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TWIN PERIODIC SOLUTIONS OF PREDATOR-PREY DYNAMIC SYSTEM ON TIME SCALES

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ABSTRACT. In this article, we consider a delayed predator-prey dynamic system with type IV functional responses on time scales. Sufficient criteria for the existence of at least two periodic solutions are established by using the well-known continuation theorem due to Mawhin. An example is given to illustrate the main result.

1. INTRODUCTION

In studying the interaction between predators and their prey, it is crucial to determine what specific form of the functional response that describes the mount of prey consumed per predator per unit of time is biological plausible provides a sound basis for theoretical development.

In this paper, we consider the following delay predator-prey dynamic equation with type IV functional responses on time scale $\mathbb{T}$:

$$\begin{aligned} y_1^{\Delta}(t) &= b_1(t) - a_1(t)\exp\{y_1(t-\tau_1(t))\} - \frac{c(t)\exp\{y_2(t-\gamma(t))\}}{\exp\{2y_1(t)\}/n + \exp\{y_1(t)\} + a}, \\ y_2^{\Delta}(t) &= -b_2(t) + \frac{a_2(t)\exp\{y_1(t-\tau_2(t))\}}{\exp\{2y_1(t)(t-\tau_2(t))\}/n + \exp\{y_1(t-\tau_2(t))\} + a}, \end{aligned} \tag{1.1}$$

where for $i = 1,2$; $c, \gamma, a_i, b_i, \tau_i \in C_{rd}(\mathbb{T})$ are ω-priodic functions with $c(t) \geq 0, \gamma(t) \geq 0, a_i(t) \geq 0, \tau_i(t) \geq 0, \bar{c} > 0$, and $\bar{b}_i > 0$, n and a are positive constants, $C_{rd}(\mathbb{T})$ will be defined later.

Calculus on time scales was initiated by Stefan Hilger in 1990 with the motivation of providing a unified approach to continuous and discrete analysis. Since then the theory of dynamic equations on time scales has become a new important mathematical branch, and it has been applied in various directions (see, eg., [1, 2, 3, 4, 5, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20] and the refs cited therein).

On the other hand, the Mawhin's continuation is a powerful tool when deal with the existence of periodic solutions for population models, and much work have been done (see, e.g., [6, 10, 13, 17, 21] and the references cited therein). However, to the best of our knowledge, the study on the existence of multiple periodic solutions for population models on time scales are scarce.

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Motivated and inspired by the above excellent work, in this paper, we establish some sufficient criteria for the existence of at least two periodic solutions for system (1.1) by using Mawhin technique.

2. Preliminaries

We first provide without proof several definitions and results from the calculus on time scales which are useful in the following argument. For more details, we refer the authors to [5].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers, and it inherits the topology from the real numbers with the standard topology. Let $\omega > 0$ is a constant. Throughout this paper, the time scale we considered is always assumed to be ω-priodic (i.e., $t \in \mathbb{T}$ implies $t \pm \omega \in \mathbb{T}$) and unbounded above and below. Set

$$\kappa = \min\{\mathbb{R}^+ \cap \mathbb{T}\}, \quad \mathbb{I}_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}.$$

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s \geq t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s \leq t\},$$

respectively, for any $t \in \mathbb{T}$. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left–dense (otherwise left-scattered).

Definition 2.2. Assume that $f : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$. Then f is called differential at $t \in \mathbb{T}$ if there exists $c \in \mathbb{R}$ such that given any $\varepsilon > 0$, there is an open neighborhood U of t satisfying

$$|[f(\sigma(t)) - f(s)] - c[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

In this case, c is called the delta (or Hilger) derivative of f at $t \in \mathbb{T}$, and is denoted by $c = f^\Delta(t)$.

Remark 2.3. We say that f is delta (Hilger) differential on $\mathbb{T}$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r), \quad r, s \in \mathbb{T}.$$

Definition 2.4. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}[\mathbb{T}, \mathbb{R})$.

Remark 2.5. Every rd-continuous function has an antiderivative. Every continuous function is rd-continuous.

Lemma 2.6. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$, then*

(C1) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;
(C2) *if* $f(t) \geq 0$ *for all* $a \leq t \leq b$, *then* $\int_a^b f(t)\Delta t \geq 0$;
(C3) *if* $|f(t)| \leq g(t)$ *on* $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, *then* $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.

Lemma 2.7 ([3]). *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R}$ is $\omega-$periodic, then*

$$g(t) \le g(t_1) + \int_{\mathbb{I}_\omega} |g^\Delta(s)|\Delta s \quad \text{and} \quad g(t) \ge g(t_2) - \int_{\mathbb{I}_\omega} |g^\Delta(s)|\Delta s.$$

In the remainder of this section we list well known elements and result which can be found in [8]. Let X and Z are two Banach spaces. Consider a operator equation:

$$Lx = \lambda Nx, \quad \lambda \in (0, 1), \tag{2.1}$$

where $L : \operatorname{Dom} L \cap X \to Z$ is a linear operator, $N : X \to Z$ is a continuous operator and λ is a parameter. Let P and Q denote two projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of this map by Kp. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $Kp(I - Q)N : \overline{\Omega} \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Recall that operator L will be called a Fredholm operator of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$, and $\operatorname{Im} L$ is closed in Z.

Lemma 2.8 (Continuation Theorem [8]). *Let L be a Fredholm mapping of index zero and let N be $L-$compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) *$\deg\{JQN, \Omega, \theta\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

To facilitate the discussion below, throughout this paper we adopt the following notation

$$\bar{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s, \quad |\bar{b}_i| = \frac{1}{\omega}\int_{I_\omega} |b_i(s)|\Delta s,$$

where i=1, 2; $g \in C_{rd}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$. And the other symbols appearing in the sequel are denoted accordingly. Set

$$\begin{aligned}
u_{1\pm} &:= \frac{n(\bar{a}_2 - \bar{b}_2) \pm \sqrt{n^2(\bar{a}_2 - \bar{b}_2)^2 - 4\bar{b}_2^2 an}}{2\overline{b_2}},\\
l_\pm &:= \frac{1}{2\bar{b}_2}\Big(n[\bar{a}_2 \exp\{(\bar{b}_1 + |\bar{b}_1|)\omega\} - \bar{b}_2]\\
&\quad \pm \sqrt{n^2[\bar{a}_2 \exp\{(\bar{b}_1 + |\bar{b}_1|)\omega\} - \bar{b}_2]^2 - 4\bar{b}_2^2 an}\Big),\\
v_\pm &:= \frac{1}{2\bar{b}_2 \exp\{\omega(\bar{b}_1 + |\bar{b}_1|)\}}\Big(n[\bar{a}_2 - \bar{b}_2 \exp\{(\bar{b}_1 + |\bar{b}_1|)\omega\}]\\
&\quad \pm \sqrt{n^2[\bar{a}_2 - \bar{b}_2 \exp\{(\bar{b}_1 + |\bar{b}_1|)\omega\}]^2 - 4\bar{b}_2^2 an \exp\{2\omega(|\bar{b}_1| + \bar{b}_1)\}}\Big).
\end{aligned}$$

From the above six positive numbers, one obtains

$$l_- < u_{1-} < v_- < v_+ < u_{1+} < l_+. \tag{2.2}$$

3. Main result

In this section, our emphasis is focused on the existence of at least two periodic solutions for (1.1). Before formulate the main result, we first embed our problem into the frame of Lemma 2.8. Set

$$X = Z = \{y = (y_1(t), y_2(t))^T \in C(\mathbb{T}, \mathbb{R}^2) | y_i(t+\omega) = y_i(t),\ i = 1, 2,\ t \in \mathbb{T}\}.$$

Then X, Z are Banach spaces endowed with the norm $\|y\| = \sum_{i=1}^{2} \max_{t\in I_\omega} |y_i(t)|$. Define

$$Ny(t) = (\Phi_1(t), \Phi_2(t))^T, \quad Ly(t) = (y_1^\Delta(t), y_2^\Delta(t))^T,$$
$$Py = Qy = \Big(\frac{1}{\omega}\int_{\mathbb{I}_\omega} y_1(t)\Delta t, \frac{1}{\omega}\int_{\mathbb{I}_\omega} y_2(t)\Delta t\Big)^T,$$

where $y \in X$,

$$\Phi_1(t) = b_1(t) - a_1(t)\exp\{y_1(t-\tau_1(t))\} - \frac{c(t)\exp\{y_2(t-\gamma(t))\}}{\exp\{2y_1(t)\}/n + \exp\{y_1(t)\} + a},$$
$$\Phi_2(t) = -b_2(t) + \frac{a_2(t)\exp\{y_1(t-\tau_2(t))\}}{\exp\{2y_1(t-\tau_2(t))\}/n + \exp\{y_1(t-\tau_2(t))\} + a}.$$

Obviously,

$$\operatorname{Ker} L = \{y \in X : y = h = (h_1, h_2)^T \in \mathbb{R}^2,\ t \in \mathbb{T}\},$$
$$\operatorname{Im} L = \Big\{y \in Z : \int_{\mathbb{I}_\omega} y_i(t)\Delta t = 0, t \in \mathbb{T},\ i = 1, 2\Big\},$$

P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q),$$

the set $\operatorname{Im} L$ is closed in Z, and

$$dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L.$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ exists and is given by

$$K_p y = \Big(\int_\kappa^t y_1(s)\Delta s - \frac{1}{\omega}\int_{\mathbb{I}_\omega}\int_\kappa^t y_1(s)\Delta s\Delta t, \int_\kappa^t y_2(s)\Delta s - \frac{1}{\omega}\int_{\mathbb{I}_\omega}\int_\kappa^t y_2(s)\Delta s\Delta t\Big)^T.$$

Thus

$$QNy = \Big(\frac{1}{\omega}\int_{\mathbb{I}_\omega} \Phi_1(s)\Delta s, \frac{1}{\omega}\int_{\mathbb{I}_\omega} \Phi_2(s)\Delta s\Big)^T,$$
$$K_p(I - Q)Ny = \Big(\Theta_1(t), \Theta_2(t)\Big)^T,$$

where,

$$\Theta_1(t) = \int_\kappa^t \Phi_1(s)\Delta s - \frac{1}{\omega}\int_{\mathbb{I}_\omega}\int_\kappa^t \Phi_1(s)\Delta s\Delta t - \Big(t - \kappa - \frac{1}{\omega}\int_{\mathbb{I}_\omega}(t-\kappa)\Delta t\Big)\bar{\Phi}_1,$$
$$\Theta_2(t) = \int_\kappa^t \Phi_2(s)\Delta s - \frac{1}{\omega}\int_{\mathbb{I}_\omega}\int_\kappa^t \Phi_2(s)\Delta s\Delta t - \Big(t - \kappa - \frac{1}{\omega}\int_{\mathbb{I}_\omega}(t-\kappa)\Delta t\Big)\bar{\Phi}_2.$$

It is easy to show that QN and $K_p(I-Q)$ are continuous. By using the Arzela-Ascoli theorem, one can show that $K_p(I-Q)(\bar{\Omega})$ is relatively compact for any open bounded set $\Omega \in X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

After the above preparations, we now state and prove our main result.

Theorem 3.1. *System* (1.1) *has at least two ω-periodic solutions if the following conditions hold.*

(i) $\bar{a}_2 > \bar{b}_2(1 + 2\sqrt{\frac{a}{n}}) \exp\{(|\bar{b}_1| + \bar{b}_1)\omega\}$;
(ii) $\bar{b}_1 > \bar{a}_1 l_+ \exp\{(|\bar{b}_1| + \bar{b}_1)\omega\}$.

Proof. Corresponding to the operator equation (2.1) we have

$$(y_1^{\Delta}(t), y_2^{\Delta}(t))^T = \lambda(\Phi_1(t), \Phi_2(t))^T. \tag{3.1}$$

Suppose that $y \in X$ is a solution of system (3.1) for a certain $\lambda \in (0,1)$. Integrating (3.1) over set $\mathbb{I}_\omega$, we obtain

$$\bar{b}_1\omega = \int_{\mathbb{I}_\omega} \big[a_1(t)\exp\{y_1(t-\tau_1(t))\} + \frac{c(t)\exp\{y_2(t-\gamma(t))\}}{\exp\{2y_1(t)\}/n + \exp\{y_1(t)\} + a}\big]\Delta t, \tag{3.2}$$

$$\bar{b}_2\omega = \int_{\mathbb{I}_\omega} \big[\frac{a_2(t)\exp y_1(t-\tau_2(t))}{\exp\{2y_1(t-\tau_2(t))\}/n + \exp y_1(t-\tau_2(t)) + a}\big]\Delta t. \tag{3.3}$$

From (3.1)–(3.3), we have

$$\int_{\mathbb{I}_\omega} |y_1^{\Delta}(t)|\Delta t \le \lambda\big\{\int_{\mathbb{I}_\omega} |b_1(t)|\Delta t + \int_{\mathbb{I}_\omega} [-\Phi_1(t) + b_1(t)]\Delta t\big\} < (|\bar{b}_1| + \bar{b}_1)\omega, \tag{3.4}$$

$$\int_{\mathbb{I}_\omega} |y_2^{\Delta}(t)|\Delta t \le \lambda\big\{\int_{\mathbb{I}_\omega} |b_2(t)|\Delta t + \int_{\mathbb{I}_\omega} [\Phi_2(t) + b_2(t)]\Delta t\big\} < (|\bar{b}_2| + \bar{b}_2)\omega. \tag{3.5}$$

Because $y = (y_1(t), y_2(t))^T \in X$, there exist $\xi_i, \eta_i \in \mathbb{I}_\omega$, $i = 1,2$ such that

$$y_i(\xi_i) = \min_{t\in\mathbb{I}_\omega}\{y_i(t)\}, \quad y_i(\eta_i) = \max_{t\in\mathbb{I}_\omega}\{y_i(t)\}. \tag{3.6}$$

From (3.3) and (3.6) we get

$$\begin{aligned}
\bar{b}_2\omega &\le \int_{\mathbb{I}_\omega} \big[\frac{a_2(t)\exp\{y_1(\eta_1)\}}{\exp\{2y_1(\xi_1)\}/n + \exp\{y_1(\xi_1)\} + a}\big]\Delta t \\
&= \frac{\bar{a}_2\omega\exp\{y_1(\eta_1)\}}{\exp\{2y_1(\xi_1)\}/n + \exp\{y_1(\xi_1)\} + a},
\end{aligned}$$

which implies

$$y_1(\eta_1) \ge \ln\big[\frac{\bar{b}_2}{\bar{a}_2}(\exp\{2y_1(\xi_1)\}/n + \exp y_1(\xi_1) + a)\big]. \tag{3.7}$$

By virtue of (3.4), (3.7) and Lemma 2.7, we get

$$\begin{aligned}
y_1(t) &\ge y_1(\eta_1) - \int_{\mathbb{I}_\omega} |y_1^{\Delta}(t)|\Delta t \\
&> \ln[\frac{\bar{b}_2}{\bar{a}_2}(\exp\{2y_1(\xi_1)\}/n + \exp y_1(\xi_1) + a)] - (|\bar{b}_1| + \bar{b}_1)\omega.
\end{aligned}$$

In particular,

$$y_1(\xi_1) > \ln\big[\frac{\bar{b}_2}{\bar{a}_2}(\exp\{2y_1(\xi_1)\}/n + \exp y_1(\xi_1) + a)\big] - (|\bar{b}_1| + \bar{b}_1)\omega.$$

or

$$\frac{\bar{b}_2}{n}\exp\{2y_1(\xi_1)\} - \big(\bar{a}_2\exp\{(|\bar{b}_1|+\bar{b}_1)\omega\} - \bar{b}_2\big)\exp\{y_1(\xi_1)\} + \bar{b}_2 a < 0. \tag{3.8}$$

According to (i), we have

$$\ln l_- < y_1(\xi_1) < \ln l_+. \tag{3.9}$$

From (3.3), we have

$$\begin{aligned}\bar{b}_2\omega &\ge \int_{\mathbb{I}_\omega}\Big[\frac{a_2(t)\exp\{y_1(\xi_1)\}}{\exp\{2y_1(\eta_1)\}/n+\exp\{y_1(\eta_1)\}+a}\Big]\Delta t\\ &= \frac{\bar{a}_2\omega\exp\{y_1(\xi_1)\}}{\exp\{2y_1(\eta_1)\}/n+\exp\{y_1(\eta_1)\}+a};\end{aligned}$$

that is,

$$y_1(\xi_1) < \ln\Big[\frac{\bar{b}_2}{\bar{a}_2}(\exp\{2y_1(\eta_1)\}/n+\exp y_1(\eta_1)+a)\Big].$$

Which together with (3.4) and Lemma 2.7 lead to

$$\begin{aligned}y_1(t) &\le y_1(\xi_1) + \int_{\mathbb{I}_\omega}|y_1^\Delta(t)|\Delta t\\ &< \ln\Big[\frac{\overline{b_2}}{\overline{a_2}}(\exp\{2y_1(\eta_1)\}/n+\exp y_1(\eta_1)+a)\Big] + (|\bar{b}_1|+\bar{b}_1)\omega.\end{aligned}$$

Hence, we have

$$y_1(\eta_1) < \ln\Big[\frac{\bar{b}_2}{\bar{a}_2}(\exp\{2y_1(\eta_1)\}/n+\exp y_1(\eta_1)+a)\Big] + (|\bar{b}_1|+\bar{b}_1)\omega.$$

or

$$\frac{\bar{b}_2}{n}\exp\{2y_1(\eta_1)\} - \big(\bar{a}_2\exp\{-(|\bar{b}_1|+\bar{b}_1)\omega\} - \bar{b}_2\big)\exp\{y_1(\eta_1)\} + \bar{b}_2 a > 0.$$

Similarly, we can show that

$$y_1(\eta_1) < \ln v_- \quad \text{or} \quad y_1(\eta_1) > \ln v_+. \tag{3.10}$$

From (3.4), (3.9) and Lemma 2.7,

$$y_1(t) \le y_1(\xi_1) + \int_{\mathbb{I}_\omega}|y_1^\Delta(t)|\Delta t < \ln l_+ + (|\bar{b}_1|+\bar{b}_1)\omega := P_1. \tag{3.11}$$

On the other hand, (3.2) and (3.6) yield

$$\bar{b}_1\omega \ge \frac{\bar{c}\omega\exp\{y_2(\xi_2)\}}{\exp\{2P_1\}/n+\exp\{P_1\}+a}, \tag{3.12}$$

$$\bar{b}_1\omega \le \bar{a}_1\omega\exp\{P_1\} + \frac{\bar{c}\omega\exp\{y_2(\eta_2)\}}{a}. \tag{3.13}$$

It follows from (3.12) that

$$y_2(\xi_2) \le \ln\Big[\frac{\bar{b}_1}{\bar{c}}(\exp\{2P_1\}/n+\exp\{P_1\}+a)\Big].$$

This, together with (3.5) and Lemma 2.7, yields

$$\begin{aligned}y_2(t) &\le y_2(\xi_2) + \int_{\mathbb{I}_\omega}|y_2^\Delta(t)|\Delta t\\ &< \ln\Big[\frac{\bar{b}_1}{\bar{c}}(\exp\{2P_1\}/n+\exp\{P_1\}+a)\Big] + (|\bar{b}_2|+\bar{b}_2)\omega := P_2.\end{aligned} \tag{3.14}$$

Moreover, because of (ii), it follows from (3.13) that

$$y_2(\eta_2) \geq \ln\big[\frac{a}{\bar{c}}(\bar{b}_1 - \exp\{(\bar{B}_1 + \bar{b}_1)\omega\bar{a}_1 l_+\})\big],$$

which, combined with Lemma 2.7, gives

$$\begin{aligned} y_2(t) &\geq y_2(\eta_2) - \int_{\mathbb{I}_\omega} |y_2^\Delta(t)|\Delta t \\ &> \ln[\frac{a}{\bar{c}}(\bar{b}_1 - \exp\{(\bar{B}_1 + \bar{b}_1)\omega\bar{a}_1 l_+\})] - (|\bar{b}_2| + \bar{b}_2)\omega := P_3. \end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\max_{t\in\mathbb{I}_\omega} |y_2(t)| < \max\{|P_2|, |P_3|\} := P. \tag{3.16}$$

Obviously, $\ln l_\pm$, $\ln v_\pm$, P_1 and P are independent of the choice of $\lambda \in (0,1)$.

Now, let's consider QNh with $h = (h_1, h_2)^T \in \mathbb{R}^2$. Note that

$$QNh = \begin{pmatrix} \bar{b}_1 - \bar{a}_1\exp\{h_1\} - \frac{\bar{c}\exp\{h_2\}}{\exp\{2h_1\}/n + \exp\{h_1\} + a} \\ -\bar{b}_2 + \frac{\bar{a}_2\exp\{h_1\}}{\exp\{2h_1\}/n + \exp\{h_1\} + a} \end{pmatrix}$$

In view of (i) and (ii), one can show that $QNh = 0$ has two distinct constant solutions:

$$z^\dagger = \Big(\ln u_{l-}, \ln\frac{(\bar{b}_1 - \bar{a}_1 u_{l-})g(u_{l-})}{\bar{c}}\Big)^T,$$
$$z^\ddagger = \Big(\ln u_{l+}, \ln\frac{(\bar{b}_1 - \bar{a}_1 u_{l+})g(u_{l+})}{\bar{c}}\Big)^T,$$

where, $g(u_{l-}) = u_{l-}^2/n + u_{l-} + a$, and $g(u_{l+}) = u_{l+}^2/n + u_{l+} + a$. Chose $M > 0$ such that

$$M > \max\big\{|\ln\frac{(\bar{b}_1 - \bar{a}_1 u_{l-})g(u_{l-})}{\bar{c}}|,\ |\ln\frac{(\bar{b}_1 - \bar{a}_1 u_{l+})g(u_{l+})}{\bar{c}}|\big\}. \tag{3.17}$$

And set

$$\Omega_1 = \big\{y = (y_1(t), y_2(t))^T \in X : y_1(t) \in (\ln l_-, \ln v_-),\ \max_{t\in I_\omega} |y_2(t)| < P + M\big\},$$
$$\begin{aligned} \Omega_2 = \big\{y = (y_1(t), y_2(t))^T \in X : \min_{t\in I_\omega} y_1(t) \in (\ln l_-, \ln l_+), \\ \max_{t\in I_\omega} y_1(t) \in (\ln v_+, P_1),\ \max_{t\in I_\omega} |y_2(t)| < P + M\big\}. \end{aligned}$$

Then both Ω_1 and Ω_2 are open bounded subset of X. It follows from (2.2) and (3.17) that $z^\dagger \in \Omega_1, z^\ddagger \in \Omega_2$. With the help of (2.2), (3.9)–(3.11), and (3.16)–(3.17), it is not difficult to show that $\Omega_1 \cap \Omega_2 = \emptyset$ and Ω_i verifies the requirements (a) of Lemma 2.8 for $i = 1,2$. When $y \in \partial\Omega_i \cap R^2$, y is a constant vector in R^2, then $QNy \neq 0$. Moreover, after direct calculation we get the Brouwer degree

$$\deg(JQN, \Omega_i \cap \ker L, \theta) = (-1)^{i+1} \neq 0,$$

for $i = 1,2$, where the isomorphism J can be chosen to be the identity mapping, since $\operatorname{Im} Q = \operatorname{Ker} L$. Up to now, we have proved that Ω_i verify all the requirements of Lemma 2.8. Therefore, by Lemma 2.8, we derive that (1.1) has at least two ω-periodic solutions lying in $\operatorname{Dom} L \cap \overline{\Omega_i}$. The proof is complete. □

Remark 3.2. Assume $\mathbb{T} = \mathbb{Z}$, then (1.1) becomes a discrete analogue of (1.1) which has been discussed in [21]. Therefore, our result obtained generalized the result in the literature.

4. Example

Consider the following periodic predator-prey system with a IV functional response

$$
\begin{aligned}
y_1^{\Delta}(t) &= (\frac{1}{20}+\sin t) - (\frac{1}{60}+\cos t)\exp\{y_1(t-\sin t)\} \\
&\quad - \frac{\exp\{y_2(t-\cos t\}}{2\exp\{y_1(t)\}+2\exp\{y_1(t)\}+1}, \\
y_2^{\Delta}(t) &= -(\frac{1}{3}+\cos t) + \frac{2(1+\cos t)\exp\{y_1(t-\cos t)\}}{2\exp\{y_1(t-\cos t)\}+2\exp\{y_1(t-\cos t)\}+1},
\end{aligned}
\tag{4.1}
$$

by choosing the 2π-periodic time scale

$$
\mathbb{T} = \bigcup_{s\in\mathbb{Z}} [2(s-1)\pi, 2s\pi].
$$

Then, system (4.1) has at least two 2π-periodic solutions.

Direct calculations lead to

$$
\kappa = \min\{\mathbb{R}^+ \cap \mathbb{T}\} = 0,
$$

$I_\omega = [\kappa, \kappa+\omega]\cap\mathbb{T} = [0, 2\pi]$, $\bar{a}_1 = 1/60, \bar{a}_2 = 1, \bar{b}_1 = 1/20, \bar{b}_2 = 1/3, \exp\{\bar{b}_1+|\bar{b}_1|\} < 1.135$, $l_+ < 2.302$. It is straight forward to check that

$$
\begin{gathered}
\bar{a}_2 = 1.000 > 0.757 > \bar{b}_2\big(1+2\sqrt{a/n}\big)\exp\{(|\bar{b}_1|+\bar{b}_1)\omega\}, \\
\bar{b}_1 = 0.050 > 0.045 > \bar{a}_1 l_+ \exp\{(|\bar{b}_1|+\bar{b}_1)\omega\},
\end{gathered}
$$

which show that all the conditions in Theorem 3.1 are fulfilled. By Theorem 3.1 we derive that (4.1) has at least two 2π-periodic solutions.

Remark 4.1. System (4.1) models two populations (one is the predator and the other is the prey) that are both continuous in one period of the year, die out in other period of the year, and their offspring are renascent after incubating or dormant in another period of the year, both of them giving rise to non-overlapping populations.

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