda I & I \end{bmatrix},$$

$$\mathscr{E}^{-1} = \begin{bmatrix} I & -\sum_{k=1}^{2m} D_k\lambda^{k-1} & -\sum_{k=2}^{2m} D_k\lambda^{k-2} & \dots & -(D_{2m-1} + D_{2m}\lambda) \\ & I & & & \\ & & \ddots & & \\ & & & & I \end{bmatrix}.$$

It follows from the assumption on the analyticity of coefficients of differential operators $L(x, t, \partial_x)$ and $B_j(x, t, \partial_x)$, $j = 1, \dots, m$, that $\mathscr{V}(\lambda, t)$ is of the class $C^{\mathfrak{a}}([0, T]; \mathcal{B}(\mathcal{X}))$ and $\mathscr{A}(t)$ is of the class $C^{\mathfrak{a}}([0, T]; \mathcal{B}(\mathcal{X}^{2m}))$.

It is obvious that $\mathscr{U}(\lambda, t)$ and $\mathscr{V}(\lambda, t)$ have the same eigenvalues with the same multiplicities and the same corresponding eigenvectors. It follows from (3.5) that the spectra except the zero of the pencil $\mathscr{V}(\lambda, t)$ and the operator $\mathscr{A}(t)$ coincide for all $t \in [0, T]$. We now show that for each $t \in [0, T]$ all eigenvalues of $\mathscr{V}(\lambda, t)$ and $\mathscr{A}(t)$ are nonzero. It is obvious for these of $\mathscr{V}(\lambda, t)$. Suppose $\varphi = (\varphi^{(1)}, \dots, \varphi^{(2m)}) \in \mathcal{X}^{2m}, \varphi \neq 0$ such that $\mathscr{A}(t)\varphi = 0$ for some $t \in [0, T]$. Then $\varphi^{(1)} = \dots = \varphi^{(2m-1)} = 0$ and $D_{2m}(t)\varphi^{(2m)} = 0$. This implies $A_{2m}(t)\varphi^{(2m)} = 0$, but this do not occur since $\ker A_{2m}(t) = \{0\}$ which follows from (3.2).

Now we can apply [10, Le. 12.5, 12.8] to conclude that the complex number λ_0 is an eigenvalue of the pencil $\mathscr{V}(\lambda, t)$ (for some $t \in [0, T]$) if and only if $\sigma_0 = (\lambda_0)^{-1}$ is an eigenvalue of the operator $\mathscr{A}(t)$ with the same multiplicities. Hence for each $t \in [0, T]$ the spectrum of the operator $\mathscr{A}(t)$ is a bounded set consisting nonzero eigenvalues with finite multiplicities.

Lemma 3.1. *Let γ_1, γ_2 be real numbers, $\gamma_1 < \gamma_2$ such that the lines $\operatorname{Re}\lambda = \gamma_j, j = 1, 2$, do not contain any eigenvalue of $\mathscr{U}(\lambda, t)$ and all eigenvalues of this pencil in the strip*

$$\mathcal{D}_1 := \{\lambda \in \mathbb{C} : \gamma_1 < \operatorname{Re}\lambda < \gamma_2\} \tag{3.6}$$

are simple for all $t \in [0, T]$. Then there are complex-valued functions $\lambda_k(t)$ and $\mathcal{X}$-valued functions $\varphi_k(t)$, $k = 1, \dots, N$, which are analytic on $[0, T]$ such that, for each $t \in [0, T]$, $\{\lambda_1(t), \dots, \lambda_N(t)\}$ is the set of all eigenvalues of $\mathscr{U}(\lambda, t)$ in $\mathcal{D}_1$ and $\varphi_k(t)$ are eigenvectors corresponding to the eigenvalues $\lambda_k(t)$, $k = 1, \dots, N$, respectively.

Proof. Since the set $\mathcal{D}$ defined in (3.3) does not contains eigenvalues of $\mathscr{U}(\lambda, t)$ for all $t \in [0, T]$, the eigenvalues of this pencil in the strip $\mathcal{D}_\infty$ actually are located in the bounded domain $\mathcal{D}_2 = (\mathbb{C} \setminus \mathcal{D}) \cap \mathcal{D}_1$. Moreover, the boundary $\partial\mathcal{D}_2$ of $\mathcal{D}_2$ contains no eigenvalues of $\mathscr{U}(\lambda, t)$ for all $t \in [0, T]$. Let M be a positive number such that $\|\mathscr{A}(t)\| < M$ for all $t \in [0, T]$. Put $\mathcal{D}_0 = \{\sigma \in \mathbb{C} : (\sigma)^{-1} \in \mathcal{D}_2, |\sigma| < M\}$. Then $\mathcal{D}_0$ is a connected bounded domain in $\mathbb{C}$ and for each $t \in [0, T]$ the spectrum of the operator $\mathscr{A}(t)$ consists a finite set of its simple eigenvalues and does not intersect with the boundary $\partial\mathcal{D}_0$.

Now let $t_0 \in [0, T]$ and $\sigma_0 \in \mathcal{D}_0$ be a simple eigenvalue of the operator $\mathscr{A}(t_0)$. Then according to the results on analytic perturbation of linear operators (see [11, Th. XII.8]), there exists a complex-valued $\sigma(t)$ defined and analytic on a subinterval containing t_0 of $[0, T]$ such that $\sigma(t)$ is a simple eigenvalue of $\mathscr{A}(t)$ for all t in such subinterval. We show now that $\sigma(t)$ may be continued to be defined on $[0, T]$.

To see this, let I_0 be the maximal interval of $\sigma(t)$ considered and suppose that t_1 is the right end of I_0 and $0 < t_1 < T$. Since $\sigma(t)$ does not go out of the domain $\mathcal{D}_0$ and the spectrum of $\mathscr{A}(t_1)$ consists only a finite set of its eigenvalues, $\sigma(t)$ must converge to an eigenvalue $\widehat{\sigma}_0 \in \mathcal{D}_2$ of $\mathscr{A}(t_1)$ as $t \uparrow t_1$ (see [1, VII.3.5]). Thus, $\sigma(t)$ must coincide with the analytic function $\widehat{\sigma}(t)$ representing eigenvalues of $\mathscr{A}(t)$ in a subinterval containing t_1, $\widehat{\sigma}(t_1) = \widehat{\sigma}_0$. This implies that $\sigma(t)$ admits an analytic continuation beyond t_1, contradicting the supposition that t_1 is the right end of the maximal interval I_0 of $\sigma(t)$.

Treating the other eigenvalues of $\mathscr{A}(t_0)$ in $\mathcal{D}_0$ in the same way, we receive functions $\sigma_1(t), \dots, \sigma_N(t)$ analytic on $[0, T]$ such that $\sigma_k(t)$, $k = 1, \dots, N$, are simple eigenvalues of $\mathscr{A}(t)$ for all $t \in [0, T]$. One can also choose $\mathcal{X}^{2m}$-valued functions $\eta_k(t)$, $k = 1, \dots, N$, analytic on $[0, T]$ such that $\eta_k(t)$ are eigenvectors corresponding to the eigenvalues $\sigma_k(t)$ (see [11, Th. XII.8]). Set $\lambda_k(t) = (\delta_k(t))^{-1}$, $k = 1, \dots, N$. Then these functions are analytic functions on $[0, T]$ and $\{\lambda_1(t), \dots, \lambda_N(t)\}$ is the set of all eigenvalues of $\mathscr{U}(\lambda, t)$ in the strip $\mathcal{D}_1$ for each $t \in [0, T]$.

Rewrite the function $\eta_k(t)$ in the form of column vector $(\eta_k^{(1)}(t), \dots, \eta_k^{(2m)}(t))$ $(k = 1, \dots, N)$. Then $\mathcal{X}$-valued function $\varphi_k(t) = \eta_k^{(1)}(t)$ is analytic on $[0, T]$ and $\varphi_k(t)$ is an eigenvector of $\mathscr{U}(\lambda, t)$ corresponding to eigenvalues $\lambda_k(t)$ for each $t \in [0, T]$. Remember that $\mathcal{X} = W_2^l(\Omega)$, l is an arbitrary nonnegative integer. Thus, by Sobolev imbedding theorem, we have $\eta_k^{(1)}(t) \in C^{\infty, \mathfrak{a}}(\Omega_T)$. The proof is complete. □

From the assumption on the coefficients of the operators B_j and the assumption (2.4), we have

$$|B_j^\circ(0, t, \nu(x))| \ge |B_j^\circ(x, t, \nu(x))| - |B_j^\circ(0, t, \nu(x)) - B_j^\circ(x, t, \nu(x))| > 0 \tag{3.7}$$

$(j=1,\dots,m)$, for all $x\in S$ sufficiently near the origin and for all $t\in[0,T]$. $\mathfrak{B}_j(t,\nu(x))$ can be regarded as defined on K_T, and $\mathfrak{B}_j(t,\nu(x))\neq 0$ for all $x\in\partial K_T$ and for all $t\in[0,T]$ since $\nu(x)$ are the same on each axis of the cone K. Thus, the system $\{\mathfrak{B}_j(t,\partial_x)\}_{j=1}^m$ is normal on ∂K for each $t\in[0,T]$. Therefore, there are boundary operators $\mathfrak{B}_j(t,\partial_x)$, $\operatorname{ord}\mathfrak{B}_j(t,\partial_x)=\mu_j<2m$, $j=m+1,\dots 2m$, such that the system $\{\mathfrak{B}_j(x,t,\partial_x)\}_{j=1}^{2m}$ is a Dirichlet system of order $2m$ (for definition see [3], p. 63) on ∂K for each $t\in[0,T]$, and the following classical Green formula

$$\int_K \mathfrak{L}u\overline{v}dx+\sum_{j=1}^m\int_{\partial K}\mathfrak{B}_ju\overline{\mathfrak{B}'_{j+m}v}ds=\int_K u\overline{\mathfrak{L}^+v}dx+\sum_{j=1}^m\int_{\partial K}\mathfrak{B}_{j+m}u\overline{\mathfrak{B}'_jv}ds \quad (3.8)$$

holds for $u,v\in C_0^\infty(\overline{K}\setminus\{0\})$ and for each $t\in[0,T]$. Here $\mathfrak{L}^+=\mathfrak{L}^+(t,\partial_x)$ is the formal adjoint operator of $\mathfrak{L}$, i.e,

$$\mathfrak{L}^+u=(-1)^{2m}\sum_{|\alpha|=2m}\overline{a}_\alpha(0,t)\partial_x^\alpha u,$$

and $\mathfrak{B}'_j=\mathfrak{B}'_j(t,\partial_x)$ are boundary operators of order $\mu'_j=2m-1-\mu_{j+m}$ if $j\le m$, and of order $\mu'_j=2m-1-\mu_{j-m}$ if $j\ge m+1$. The coefficients of $\mathfrak{B}'_j=\mathfrak{B}'_j(t,\partial_x)$, $j=1,\dots,2m$, are independent of the variable x and dependent on t analytically on $[0,T]$.

The operators $\mathfrak{L}^+(t,\partial_x),\mathfrak{B}'_j(t,\partial_x)$ can be written in the form

$$\mathfrak{L}^+(t,\partial_x)=r^{-2m}\mathscr{L}^+(\omega,t,\partial_\omega,r\partial_r), \quad (3.9)$$

$$\mathfrak{B}'_j(t,\partial_x)=r^{-\mu'_j}\mathscr{B}'_j(\omega,t,\partial_\omega,r\partial_r). \quad (3.10)$$

From the Green formula (3.8) we get the following Green formula

$$\begin{aligned}&\int_\Omega\mathscr{L}(t,\lambda)\widetilde{u}\overline{\widetilde{v}}dx+\sum_{j=1}^m\int_{\partial\Omega}\mathscr{B}_j(t,\lambda)\widetilde{u}\overline{\mathscr{B}'_{j+m}(t,-\overline{\lambda}+2m-n)\widetilde{v}}ds\\ &=\int_\Omega\widetilde{u}\overline{\mathscr{L}^+(t,-\overline{\lambda}+2m-n)\widetilde{v}}dx+\sum_{j=1}^m\int_{\partial\Omega}\mathscr{B}_{j+m}(t,\lambda)\widetilde{u}\overline{\mathscr{B}'_j(t,-\overline{\lambda}+2m-n)\widetilde{v}}ds\end{aligned}$$

for $\widetilde{u},\widetilde{v}\in C^\infty(\overline{\Omega})$ and for all $t\in[0,T]$ (see [3, p. 206]). Here for the sake of brevity, we have omitted the arguments ω and ∂_ω in the operators of this formula.

We denote by $\mathcal{U}^+(\lambda,t)$ the operator of the boundary-value problem

$$\mathscr{L}^+(\omega,t,\partial_\omega,-\lambda+2m-n)v=f \quad \text{in } \Omega, \quad (3.11)$$

$$\mathscr{B}_j^+(\omega,t,\partial_\omega,-\lambda+2m-n)v=g_j \quad \text{on } \partial\Omega,\quad j=1,\dots,m. \quad (3.12)$$

Let $\lambda_0(t)$ be an analytic function on $[0,T]$ such that $\lambda_0(t)$ be a simple eigenvalue of $\mathscr{U}(\lambda,t)$ for each $t\in[0,T]$ and let $\varphi\in C^{\infty,\mathfrak{a}}(\Omega_T)$ such that $\varphi(t)$ be an eigenvector of $\mathscr{U}(\lambda,t)$ corresponding to the eigenvalue $\lambda_0(t)$ for each $t\in[0,T]$. Then $\overline{\lambda}_0(t)$ are simple eigenvalues of the pencil $\mathcal{U}^+(\lambda,t)$ for all $t\in[0,T]$ (see [3, 6.1.6]). Moreover, there exists a function $\psi\in C^{\infty,\mathfrak{a}}(\Omega_T)$ such that $\psi(t)$ is an eigenvector of $\mathscr{U}^+(\lambda,t)$ corresponding to the eigenvalues $\overline{\lambda}_0(t)$ for each $t\in[0,T]$ which is analogous to the case of the pencil $\mathscr{U}(\lambda,t)$. We claim that

$$\begin{aligned}&(\mathscr{L}^{(1)}(\lambda_0(t),t)\varphi(t),\psi(t))_\Omega\\ &+\sum_{j=1}^m\big(\mathscr{B}_j^{(1)}(\lambda_0(t),t)\varphi(t),\mathscr{B}'_{j+m}(-\overline{\lambda}_0(t)+2m-n,t)\psi(t)\big)_{\partial\Omega}\neq 0,\end{aligned} \quad (3.13)$$

for all $t \in [0,T]$, where

$$\mathscr{L}^{(1)}(\lambda,t) = \frac{d}{d\lambda}\mathscr{L}(\lambda,t), \mathscr{B}_j^{(1)}(\lambda,t) = \frac{d}{d\lambda}\mathscr{B}_j(\lambda,t), j = 1,\dots,m.$$

We prove this by contradiction. If (3.13) is not true for some $t_0 \in [0,T]$, then one can solve with respect to u the following elliptic boundary-value problem

$$\begin{aligned}\mathscr{U}(\lambda,t_0)u &= \mathscr{U}^{(1)}(\lambda,t_0)\varphi(t_0)\\ &\equiv (\mathscr{L}^{(1)}(\lambda,t_0)\varphi(t_0), \mathscr{B}_1^{(1)}(\lambda,t_0)\varphi(t_0),\dots,\mathscr{B}_m^{(1)}(\lambda,t_0)\varphi(t_0)).\end{aligned}$$

This implies that the eigenvalue $\lambda_0(t_0)$ is not simple which is not possible. It follows from (3.13) that we can choose $\psi(t) \in C^{\infty,\mathfrak{a}}(\Omega_T)$ such that

$$\begin{aligned}&(\mathscr{L}^{(1)}(\lambda_0(t),t)\varphi(t),\psi(t))_\Omega\\ &+\sum_{j=1}^m \big(\mathscr{B}_j^{(1)}(\lambda_0(t),t)\varphi(t), \mathscr{B}'_{j+m}(-\overline{\lambda}_0(t)+2m-n,t)\psi(t)\big)_{\partial\Omega} = 1\end{aligned} \tag{3.14}$$

for all $t \in [0,T]$. Moreover, applying [3, Th. 5.1.1], we assert that there exists a neighborhood U of the origin O such that in $U_T = U \times [0,T]$ the inverse $\mathscr{U}^{-1}(\lambda,t)$ has the following representation

$$\mathscr{U}^{-1}(\lambda,t) = \frac{P_{-1}(t)}{\lambda-\lambda_0(t)} + \mathscr{P}(\lambda,t), \tag{3.15}$$

where $P_{-1}(t)$ is a 1-dimensional operator from from $\mathcal{Y}$ into $\mathcal{X}$ depending analytically on $t \in [0,T]$ defined by

$$P_{-1}(t)v = \langle\langle v,\psi(t)\rangle\rangle\,\varphi(t), \quad v \in \mathcal{Y}, \tag{3.16}$$

and $\mathscr{P}(\lambda,t)$ is a pencil of continuous operators from $\mathcal{Y}$ into $\mathcal{X}$ depending analytically on both $\lambda \in \mathbb{C}$ and $t \in [0,T]$. Here

$$\langle\langle v,\psi(t)\rangle\rangle := (v_0,\psi(t))_\Omega + \sum_{j=1}^m \big(v_j, B'_{j+m}(-\lambda_0(t)+2m-n,t)\psi(t)\big)_{\partial\Omega}$$

for $v = (v_0,v_1,\dots,v_m) \in \mathcal{Y}$.

4. Asymptotic behaviour of the solutions

In this section we investigate the behaviour of the solutions of the problem (2.5), (2.6) in a neighborhood of the conical point. First, let us introduce some needed lemmas.

Lemma 4.1. *Let $u \in V^{l_1,h}_{2,\beta_1}(G_T)$ be a solution of the problem*

$$\mathfrak{L}(t,\partial_x)u = f \quad \text{in } G_T, \tag{4.1}$$

$$\mathfrak{B}_j(t,\partial_x)u = g_j \text{ on } S_T,\ j = 1,\dots,m, \tag{4.2}$$

where $f \in V^{l_2-2m,h}_{2,\beta_2}(G_T)$, $g_j \in V^{l_2-\mu_j-\frac12,h}_{2,\beta_2}(S_T)$, $l_1,l_2 \ge 2m$, $\beta_1 - l_1 > \beta_2 - l_2$. Suppose that the lines $\operatorname{Re}\lambda = -\beta_i + l_i - \frac{n}{2}$ $(i = 1,2)$ do not contain eigenvalues of the pencil $\mathscr{U}(\lambda,t)$, and all eigenvalues of this pencil in the strip $-\beta_1 + l_1 - \frac{n}{2} < \operatorname{Re}\lambda < -\beta_2 + l_2 - \frac{n}{2}$ are simple for all $t \in [0,T]$ which are chosen to be analytic functions $\lambda_1(t),\lambda_2(t),\dots,\lambda_N(t)$ defined on $[0,T]$ as the result of Lemma 3.1. Then *there exists a neighborhood V of the origin of $\mathbb{R}^n$ such that in V_T the solution u has representation*

$$u(x,t)=\sum_{k=1}^{N}c_k(t)r^{\lambda_k(t)}\varphi_k(\omega,t)+w(x,t), \tag{4.3}$$

where $w\in V^{l_2,h}_{2,\beta_2}(K_T)$, $c_k(t)\in W^h_2((0,T))$ and $\varphi_k\in C^{\infty,\mathfrak{a}}(\Omega_T)$ are eigenvectors of $\mathscr{U}(\lambda,t)$ corresponding to the eigenvalues $\lambda_k(t)$, $k=1,\dots,N$.

Proof. For each $k=1,\dots,N$, let $\psi_k(t)$ be eigenvectors the pencil $\mathscr{U}^+(\lambda,t)$ corresponding to the eigenvalues $\overline{\lambda}_k(t)$ $(k=1,\dots,N)$ having the properties as in (3.14). Set $v_k=r^{-\overline{\lambda}_k(t)+2m-n}\psi_k$ for $k=1,\dots,N$.

(i) First, we assume that the function u has the support contained in U_T, where U is a certain neighborhood of $0\in\mathbb{R}^n$ in which the domain G coincides with the cone K. By extension by zero to K_T (respectively, ∂K_T) we can regard u, f (respectively, g_j) as functions defined in K_T (respectively, ∂K_T).

For each $t\in[0,T]$ fixed, according to results for elliptic boundary problem in a cone (see, e.g, [3, Th. 6.1.4, Th. 6.1.7]), the solution $u(x,t)$ admits the representation (4.3) in K with

$$w(x,t)=\frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=-\beta_2+l_2-\frac{n}{2}}r^{\lambda}\mathscr{U}^{-1}(\lambda,t)\widetilde{\mathcal{F}}(\omega,\lambda,t)d\lambda \tag{4.4}$$

and

$$\begin{aligned}c_k(t)&=\big(f(.,t),v_k(.,t)\big)_K+\sum_{j=1}^{m}\big(g_j(.,t),B'_{j+m}v_k(.,t)\big)_{\partial K}\\&=\big(f(.,t),v_k(.,t)\big)_G+\sum_{j=1}^{m}\big(g_j(.,t),B'_{j+m}v_k(.,t)\big)_S\end{aligned}$$

for $k=1,\dots,N$, where $\widetilde{\mathcal{F}}=(\widetilde{r^{2m}f},\widetilde{r^{\mu_1}g_1},\dots,\widetilde{r^{\mu_m}g_m})$. Here $\widetilde{g}(\omega,\lambda,t)$ denotes the Mellin transformation with respect to the variable r of $g(\omega,r,t)$; i.e,

$$\widetilde{g}(\omega,\lambda,t)=\int_0^{+\infty}r^{-\lambda-1}g(\omega,r,t)dr.$$

We will prove below that $w\in V^{l_2,h}_{2,\beta_2}(K_T)$, $c_k(t)\in W^h_2((0,T))$.

Now we make clear the first one. Since there are no eigenvalues of the operator pencil $\mathscr{U}(\lambda,t)$ on the line $\operatorname{Re}\lambda=-\beta_2+l_2-\frac{n}{2}$, from the proof of [3, Th. 3.6.1] we have the estimate

$$\|\mathscr{U}^{-1}(\lambda,t)\widetilde{\Psi}\|^2_{W^l_2(\Omega,\lambda)}\le C\Big(\|\widetilde{\eta}\|^2_{W^{l-2m}_2(\Omega,\lambda)}+\sum_{j=1}^{m}\|\widetilde{\eta}_j\|^2_{W_2^{l-\mu_j-\frac12}(\partial\Omega,\lambda)}\Big) \tag{4.5}$$

for all λ on the line $\operatorname{Re}\lambda=-\beta_2+l_2-\frac{n}{2}$, $t\in[0,T]$, and all $\widetilde{\Psi}=(\widetilde{\eta},\widetilde{\eta}_1,\dots,\widetilde{\eta}_m)$ $\in W^{l-2m}_2(\Omega)\times\prod_{k=1}^{m}W_2^{l-\mu_k-\frac12}(\partial\Omega)$, where the constant C is independent of λ, t and $\widetilde{\Psi}$. Here

$$\|u\|_{W^l_2(\Omega,\lambda)}=\|u\|_{W^l_2(\Omega)}+|\lambda|^l\|u\|_{L_2(\Omega)},$$
$$\|u\|_{W^l_2(\partial\Omega,\lambda)}=\|u\|_{W_2^{l-\frac12}(\partial\Omega)}+|\lambda|^{l-\frac12}\|u\|_{L_2(\partial\Omega)},$$

which are equivalent to the norms in $W_2^l(\Omega)$, $W_2^{l-\frac{1}{2}}(\partial\Omega)$, respectively, for arbitrary fixed complex number λ.

We will prove by induction on h that

$$\|(\mathscr{U}^{-1})_{t^h}(\lambda,t)\widetilde{\Psi}\|^2_{W_2^l(\Omega,\lambda)} \le C(h)\Big(\|\widetilde{\eta}\|^2_{W_2^{l-2m}(\Omega,\lambda)} + \sum_{j=1}^m \|\widetilde{\eta}_j\|^2_{W_2^{l-\mu_j-\frac{1}{2}}(\partial\Omega,\lambda)}\Big). \quad (4.6)$$

It holds for $h=0$ by (4.5). Assume that it holds for $h-1$. From the equality

$$\mathscr{U}(\lambda,t)\mathscr{U}^{-1}(\lambda,t) = I,$$

differentiating both sides of it h ($h \ge 1$) times with respect to t we obtain

$$\sum_{k=0}^{h-1} \binom{h-1}{k} \mathscr{U}_{t^{h-k}}(\lambda,t)(\mathscr{U}^{-1})_{t^k}(\lambda,t) + \mathscr{U}(\lambda,t)(\mathscr{U}^{-1})_{t^h}(\lambda,t) = 0.$$

Rewrite this equality in the form

$$(\mathscr{U}^{-1})_{t^h}(\lambda,t) = -\mathscr{U}^{-1}(\lambda,t)\sum_{k=0}^{h-1} \binom{h-1}{k} \mathscr{U}_{t^{h-k}}(\lambda,t)(\mathscr{U}^{-1})_{t^k}(\lambda,t).$$

Then (4.6) follows from this equality and the inductive assumption. It is well-known (see [3, Le. 6.1.4]) that the norm (2.1) is equivalent to

$$|||u|||_{V^l_{2,\beta}(K)} = \Big(\frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=-\beta+l-\frac{n}{2}} \|\widetilde{u}(.,\lambda)\|^2_{W_2^l(\Omega,\lambda)}d\lambda\Big)^{1/2},$$

and the norm (2.2) is equivalent to

$$|||u|||_{V_{2,\beta}^{l-\frac{1}{2}}(\partial K)} = \Big(\frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=-\beta+l-\frac{n}{2}} \|\widetilde{u}(.,\lambda)\|^2_{W_2^{l-\frac{1}{2}}(\partial\Omega,\lambda)}d\lambda\Big)^{1/2}.$$

Using these with noting

$$\widetilde{w}(\omega,\lambda,t) = \mathscr{U}^{-1}(\lambda,t)\widetilde{\mathcal{F}}(.,\lambda,t)$$

(see [3, Le. 6.1.3]) and (4.5), we get from (4.4) that

$$\begin{aligned}
&\|w(.,t)\|^2_{V^{l_2}_{2,\beta_2}(K)}\\
&\le \frac{C}{2\pi i}\int_{\operatorname{Re}\lambda=-\beta_2+l_2-\frac{n}{2}} \|\widetilde{w}(.,\lambda,t)\|^2_{W_2^{l_2}(\Omega,\lambda)}d\lambda\\
&= \frac{C}{2\pi i}\int_{\operatorname{Re}\lambda=-\beta_2+l_2-\frac{n}{2}} \|\mathscr{U}^{-1}(\lambda,t)\widetilde{\mathcal{F}}(.,\lambda,t)\|_{W_2^{l_2}(\Omega,\lambda)}d\lambda\\
&\le \frac{C}{2\pi i}\int_{\operatorname{Re}\lambda=-\beta_2+l_2-\frac{n}{2}} \Big(\|\widetilde{r^{2m}f(.,t)}\|^2_{W_2^{l-2m}(\Omega,\lambda)}\\
&\quad + \sum_{j=1}^m \|\widetilde{r^{\mu_j}g_j(.,t)}\|^2_{W_2^{l-\mu_j-\frac{1}{2}}(\partial\Omega,\lambda)}\Big)d\lambda \qquad (4.7)\\
&\le C\Big(\|r^{2m}f(.,t)\|^2_{V^{l_2-2m}_{2,\beta_2-2m}(K)} + \sum_{j=1}^m \|r^{\mu_j}g_j(.,t)\|^2_{V^{l_2-\mu_j-\frac{1}{2}}_{2,\beta_2-\mu_j}(\partial K)}\Big)\\
&\le C\Big(\|f(.,t)\|^2_{V^{l_2-2m}_{2,\beta_2}(K)} + \sum_{j=1}^m \|g_j(.,t)\|^2_{V^{l_2-\mu_j-\frac{1}{2}}_{2,\beta_2}(\partial K)}\Big)
\end{aligned}$$

$$= C\Big(\|f(.,t)\|^2_{V^{l_2-2m}_{2,\beta_2}(G)} + \sum_{j=1}^m \|g_j(.,t)\|^2_{V^{l_2-\mu_j-\frac12}_{2,\beta_2}(S)}\Big)$$

for all $t \in [0,T]$. Here, and sometimes later, for convenience, we denote different constants by the same symbol C. Integrating the last inequality with respect to t from 0 to $+\infty$, we obtain $w \in V^{l_2}_{2,\beta_2}(K_T)$ and

$$\|w\|^2_{V^{l_2,0}_{2,\beta_2}(K_T)} \le C\Big(\|f\|^2_{V^{l_2-2m,0}_{2,\beta_2}(G_T)} + \sum_{j=1}^m \|g_j\|^2_{V^{l_2-\mu_j-\frac12,0}_{2,\beta_2}(S_T)}\Big).$$

Differentiating (4.4) h times with respect to t we have

$$w_{t^h}(x,t) = \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=-\beta_2+l_2-\frac n2} r^\lambda \sum_{k=0}^h \binom{h}{k} (\mathscr{U}^{-1})_{t^k}(\lambda,t)\widetilde{\mathcal{F}}_{t^{h-k}}(\omega,\lambda,t)d\lambda.$$

Now using (4.6) and arguments the same as in (4.7) we arrive at

$$\|w_{t^h}\|^2_{V^{l_2,0}_{2,\beta_2}(K_T)} \le C\sum_{k=0}^h \Big(\|f_{t^k}\|^2_{V^{l_2-2m,0}_{2,\beta_2}(G_T)} + \sum_{k=1}^m \|(g_j)_{t^k}\|^2_{V^{l_2-\mu_j-\frac12,0}_{2,\beta_2}(S_T)}\Big). \tag{4.8}$$

Therefore, $w \in V^{l_2,h}_{2,\beta_2}(K_T)$ and

$$\|w\|^2_{V^{l_2,h}_{2,\beta_2}(K_T)} \le C\Big(\|f\|^2_{V^{l_2-2m,h}_{2,\beta_2}(G_T)} + \sum_{j=1}^m \|g_j\|^2_{V^{l_2-\mu_j-\frac12,h}_{2,\beta_2}(S_T)}\Big). \tag{4.9}$$

Now we verify that $c_k(t) \in W^h_2((0,T))$ for $k=1,\dots,N$. For some such k put

$$v(x,t) = r^{-\overline{\lambda}_k(t)+2m-n}\psi_k(\omega,t). \tag{4.10}$$

Using formula (3.1), we have

$$\begin{aligned}\partial^\alpha v &= r^{-|\alpha|}\sum_{p=0}^{|\alpha|}(r\partial r)^p r^{-\overline{\lambda}_k(t)+2m-n}P_{\alpha,p}\psi_k\\ &= r^{-|\alpha|-\overline{\lambda}_k(t)+2m-n}\sum_{p=0}^{|\alpha|}(-\overline{\lambda}_k(t)+2m-n)^p P_{\alpha,p}\psi_k.\end{aligned} \tag{4.11}$$

Since $\operatorname{Re}\lambda_k(t) < -\beta_2 + l_2 - \frac n2$ for all $t\in[0,T]$ and $\lambda_k(t)$ is analytic on $[0,T]$, then there is a real number $\epsilon > 0$ such that $\operatorname{Re}\lambda_k(t) \le -\beta_2 + l_2 - \frac n2 - 2\epsilon$ for all $t \in [0,T]$. Thus, it follows from (4.11) that

$$|r^{-\gamma_2+l_2-2m+|\alpha|}\partial^\alpha v(x,t)| \le Cr^{-\frac n2+\epsilon}\sum_{p=0}^{|\alpha|}|P_{\alpha,p}\psi_k(\omega,t)|$$

for all $(x,t) \in G_T$ and all multi-index α. This implies $v(.,t) \in V^l_{2,-\beta_2+l_2-2m+l}(G)$ and

$$\|v(.,t)\|_{V^l_{2,-\gamma_2+l_2-2m+l}(G)} \le C\|\psi_k(.,t)\|_{W^l_2(\Omega)}$$

for an arbitrary integer l. Using Faà Di Bruno's Formula for the higher order derivatives of composite functions (see, e.g, [7]), we have

$$v_{t^p} = \sum_{q=0}^p \binom{p}{q}\big(r^{-\overline{\lambda}_k(t)+2m-n}\big)_{t^{p-q}}(\psi_k)_{t^q}$$

$$
\begin{aligned}
&= r^{-\overline{\lambda}_k(t)+2m-n}\sum_{q=0}^{p}\binom{p}{q}\sum\frac{n!}{m_1!\dots m_n!}(\ln r)^{m_1+\dots+m_n}\\
&\quad\times\prod_{s=1}^{n}\Big(\frac{-\lambda_k^{(s)}(t)}{s!}\Big)^{m_s}(\psi_k)_{t^q},
\end{aligned}
$$

where the second sum is over all $n-$tuples $(m_1,\dots,m_n)$ satisfying the condition

$$m_1+2m_2+\dots+nm_n=n.$$

According to Lemma 3.1, $\lambda_k(t)$ is analytic on $[0,T]$. Therefore, it together with its derivatives are bounded on $[0,T]$. Repeating the arguments as above, we get

$$\|v_{t^p}(.,t)\|_{V^l_{2,-\gamma_2+l_2-2m+l}(G)}\le C\sum_{q=0}^{p}\|(\psi_k)_{t^q}\|_{W_2^l(\Omega)}.$$

Thus, we have

$$\sup_{t\in[0,T]}\|v\|_{V^{l,p}_{2,-\gamma_2+l_2-2m+l}(G)}\le C\sum_{q=0}^{p}\sup_{t\in[0,T]}\|(\psi_k)_{t^q}\|_{W_2^l(\Omega)}<+\infty \tag{4.12}$$

for arbitrary nonnegative integers l,p.

Set $c(t)=(f(.,t),v(.,t))_G$. For $p\le h$, using (4.12), we have

$$
\begin{aligned}
|c_{t^p}(t)|^2&=\Big|\sum_{q=0}^{p}\tbinom{p}{q}(f_{t^{p-q}}(.,t),v_{t^q}(.,t))_G\Big|^2\\
&\le C\Big(\sum_{q=0}^{p}\|r^{\beta_2-l_2+2m}f_{t^q}\|^2_{L_2(G)}\Big)\Big(\sum_{q=0}^{p}\|r^{-\beta_2+l_2-2m}v_{t^q}\|^2_{L_2(G)}\Big)\\
&\le C\sum_{q=0}^{p}\|f_{t^q}\|^2_{V^{l_2-2m}_{2,\beta_2}(G)}.
\end{aligned}
$$

This implies $c(t)\in W_2^h((0,T))$ and

$$\|c\|_{W_2^h((0,T))}\le C\|f\|_{V^{l_2-2m,h}_{2,\beta_2}(G_T)}. \tag{4.13}$$

Now set $c_j(t)=(g_j,B'_{j+m}v)_S$, $j=1,\dots,m$. Then also using (4.12), we have

$$
\begin{aligned}
&|(c_j)_{t^p}(t)|^2\\
&=\Big|\sum_{q=0}^{p}\tbinom{p}{q}((g_j)_{t^{p-q}}(.,t),v_{t^q}(.,t))_S\Big|^2\\
&\le C\Big(\sum_{q=0}^{p}\|r^{\beta_2-l_2+\mu_j+\frac12}(g_j)_{t^q}\|^2_{L_2(G)}\Big)\Big(\sum_{q=0}^{p}\|r^{-\beta_2+l_2-\mu_j-\frac12}(B'_{j+m}v)_{t^q}\|^2_{L_2(G)}\Big)\\
&\le C\Big(\sum_{q=0}^{p}\|(g_j)_{t^q}\|^2_{V^{l_2-\mu_j-\frac12}_{2,\beta_2}(S)}\Big).\Big(\sum_{q=0}^{p}\|v_{t^q}\|^2_{V^{2m-\mu_j}_{2,-\beta_2+l_2-\mu_j}(G)}\Big)\\
&\le C\sum_{q=0}^{p}\|(g_j)_{t^q}\|^2_{V^{l_2-\mu_j-\frac12}_{2,\beta_2}(S)}(p\le h).
\end{aligned}
$$

This implies $c_j \in W_2^h((0,T))$ and

$$\|c_j\|_{W_2^h((0,T))} \le C\|g_j\|_{V_{2,\beta_2}^{l_2-\mu_j-\frac{1}{2},h}(S_T)}. \tag{4.14}$$

From (4.13) and (4.14), we can conclude that $c_k(t) \in W_2^h((0,T))$ and

$$\|c_k\|_{W_2^h((0,T))} \le C\Big(\|f\|_{V_{2,\beta_2}^{l_2-2m,h}(G_T)} + \sum_{j=1}^m \|g_j\|_{V_{2,\beta_2}^{l_2-\mu_j-\frac{1}{2},h}(S_T)}\Big). \tag{4.15}$$

(ii) Now we consider the case $u \in V_{2,\beta_1}^{l_1,h}(G_T)$ is arbitrary. Let η be an infinitely differential function with support in U, equal to one in a neighborhood V of the origin. Denote by $\mathfrak{G}$ the set of all subdomain G' of G with the smooth boundary such that $G \cap U \setminus V \subset G'$. We will show that $u \in W_2^{l_2,h}(G'_T)$ for all $G' \in \mathfrak{G}$. To this end, we will prove by induction on h that

$$u_{t^k} \in W_2^{l_2,0}(G'_T) \quad \text{for } k = 0, \dots, h \text{ and } G' \in \mathfrak{G}. \tag{4.16}$$

According to the results on the regularity of solutions of elliptic boundary problems in smooth domains, we can conclude from (4.1), (4.2) that $u(.,t) \in W_2^{l_2}(G')$ for each $t \in [0,T]$ and

$$\begin{aligned}\|u(.,t)\|_{W_2^{l_2}(G')} \le C\Big(&\|u(.,t)\|_{W_2^{l_1}(G'')} + \|f(.,t)\|_{W_2^{l_2-2m}(G'')}\\ &+ \sum_{j=1}^m \|g_j(.,t)\|_{W_2^{l_2-\mu_j-\frac{1}{2}}(S\cap\partial G'')}\Big),\end{aligned}$$

where $G'' \in \mathfrak{G}$ such that $\overline{G'} \subset S \cup G''$ and C is a constant independent of u, f, g_j and t. Integrating this inequality with respect to t from 0 to T we get $u \in W_2^{l_2,0}(G'_T)$. Thus (4.16) holds for $h = 0$. Assume that it holds for $h-1$. Differentiating equalities (4.1), (4.2) with respect to t h times and using the inductive assumption, we have

$$Lu_{t^h} = f_{t^h} - \sum_{k=0}^{h-1}\binom{h}{k} L_{t^{h-k}} u_{t^k} \in W_2^{l_2-2m,0}(G''_T),$$

$$B_j u_{t^h} = (g_j)_{t^h} - \sum_{k=0}^{h-1}\binom{h}{k}(B_j)_{t^{h-k}} u_{t^k} \in W_{2,\beta_2}^{l_2-\mu_j-\frac{1}{2},0}(S_T \cap \partial G''_T),$$

where $G', G'' \in \mathfrak{G}$, $\overline{G'} \subset S \cup G''$. Applying the arguments above for u_{t^h}, we get $u_{t^h} \in W_2^{l_2,0}(G'_T)$.

From (4.1) we have

$$\mathfrak{L}(\eta u) = \eta f + [\mathfrak{L}, \eta]u \text{ in } G_T, \tag{4.17}$$

where $[\mathfrak{L}, \eta] = \mathfrak{L}\eta - \eta\mathfrak{L}$ is the commutator of $\mathfrak{L}$ and η. Noting that $u \in W_2^{l_2,h}(G'_T)$ for all $G' \in \mathfrak{G}$ and $[\mathfrak{L}, \eta]$ is a differential expression (acting on u) of order $\le 2m-1$ with coefficients having the supports contained in $U \setminus V$, we have $[\mathfrak{L}, \eta]u$ is in $W_{2,\beta_2}^{l_2-2m,h}(G_T)$. So is the right-hand side of (4.17). Similarly, we have

$$\mathfrak{B}_j(\eta u) = \eta g_j + [\mathfrak{B}_j, \eta]u \in W_{2,\beta_2}^{l_2-\mu_j-\frac{1}{2},h}(S_T)(j = 1, \dots, m). \tag{4.18}$$

Applying the the part (i) above for the function ηu, we conclude from (4.17) and (4.18) that u admits the decomposition (4.3) in V_T. The theorem is proved. $\square$

Lemma 4.2. *Let*

$$f = r^{\lambda_0(t)-2m} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\ln r)^\sigma f_{s-\sigma}, \tag{4.19}$$

$$g_j = r^{\lambda_0(t)-\mu_j} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\ln r)^\sigma g_{j,s-\sigma}, \quad j = 1, \dots, m, \tag{4.20}$$

where $f_\sigma \in W_2^{l-2m,h}(\Omega_T)$, $g_{j,\sigma} \in W_2^{l-\mu_j-\frac{1}{2},h}(\partial\Omega_T)$, $\sigma = 0, \dots, s$, $j = 1, \dots, m$ and $\lambda_0(t)$ be a complex-valued function defined on $[0,T]$. Suppose that if $\lambda_0(t)$ is an eigenvalue of $\mathscr{U}(\lambda, t)$ for some t, then $\lambda_0(t)$ are simple eigenvalues of $\mathscr{U}(\lambda, t)$ for all $t \in [0,T]$. Then there exists a solution u of (4.1), (4.2) which has the form

$$u = r^{\lambda_0(t)} \sum_{\sigma=0}^{s+\kappa} \frac{1}{\sigma!} (\ln r)^\sigma u_{s+\kappa-\sigma} \tag{4.21}$$

where $u_\sigma \in W_2^{l,h}(\Omega_T), \sigma = 0, \dots, s+\kappa$. Here $\kappa = 1$ or $\kappa = 0$ according as $\lambda_0(t)$ are simple eigenvalues of $\mathscr{U}(\lambda, t)$ or not.

Proof. According to (3.15), the inverse of $\mathscr{U}(\lambda, t)$ admits the representation

$$\mathscr{U}^{-1}(\lambda, t) = \sum_{k=-\kappa}^{+\infty} P_k(t) \big(\lambda - \lambda_0(t)\big)^k,$$

where $P_-1(t)$ is defined in (3.16) for the case $\kappa = 1$, and

$$P_k(t) = \frac{1}{n!} \frac{\partial^k \mathscr{P}}{\partial \lambda^k} (\lambda_0(t), t) \tag{4.22}$$

for $k = 0, 1, \dots$. It is obvious that $P_k(t)$, $k = -\kappa, -\kappa+1, \dots$, are continuous operators from $\mathcal{Y}$ into $\mathcal{X}$ depending analytically on t on $[0,T]$. From the equality

$$\mathscr{U}(\lambda, t)\mathscr{U}^{-1}(\lambda, t) = \sum_{k=-\kappa}^{+\infty} \Big(\sum_{q=0}^{\kappa+k} \frac{1}{q!} \mathscr{U}^{(q)}(\lambda_0(t), t) P_{k-q}(t) \Big) (\lambda - \lambda_0(t))^k = I$$

it follows that

$$\sum_{q=0}^{\kappa+k} \frac{1}{q!} \mathscr{U}^{(q)}(\lambda_0(t), t) P_{k-q}(t) = \delta_{k,0}, \quad k = -\kappa, -\kappa+1, \dots, \tag{4.23}$$

where $\delta_{k,l}$ is Kronecker symbol. Let u be the function given in (4.21). Then

$$\begin{aligned}
\mathscr{U}(r\partial_r, t)u &= r^{\lambda_0(t)} \mathscr{U}(\lambda_0(t) + r\partial_r, t) \sum_{\sigma=0}^{s+\kappa} \frac{1}{\sigma!} (\ln r)^\sigma u_{s+\kappa-\sigma} \\
&= r^{\lambda_0(t)} \sum_{q=0}^{2m} \frac{1}{q!} \mathscr{U}^{(q)}(\lambda_0(t), t)(r\partial_r)^q \sum_{\sigma=0}^{s+\kappa} \frac{1}{\sigma!} (\ln r)^\sigma u_{s+\kappa-\sigma} \\
&= r^{\lambda_0(t)} \sum_{\sigma=0}^{s+\kappa} \frac{1}{\sigma!} (\ln r)^\sigma \sum_{q=0}^{s+\kappa-\sigma} \frac{1}{q!} \mathscr{U}^{(q)}(\lambda_0(t), t) u_{s+\kappa-\sigma-q}
\end{aligned}$$

Setting $v_\sigma = (f_\sigma, g_{1,\sigma}, \dots, g_{m,\sigma})$, $\sigma = 0, \dots, s$, and

$$u_k = \sum_{p=0}^{\min(k,s)} P_{-\kappa+k-p}(t) v_p, k = 1, \dots, s+\kappa,$$

we get $u_k \in W_2^{l,h}(\Omega_T), k = 0, \dots, s+\kappa$. Using the equality (4.23) we have

$$\begin{aligned}
&\mathscr{U}(r\partial_r, t)u \\
&= r^{\lambda_0(t)} \sum_{\sigma=0}^{s+\kappa} \frac{1}{\sigma!}(\ln r)^\sigma \sum_{q=0}^{s+\kappa-\sigma} \frac{1}{q!}\mathscr{U}^{(q)}(\lambda_0(t), t) \sum_{p=0}^{\min(s+\kappa-\sigma-q,s)} P_{s-\sigma-q-p}(t)v_p \\
&= r^{\lambda_0(t)} \sum_{\sigma=0}^{s+\kappa} \frac{1}{\sigma!}(\ln r)^\sigma \sum_{p=0}^{\min(s+\kappa-\sigma,s)} \sum_{q=0}^{s+\kappa-\sigma-p} \frac{1}{q!}\mathscr{U}^{(q)}(\lambda_0(t), t)P_{s-\sigma-p-q}(t)v_p \\
&= r^{\lambda_0(t)} \sum_{p=0}^{s} \sum_{\sigma=0}^{s+\kappa-p} \frac{1}{\sigma!}(\ln r)^\sigma \Big(\sum_{q=0}^{s+\kappa-\sigma-p} \frac{1}{q!}\mathscr{U}^{(q)}(\lambda_0(t), t)P_{s-\sigma-p-q}(t)\Big)v_p \\
&= r^{\lambda_0(t)} \sum_{p=0}^{s} \sum_{\sigma=0}^{s+\kappa-p} \frac{1}{\sigma!}(\ln r)^\sigma \delta_{s+\kappa-\sigma-p,0} v_p \\
&= r^{\lambda_0(t)} \sum_{p=0}^{s} \frac{1}{(s-p)!}(\ln r)^{s-p} v_p = r^{\lambda_0(t)} \sum_{p=0}^{s} \frac{1}{\sigma!}(\ln r)^\sigma v_{s-\sigma}.
\end{aligned}$$

Rewrite this equality in the form

$$\begin{gathered}
\mathscr{L}(\omega, t, \partial_\omega, r\partial_r)u = r^{2m} f \quad \text{in } G_T, \\
\mathscr{B}_j(\omega, t, \partial_\omega, r\partial_r)u = r^{\mu_j} g_j \quad \text{on } S_T,\ j = 1, \dots, m.
\end{gathered}$$

This implies u is a solution of (4.1), (4.2), and the lemma is proved. □

Lemma 4.3. *Let $u \in V_{2,\beta}^{l,h}(G_T)$ be a solution of Problem* (2.5), (2.6), *where $f \in V_{2,\beta+s}^{l-2m+s,h}(G_T)$, $g_j \in V_{2,\beta+s}^{l_1-\mu_j+s-\frac{1}{2},h}(S_T)$, l, s, h are nonnegative integers, $l \geq 2m$. Then $u \in V_{2,\beta+s}^{l+s,h}(G_T)$ and*

$$\|u\|^2_{V_{2,\beta+s}^{l+s,h}(G_T)} \leq C(\|f\|^2_{V_{2,\beta+s}^{l-2m+s,h}(G_T)} + \sum_{j=1}^m \|g_j\|^2_{V_{2,\beta+s}^{l_1-\mu_j+s-\frac{1}{2},h}(S_T)}) \tag{4.24}$$

with the constant C independent of u, f and g_j.

Proof. It is only needed to show that $u_{t^k} \in V_{2,\beta+s}^{l+s,0}(G_T)$ and

$$\|u_{t^k}\|^2_{V_{2,\beta+s}^{l+s,0}(G_T)} \leq C(\|f\|^2_{V_{2,\beta+s}^{l-2m+s,k}(G_T)} + \sum_{j=1}^m \|g_j\|^2_{V_{2,\beta+s}^{l_1-\mu_j+s-\frac{1}{2},k}(S_T)}) \tag{4.25}$$

for $k = 0, \dots, h$, where C is a constant independent of u, f and g_j. We will prove this by induction on h.

First, we fix some $t \in [0, T]$ and consider (2.5), (2.6) as an elliptic boundary-value problem (without parameter). Applying Corollary 6.3.2 of [3], we conclude that $u(., t) \in V_{2,\beta+s}^{l+s}(G)$ and

$$\|u(., t)\|^2_{V_{2,\beta+s}^{l+s}(G)} \leq C(\|f(., t)\|^2_{V_{2,\beta+s}^{l-2m+s}(G)} + \sum_{j=1}^m \|g_j(., t)\|^2_{V_{2,\beta+s}^{l_1-\mu_j+s-\frac{1}{2}}(S)}), \tag{4.26}$$

where C is a constant independent of u, f, g_j and $t \in [0,T]$. Integrating both sides of this inequality with respect to t from 0 to T we get

$$\|u\|^2_{V^{l+s,0}_{2,\beta+s}(G_T)} \le C(\|f\|^2_{V^{l-2m+s,0}_{2,\beta+s}(G_T)} + \sum_{j=1}^{m} \|g_j\|^2_{V^{l_1-\mu_j+s-\frac{1}{2},0}_{2,\beta+s}(S_T)}). \tag{4.27}$$

Thus (4.25) holds for $h = 0$.

Now suppose that it is true for $h-1$ ($h \ge 1$). Differentiating both sides of (2.5) with respect to t h times and using the inductive assumption, we get

$$Lu_{t^h} = f_{t^h} - \sum_{k=0}^{h-1} \binom{h}{k} L_{t^{h-k}} u_{t^k} \in V^{l-2m+s,0}_{2,\beta+s}(G_T). \tag{4.28}$$

Similarly, we have

$$B_j u_{t^k} = (g_j)_{t^k} - \sum_{p=0}^{k-1} \binom{k}{p} (B_j)_{t^{k-p}} u_{t^p} \in V^{l_1-\mu_j+s-\frac{1}{2},0}_{2,\beta+s}(S_T) \tag{4.29}$$

for $j = 1,\dots,m$. It is the same as in (4.27), we get from (4.28) and (4.29) that $u_{t^k} \in V^{l+s,0}_{2,\beta+s}(G_T)$ and the estimate (4.25) holds. The proof is complete. $\square$

Now let us give the main result of the present paper.

Theorem 4.4. *Let $u \in V^{l_1,h}_{2,\beta_1}(G_T)$ be a solution of Problem* (2.5), (2.6), *where $f \in V^{l_2-2m,h}_{2,\beta_2}(G_T)$, $g_j \in V^{l_1-k_j-\frac{1}{2},h}_{2,\beta_2}(S_T)$, l_1, l_2, h are nonnegative integers, $l_1, l_2 \ge 2m$, $l_1 - \beta_1 < l_2 - \beta_2$. Suppose that there are real numbers $\delta_0, \delta_2, \dots, \delta_M$ such that*

$$\delta_0 = \beta_1 + l_2 - l_1, \delta_M = \beta_2, \quad 0 < \delta_{d-1} - \delta_d \le 1, \quad d = 1,\dots,M,$$

and the lines $\operatorname{Re}\lambda = -\delta_d + l_2 - \frac{n}{2}$, $d = 0,\dots,M$, do not contain eigenvalues of the pencil $\mathscr{U}(\lambda,t)$. Furthermore, suppose that all eigenvalues of this pencil in the strip $-\delta_0 + l_2 - \frac{n}{2} < \operatorname{Re}\lambda < -\delta_M + l_2 - \frac{n}{2}$ are simple for all $t \in [0,T]$ which are chosen to be analytic functions $\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t)$ defined on $[0,T]$ as the result of Lemma 3.1. If $\lambda_j(t_0) = \lambda_k(t_0) + s$ for some $j,k \in \{1,\dots,N\}$, for some integer s and $t_0 \in [0,T]$, then let $\lambda_j(t) = \lambda_k(t) + s$ for all $t \in [0,T]$. Then the solution u admits the decomposition

$$u = \sum_{k=1}^{N} \sum_{\tau=0}^{\ell_k} r^{\lambda_k(t)+\tau} P_{k,\tau}(\ln r) + w, \tag{4.30}$$

where $w \in V^{l_2,h}_{2,\beta_2}(G_T)$, $P_{k,\tau}$ are polynomials with coefficients belonging to $W^{l_2,h}_2(\overline{\Omega}_T)$, ℓ_k is the minimal integer greater than $-\delta_M - \lambda_k(t) - 1 + l_2 - \frac{n}{2}$ for all $t \in [0,T]$.

Proof. According to Lemma 4.3, $u \in V^{l_2,h}_{2,\delta_0}(G_T)$. Suppose, by renumbering if necessary, $\operatorname{Re}\lambda_1(t) < \operatorname{Re}\lambda_2(t) < \dots < \operatorname{Re}\lambda_N(t)$ for all $t \in [0,T]$. For each $d \in \{1,\dots,M\}$ denote by N_d the maximal integer in $\{0,1,\dots,N\}$ such that $\lambda_1(t), \lambda_2(t), \dots, \lambda_{N_d}(t)$ belong to the strip $-\delta_0 + l_2 - \frac{n}{2} < \operatorname{Re}\lambda < -\delta_d + l_2 - \frac{n}{2}$ and by $\ell_{k,d}$ the minimal integer greater than $-\delta_d - \lambda_k(t) - 1 + l_2 - \frac{n}{2}$ for all $t \in [0,T]$.

We will prove by induction on d ($1 \le d \le M$) that the function u can be represented in the form

$$u = \sum_{k=1}^{N_d} \sum_{\tau=0}^{\ell_{k,d}} r^{\lambda_k(t)+\tau} P_{k,\tau}^{(d)}(\ln r) + u_d, \tag{4.31}$$

where $P_{k,\tau}^{(d)}$ are polynomials with coefficients belonging to $W_2^{l_2,h}(\Omega_T)$ and $u_d \in V_{2,\delta_d}^{l,h}(G_T)$. Then (4.31) for $d = M$ proves the theorem.

We rewrite (2.5), (2.6) in the form

$$\mathfrak{L}u = f + (\mathfrak{L} - L)u \equiv f + L'u \quad \text{in } G_T, \tag{4.32}$$

$$\mathfrak{B}_j u = g_j + (\mathfrak{B}_j - B_j)u \equiv g_j + B_j' u \quad \text{on } S_T, j = 1, \dots, m. \tag{4.33}$$

We write

$$L'u = \sum_{|\alpha|=2m} (a_\alpha(0,t) - a_\alpha(x,t))\partial^\alpha u + \sum_{|\alpha|\le 2m-1} a_\alpha(x,t)\partial^\alpha u \equiv L_1 u + L_2 u.$$

Since $|a_\alpha(x,t) - a_\alpha(0,t)| \le Cr$, and $u \in V_{2,\delta_0}^{l_2,h}(G_T)$, we have $L_1 u \in V_{2,\delta_0-1}^{l_2-2m,h}(G_T)$. Otherwise, $L_2 u \in V_{2,\delta_0}^{l_2-2m+1,h}(G_T) \subset V_{2,\delta_0-1}^{l_2-2m,h}(G_T)$. Since $\delta_0 + 1 \ge \delta_1$ and $V_{2,\delta_0-1}^{l_2-2m,h}(G_T) \subset V_{2,\delta_1}^{l_2-2m,h}(G_T)$. Therefore, $f + L'u \in V_{2,\delta_1}^{l_2-2m,h}(G_T)$. Similarly, $g_j + B_j' u \in V_{2,\delta_1}^{l_2-k_j-\frac{1}{2},h}(S_T)$. Now we can apply Lemma 4.1 to conclude that

$$u = \sum_{k=1}^{N_2} r^{\lambda_k(t)} c_k(t)\varphi_k + u_1, \tag{4.34}$$

where $c_k(t) \in W_2^h((0,T))$, $u_1 \in V_{2,\delta_1}^{l_2,h}(G_T)$. Thus (4.31) holds for $d = 1$ with $P_{k,\tau}^{(1)}(\ln r) = c_k(t)\varphi_k$.

We assume now (4.31) is true for some d ($1 \le d \le M-1$). Then we can rewrite (2.5), (2.6) in the form

$$\mathfrak{L}u_d = f + (\mathfrak{L} - L)u_d - Lz \quad \text{in } G_T, \tag{4.35}$$

$$\mathfrak{B}_j u_d = g_j + (\mathfrak{B}_j - B_j)u_d - B_j z \quad \text{on } S_T, j = 1, \dots, m, \tag{4.36}$$

where

$$z = \sum_{k=1}^{N_d} \sum_{\tau=0}^{\ell_{k,d}} r^{\lambda_k(t)+\tau} P_{k,\tau}^{(d)}(\ln r).$$

Since the coefficients $a_\alpha, |\alpha| \le 2m$, belong to the class $C^{\infty,\mathfrak{a}}(\overline{G}_T)$, then, for an arbitrary nonnegative integer k, they admit representation

$$a_\alpha = a_\alpha(\omega, r, t) = \sum_{\delta=0}^{k} r^\delta a_\alpha^{(\delta)}(\omega, t) + r^{k+1} a_\alpha^{(k+1)}(\omega, r, t) \tag{4.37}$$

where $a_\alpha^{(\delta)} \in C^{\infty,\mathfrak{a}}(\overline{\Omega}_T)$ for $\delta = 0, \dots, k$, and $a_\alpha^{(k+1)} \in C^{\infty,\mathfrak{a}}(\overline{G}_T)$. Thus, we can write the operator L in the form

$$L = \sum_{\delta=0}^{\ell_{k,d+1}} r^{-2m+\delta} \mathscr{L}^{(\delta)}(\omega, t, \partial_\omega, r\partial_r) + r^{-2m+\ell_{k,d+1}+1} \mathscr{L}^{(\ell_{k,d+1}+1)}(\omega, r, t, \partial_\omega, r\partial_r) \tag{4.38}$$

where $\mathscr{L}^{(\delta)}(\omega,t,\partial_\omega,r\partial_r)$, $\delta=0,\dots,\ell_{k,d+1}$, and $\mathscr{L}^{(\ell_{k,d+1}+1)}(\omega,r,t,\partial_\omega,r\partial_r)$ are polynomials of ∂_ω and $r\partial r$ of order not greater than $2m$ with coefficients in $C^{\infty,\mathfrak{a}}(\overline{\Omega}_T)$ and $C^{\infty,\mathfrak{a}}(\overline{G}_T)$, respectively.

$$\begin{aligned} &L\Big(\sum_{\tau=0}^{\ell_{k,d}} r^{\lambda_k(t)+\tau}P_{k,\tau}^{(d)}(\ln r)\Big)\\ &=\sum_{\delta=0}^{\ell_{k,d+1}}\sum_{\tau=0}^{\ell_{k,d}} r^{-2m+\lambda_k(t)+\tau+\delta}\mathscr{L}^{(\delta)}(\omega,t,\partial_\omega,\lambda_k(t)+\tau+r\partial_r)P_{k,\tau}^{(d)}(\ln r)\\ &\quad+\sum_{\tau=0}^{\ell_{k,d}} r^{-2m+\lambda_k(t)+\ell_{k,d+1}+1+\tau}\mathscr{L}^{(\ell_{k,d+1}+1)}(\omega,r,t,\partial_\omega,\lambda_k(t)+\tau+r\partial_r)P_{k,\tau}^{(d)}(\ln r). \end{aligned} \tag{4.39}$$

Write the first term of the right-hand side of (4.39) in the form

$$\sum_{\tau=0}^{\ell_{k,d+1}} r^{-2m+\lambda_k(t)+\tau}\Psi_{k,\tau}^{(d+1)}(\ln r)+\sum_{\tau=\ell_{k,d+1}+1}^{\ell_{k,d}+\ell_{k,d+1}} r^{-2m+\lambda_k(t)+\tau}\Psi_{k,\tau}^{(d+1)}(\ln r), \tag{4.40}$$

where $\Psi_{k,\tau}^{(d+1)}$ are polynomials with coefficients belonging to $W_2^{l_2,h}(\Omega_T)$. Since $\ell_{k,d+1}>-\delta_{d+1}-\lambda_k(t)-1+l_2-\frac{n}{2}$ for all $t\in[0,T]$, then $-2m+\lambda_k(t)+\ell_{k,d+1}+1>-\delta_{d+1}+l_2-2m-\frac{n}{2}$ for all $t\in[0,T]$. Thus, the final terms in (4.39) and (4.40) belong to $V_{2,\delta_{d+1}}^{l-2m,h}(G_T)$. Hence, (4.39) can be rewritten in the form

$$L\Big(\sum_{\tau=0}^{\ell_{k,d}} r^{\lambda_k(t)+\tau}P_{k,\tau}^{(d)}(\ln r)\Big)=\sum_{\tau=0}^{\ell_{k,d+1}} r^{-2m+\lambda_k(t)+\tau}\Psi_{k,\tau}^{(d+1)}(\ln r)+w_k \tag{4.41}$$

where $w_k\in V_{2,\delta_{d+1}}^{l-2m,h}(G_T)$. Analogously, we can write

$$B_j\Big(\sum_{\tau=0}^{\ell_{k,d}} r^{\lambda_k(t)+\tau}P_{k,\tau}^{(d)}(\ln r)\Big)=\sum_{\tau=0}^{\ell_{k,d+1}} r^{-k_j+\lambda_k(t)+\tau}\Psi_{k,\tau,j}^{(d+1)}(\ln r)+w_{k,j} \tag{4.42}$$

for $j=1,\dots,m$, where $\Psi_{k,\tau,j}^{(d+1)}$ are polynomials with coefficients belonging to $W_2^{l_2,h}(\partial\Omega_T)$ and $w_{k,j}\in V_{2,\delta_{d+1}}^{l-k_j-\frac{1}{2},h}(\partial\Omega_T)$. According to Lemma 4.2, there are polynomials $\Phi_{k,\tau}^{(d+1)}$ with coefficients belonging to $W_2^{l_2,h}(\Omega_T)$ such that

$$\mathfrak{L}(t,\partial_x)\Big(r^{\lambda_k(t)+\tau}\Phi_{k,\tau}^{(d+1)}(\ln r)\Big)=r^{-2m+\lambda_k(t)+\tau}\Psi_{k,\tau}^{(d+1)}(\ln r)\ \text{in } G_T, \tag{4.43}$$

$$\mathfrak{B}_j(t,\partial_x)\Big(r^{\lambda_k(t)+\tau}\Phi_{k,\tau}^{(d+1)}(\ln r)\Big)=r^{-k_j+\lambda_k(t)+\tau}\Psi_{k,\tau,j}^{(d+1)}(\ln r)\ \text{on } S_T. \tag{4.44}$$

Set

$$v=\sum_{k=1}^{N_d}\sum_{\tau=0}^{\ell_{k,d+1}} r^{\lambda_k(t)+\tau}\Phi_{k,\tau}^{(d+1)}(\ln r). \tag{4.45}$$

Now we rewrite (4.35), (4.36) in the form

$$\mathfrak{L}(t,\partial_x)(u_d+v)=f+(\mathfrak{L}-L)u_d+\sum_{k=1}^{N_d} w_k \quad\text{in } G_T, \tag{4.46}$$

$$\mathfrak{B}_j(t,\partial_x)(u_d+v)=g_j+(\mathfrak{B}_j-B_j)u_d+\sum_{k=1}^{N_d}w_{k,j}\quad \text{on } S_T,\ j=1,\dots,m. \tag{4.47}$$

It is the same as in (4.32), (4.33) that $(\mathfrak{L}-L)u_d \in V^{l-2m,h}_{2,\delta_{d+1}}(G_T)$ and $(\mathfrak{B}_j-B_j)u_d \in V^{l-k_j-\frac12,h}_{2,\delta_{d+1}}(S_T)$. Thus the right-hand sides of (4.46), (4.47) belong to $V^{l-2m,h}_{2,\delta_{d+1}}(G_T)$, $V^{l-k_j-\frac12,h}_{2,\delta_{d+1}}(S_T)$, respectively. Now we can apply Lemma 4.1 to conclude from (4.46), (4.47) that

$$u_d+v=\sum_{k=1}^{N_{d+1}}\sum_{\tau=0}^{\ell_{k,d+1}} r^{\lambda_k(t)+\tau}\Upsilon^{(d+1)}_{k,\tau}(\ln r)+u_{d+1}, \tag{4.48}$$

where $\Upsilon^{(d+1)}_{k,\tau}$ are polynomials with coefficients belonging to $W^{l_2,h}_2(\Omega_T)$ and $u_{d+1}\in V^{l_2,h}_{2,\delta_{d+1}}(G_T)$. Setting $P^{(d+1)}_{k,\tau}=P^{(d)}_{k,\tau}-\Phi^{(d+1)}_{k,\tau}+\Upsilon^{(d+1)}_{k,\tau}$ for $k=1,\dots,N_d$, and $P^{(d+1)}_{k,\tau}(\ln r)=\Upsilon^{(d+1)}_{k,\tau}(\ln r)$ for $k=N_{d+1}$, we have

$$u=\sum_{k=1}^{N_{d+1}}\sum_{\tau=0}^{\ell_{k,d+1}} P^{(d+1)}_{k,\tau}(\ln r)+u_{d+1}.$$

This implies that (4.31) holds for $d+1$. The proof is complete. □

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