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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this article, we established the existence and uniqueness of solutions for fractional integro-differential equations with nonlocal conditions in Banach spaces. Krasnoselskii-Krein-type conditions are used for obtaining the main result.

1. Introduction

In this article, we are interesting in the existence and uniqueness of solutions for the Cauchy problem with a Caputo fractional derivative and nonlocal conditions:

$$D^q x(t) = f(t, x(t), [\theta x](t)), \quad q \in (0,1)\ t \in I := [0,1], \tag{1.1}$$

$$x(0) + g(x) = x_0, \tag{1.2}$$

where $q \in (0,1)$, $f : I \times X \times X \to X$, $g : C(I,X) \to X$, $\theta : X \to X$ defined as

$$[\theta x](t) = \int_0^t k(t,s,x(s))ds,$$

and $k : \Delta \times X \to X$, $\Delta = \{(t,s) : 0 \le s \le t \le 1\}$. Here, $(X, \|\cdot\|)$ is a Banach space and $C = C(I,X)$ denotes the Banach space of all bounded continuous functions from I into X equipped with the norm $\|\cdot\|_C$.

The study of fractional differential equations and inclusions is linked to the wide applications of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics. The theory of fractional differential equations has seen considerable development, see for example the monographs of Kilbas et al. [5] and Lakshmikantham et al. [9].

Recently, existence and uniqueness criteria for the various fractional (integro-)differential equations were considered by Ahmad and Nieto [1], Bhaskar[4], Lakshmikantham and Leela et al [7, 8]. For more information in this fields, see [2, 3] and the references therein.

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As indicated in many previous articles, the nonlocal condition $x(0) + g(x) = x_0$ generalizes the Cauchy condition $x(0) = x_0$, and can be applied in physics with better cases than the Cauchy condition. The term $g(x)$ denotes the nonlocal effects, which describe the diffusion phenomenon of the a small amount in a transparent tube, with the general form $g(x) = \sum_{i=1}^{p} c_i x(t_i)$. Also, the problem (1.1)-(1.2) includes many classical formulations. For example, $g(x) = x_0 - x(T)$ becomes a periodic boundary problem, $g(x) = x_0 + x(T)$ becomes an antiperiodic boundary problem, while $g(x) = 0$ becomes a Cauchy problem.

In [2], the authors presented some existence and uniqueness results for the problem (1.1)-(1.2), when $f(t, x(t), [\theta x](t)) = p(t, x(t)) + \int_0^t k(t, s, x(s))ds$. In [3], the authors presented some existence and uniqueness results for the problem (1.1)-(1.2), when $f(t, x(t), [\theta x](t)) = \int_0^t k(t, s, x(s))ds$. The aim of this paper is to present some existence results for the problem (1.1)-(1.2) for some Krasnoselskii-Krein-type conditions. Our methods are based on the equivalence of norms and a fixed point theorem.

2. Main results

For the next theorem, we sue the following assumptions:

(F1) f is continuous and there exist constants $\alpha, \beta \in (0, 1]$, $L_1, L_2 > 0$ such that for $t \in I$ and $x_i, y_i \in X$,
$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le L_1 \|x_1 - y_1\|^{\alpha} + L_2 \|x_2 - y_2\|^{\beta} \,;$$
(F2) k is continuous and there exist $\beta_1 \in (0, 1]$, $h \in L^1(I)$ such that
$$\|k(t, s, x) - k(t, s, y)\| \le h(s)\|x - y\|^{\beta_1}, \quad (t, s) \in \Delta,\ x, y \in X \,;$$
(G) g is bounded, continuous, and there exists a constant $b \in (0, 1)$ such that $\|g(u) - g(v)\| \le b\|u - v\|$.

Theorem 2.1. *Under Assumptions* (F1), (F2), (G), *Problem* (1.1)-(1.2) *has a unique solution.*

For special cases of f, we obtain the following corollaries.

Corollary 2.2. *Let* $f(t, x(t), [\theta x](t)) = p(t, x(t)) + \int_0^t k(t, s, x(s))ds$. *Assume* (F2), (G) *and that* p *is continuous and there exist constants* $\beta \in (0, 1]$, $L > 0$ *such that*
$$\|p(t, x) - p(t, y)\| \le L\|x - y\|^{\beta} \quad t \in I,\ x, y \in X.$$
Then (1.1)-(1.2) *has a unique solution.*

Corollary 2.3. *Assume* (F1), (G) *and that* $k(t, s, x(s)) = \gamma(t, s)x(s)$ *and* $\gamma \in C(\Delta)$. *Then* (1.1)-(1.2) *has a unique solution.*

For the next theorem, we use the assumptions:

(F1') f is continuous and there exist constants $p_1, p_2 \in [0, q)$, $L_1, L_2, C > 0$ such that
$$\|f(t, x, y)\| \le \frac{L_1}{t^{p_1}}\|x\| + \frac{L_2}{t^{p_2}}\|y\| + C, \quad t \in I,\ x, y \in X \,;$$
(F2') k is continuous and there exist $h \in L^1(I)$, $K > 0$ such that
$$\|k(t, s, x)\| \le h(s)\|x\| + K, \quad (t, s) \in \Delta,\ x, y \in X.$$

Theorem 2.4. *Assume* (F1'), F(2'), (G). *Then* (1.1)-(1.2) *has at least one solution.*

We remark that Theorem 2.1 extends [2, Theorem 2.1] and [3, Theorem 2.1].

3. Proof of Theorem 2.1

The following lemma, due to Krasnoselskii, plays an important role in the proof of the existence part of Theorem 2.1.

Lemma 3.1 ([6]). *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that (1) $Ax + By \in M$ whenever $x, y \in M$; (2) A is compact and continuous; (3) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Proof of Theorem 2.1. First, we transform the Cauchy problem (1.1)-(1.2) into fixed point problem with $F : C(I, X) \to C(I, X)$ defined by

$$Fx(t) = x_0 - g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), [\theta x](s)) ds. \tag{3.1}$$

Let $F = A + B$, with

$$Ax(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), [\theta x](s)) ds; \tag{3.2}$$

$$Bx(t) = x_0 - g(x). \tag{3.3}$$

Define the norm $\|\cdot\|_k$ in $C(I, X)$, for $u \in C(I, X)$ and for some $k \in \mathbb{N}$, by

$$\|u\|_k = \max\{e^{-kt}\|u(t)\| : t \in I\}.$$

Note that the norms $\|\cdot\|_C$ and $\|\cdot\|_k$ are equivalent.

We prove Theorem 2.1 in the following two steps.

Step 1: Existence. Let $P = \sup_{x\in X} \|g(x)\|$, $M_0 = \sup_{t\in I} \| \int_0^t k(t, s, 0) ds\|$, $M_1 = \sup_{t\in I} \|f(t, 0, 0)\|$ and $Q = \|x_0\| + P + \frac{M_1}{\Gamma(q+1)} + 3$. Choose a $k_1 \in N$ such that

$$\frac{1}{k_1^q}(L_1 Q^\alpha + L_2(\|h\|_{L^1} Q^{\beta_1} + M_0)^\beta) < 3.$$

Setting $B_Q = \{u \in C(I, X) : \|u\|_{k_1} \le Q\}$. For $u \in B_Q$, noting the assumption (F2), we have

$$\begin{aligned} \|[\theta u](t)\| &\le \int_0^t \|k(t, r, u(r)) - k(t, r, 0) + k(t, r, 0)\| dr \\ &\le \|h\|_{L^1} \sup_{r\in[0,t]} \|x(r)\|^{\beta_1} + M_0 \\ &\le \|h\|_{L^1} e^{k_1 t} Q^{\beta_1} + M_0. \end{aligned}$$

Thus

$$\|\theta u\|_{k_1} \le \|h\|_{L^1} Q^{\beta_1} + M_0.$$

By assumption (F1), for $u \in B_Q$, we obtain

$$\begin{aligned} \|Fu(t)\| \le \|x_0\| + P &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, u(s), [\theta u](s)) - f(s, u(s), 0)\| ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, u(s), 0) - f(s, 0, 0)\| ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, 0, 0)\| ds \end{aligned}$$

$$
\begin{aligned}
&\leq \|x_0\| + P + \frac{L_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|[\theta u](s)\|^{\beta} ds \\
&\quad + \frac{L_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|u(s)\|^{\alpha} ds + \frac{M_1}{\Gamma(q+1)} \\
&\leq \|x_0\| + P + \frac{M_1}{\Gamma(q+1)} + \frac{L_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} e^{\beta k_1 s} ds \|\theta u\|_{k_1}^{\beta} \\
&\quad + \frac{L_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} e^{\alpha k_1 s} ds \|u\|_{k_1}^{\alpha} \\
&\leq \|x_0\| + P + \frac{M_1}{\Gamma(q+1)} + \frac{L_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} e^{k_1 s} ds \|u\|_{k_1}^{\alpha} \\
&\quad + \frac{L_2}{\Gamma(q)} \int_0^t (t-s)^{q-1} e^{k_1 s} ds (\|h\|_{L^1} Q^{\beta_1} + M_0)^{\beta} \\
&\leq \|x_0\| + P + \frac{M_1}{\Gamma(q+1)} + e^{k_1 t} [\frac{L_1}{k_1^q} Q^{\alpha} + \frac{L_2}{k_1^q} (\|h\|_{L^1} Q^{\beta_1} + M_0)^{\beta}].
\end{aligned}
$$

Thus

$$
\|Fu\|_{k_1} \leq \|x_0\| + P + \frac{M_1}{\Gamma(q+1)} + \frac{L_1}{k_1^q} Q^{\alpha} + \frac{L_2}{k_1^q} (\|h\|_{L^1} Q^{\beta_1} + M_0)^{\beta} < Q.
$$

This implies $F(B_Q) \subset B_Q$.

On the other hand, for $u \in B_Q$ and $t_1, t_2 \in J (t_1 < t_2)$, we deduce that

$$
\begin{aligned}
&\|Au(t_2) - Au(t_1)\| \\
&= \frac{1}{\Gamma(q)} \| \int_0^{t_2} (t_2 - s)^{q-1} f(s, u(s), [\theta u](s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, u(s), [\theta u](s)) ds \| \\
&\leq \frac{M}{\Gamma(q+1)} [2(t_2 - t_1)^q + (t_1)^q - (t_2)^q] \\
&\leq \frac{2M}{\Gamma(q+1)} (t_2 - t_1)^q,
\end{aligned}
$$

where $M = \sup\{\|f(t, x, y)\| : (t, x, y) \in I \times B_Q \times \theta(B_Q)\}$. This means $A(B_Q)$ is equicontinuous set. By Ascoli-Arzela theorem, we easily deduce that $A(B_Q)$ is relatively compact set. It follows from the continuousness of f that A is complete continuous.

By Assumption (G), it is easy to see that B is contraction mapping. Following the Lemma 3.1 (Krasnoselskii's fixed point theorem), we conclude that F has a fixed point in B_Q. Thus there exists a solution of Cauchy problem (1.1)-(1.2).

Step 2: Uniqueness. Let $\varphi(t)$ and $\psi(t)$ be two solutions of Cauchy problem (1.1)-(1.2), and set $m(t) = \|\varphi(t) - \psi(t)\|$.

First, we prove that $m(0) = 0$. Indeed, by the definition of operator B and assumption (G), we see that B is contraction on $C(I, X)$. Thus there exists a unique $y(t)$ such that $By(t) = x_0 + g(y)$. On the other hand, noting that $\varphi(0) = x_0 + g(\varphi)$ and $\psi(0) = x_0 + g(\psi)$, we obtain $\varphi(0) = \psi(0)$.

Next, we prove $m(t) \equiv 0$ for $t \in I$ by contraction. If $m(t) \neq 0$ for some $t \in I$. Setting $t_* = \min\{t \in I : m(t) \neq 0\}$, then $m(t) \equiv 0$ for $t \in [0, t_*]$. Thus $m(t) \equiv 0$ for $t \in I$ if and only if $t_* = 1$. If $t_* < 1$, then we can choose positive numbers ε_0 and

$k_2 \in N$ such that

$$\frac{e^{k_2\varepsilon_0}}{k_2^q}(L_1 m_{\varepsilon_0}^{\alpha-1} + L_2\|h\|_{L^1}^{\beta} m_{\varepsilon_0}^{\beta\beta_1-1}) < 1,$$

where $m_{\varepsilon_0} = \max\{\|\varphi(t) - \psi(t)\| : t \in [t_*, t_* + \varepsilon_0]\}$.

Redefine the norm $\|\cdot\|_{k_2}$ on the interval $[t_*, t_* + \varepsilon_0]$ by

$$\|u\|_{k_2} = \sup\{e^{-k_2(t-t_*)}\|u(t)\| : t \in [t_*, t_* + \varepsilon_0]\},$$

then the norms $\|\cdot\|_{k_2}$ and $\|\cdot\|_C$ are equivalent on $[t_*, t_* + \varepsilon_0]$. Since $\varphi(0) = \psi(0)$, we claim that $g(\varphi) = g(\psi)$. Thus there exists $t_1 \in [t_*, t_* + \varepsilon_0]$ such that

$$\begin{aligned}
0 < m_{\varepsilon_0} &= \|\varphi(t_1) - \psi(t_1)\| \\
&= \|F\varphi(t_1) - F\psi(t_1)\| \\
&\le \frac{1}{\Gamma(q)}\int_{t_*}^{t_1}(t_1 - s)^{q-1}\|f(s, \varphi(s), [\theta\varphi](s)) - f(s, \psi(s), [\theta\psi](s))\|ds \\
&\le \frac{L_1}{\Gamma(q)}\int_{t_*}^{t_1}(t_1 - s)^{q-1}\|\varphi(s) - \psi(s)\|^{\alpha}ds \\
&\quad + \frac{L_2}{\Gamma(q)}\int_{t_*}^{t_1}(t_1 - s)^{q-1}\|[\theta\varphi](s) - [\theta\psi](s)\|^{\beta}ds \\
&\le \frac{1}{\Gamma(q)}\int_{t_*}^{t_1}(t_1 - s)^{q-1}[L_1 m^{\alpha}(s) + L_2\|h\|_{L^1}^{\beta}\sup_{r\in[0,s]} m^{\beta\beta_1}(r)]ds \\
&\le \frac{L_1}{\Gamma(q)}\int_{t_*}^{t_1}(t_1 - s)^{q-1}e^{\alpha k_2(s-t_*)}ds\|\varphi - \psi\|_{k_2}^{\alpha} \\
&\quad + \frac{L_2\|h\|_{L^1}^{\beta}}{\Gamma(q)}\int_{t_*}^{t_1}(t_1 - s)^{q-1}e^{\beta\beta_1 k_2(s-t_*)}ds\|\varphi - \psi\|_{k_2}^{\beta\beta_1} \\
&\le \frac{e^{k_2\varepsilon_0}}{k_2^q}(L_1 m_{\varepsilon_0}^{\alpha} + L_2\|h\|_{L^1}^{\beta} m_{\varepsilon_0}^{\beta\beta_1}) < m_{\varepsilon_0}.
\end{aligned}$$

This is impossible. Thus $t_* = 1$ and we conclude that $\varphi(t) \equiv \psi(t)$ for $t \in [0, 1]$. The proof is complete. $\square$

4. Proof of Theorem 2.4

Define an operator $H : C(I, R^+) \to C(I, R^+)$ by

$$Hx(t) = \frac{1}{\Gamma(q)}\int_0^t (t - s)^{q-1}(as^{-p_1} + bs^{-p_2})\sup_{r\in[0,s]} x(r)ds,$$

where $p_1, p_2 \in [0, q)$ are constants and $a = L_1$, $b = L_2\|h\|_{L^1}$.

Lemma 4.1. *There exist an increasing function $b \in C(I, R^+)$ and a $\delta \in (0, 1)$ such that $Hb(t) \le \delta b(t)$.*

Proof. We choose a positive number $\eta \in I$ such that

$$\frac{a\eta^{q-p_1}B(q, 1 - p_1)}{\Gamma(q)} + \frac{b\eta^{q-p_2}B(q, 1 - p_2)}{\Gamma(q)} + a\eta^{q-p_1} + b\eta^{q-p_2} < 1,$$

where $B(\cdot,\cdot)$ is the Beta function $B(x,y)=\int_0^1(1-s)^{x-1}s^{y-1}ds$. Let

$$\delta=\frac{a\eta^{q-p_1}B(q,1-p_1)}{\Gamma(q)}+\frac{b\eta^{q-p_2}B(q,1-p_2)}{\Gamma(q)}+a\eta^{q-p_1}+b\eta^{q-p_2}$$

and define an increasing function $b:I\to\mathbb{R}$ by

$$b(t)=\begin{cases}1, & \text{if } t\in[0,\eta],\\ e^{(t-\eta)/\eta}, & \text{if } t\in(\eta,1].\end{cases}$$

We claim that $Hb(t)\le\delta b(t)$ for $t\in[0,1]$. For $t\in[0,\eta]$, recalling that $B(x,y)=\int_0^1(1-s)^{x-1}s^{y-1}ds$, we have

$$\begin{aligned}
Hb(t)&=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(as^{-p_1}+bs^{-p_2})ds\\
&=\frac{a}{\Gamma(q)}t^{q-p_1}\int_0^1(1-z)^{q-1}z^{1-p_1-1}dz+\frac{b}{\Gamma(q)}t^{q-p_2}\int_0^1(1-z)^{q-1}z^{1-p_2-1}dz\\
&=\frac{aB(q,1-p_1)}{\Gamma(q)}t^{q-p_1}+\frac{bB(q,1-p_2)}{\Gamma(q)}t^{q-p_2}\\
&\le\frac{aB(q,1-p_1)}{\Gamma(q)}\eta^{q-p_1}+\frac{bB(q,1-p_2)}{\Gamma(q)}\eta^{q-p_2}<\delta b(t).
\end{aligned}$$

For $t\in(\eta,1]$, we have

$$\begin{aligned}
Hb(t)&=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(as^{-p_1}+bs^{-p_2})b(s)ds\\
&=\frac{1}{\Gamma(q)}\int_0^\eta(t-s)^{q-1}(as^{-p_1}+bs^{-p_2})ds\\
&\quad+\frac{1}{\Gamma(q)}\int_\eta^t(t-s)^{q-1}(as^{-p_1}+bs^{-p_2})e^{\frac{s-\eta}{\eta}}ds\\
&\le\frac{1}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}(as^{-p_1}+bs^{-p_2})ds\\
&\quad+\frac{1}{\Gamma(q)}\int_\eta^t(t-s)^{q-1}(as^{-p_1}+bs^{-p_2})e^{\frac{s-\eta}{\eta}}ds\\
&\le\frac{a\eta^{q-p_1}B(q,1-p_1)}{\Gamma(q)}+\frac{b\eta^{q-p_2}B(q,1-p_2)}{\Gamma(q)}\\
&\quad+\frac{1}{\Gamma(q)}\int_\eta^t(t-s)^{q-1}(as^{-p_1}+bs^{-p_2})e^{-\frac{t-s}{\eta}}dse^{\frac{t-\eta}{\eta}}\\
&\le[\frac{a\eta^{q-p_1}B(q,1-p_1)}{\Gamma(q)}+\frac{b\eta^{q-p_2}B(q,1-p_2)}{\Gamma(q)}+a\eta^{q-p_1}+b\eta^{q-p_2}]e^{\frac{t-\eta}{\eta}}\\
&=\delta b(t).
\end{aligned}$$

The proof is complete. □

Proof of Theorem 2.4. As in the proof of Theorem 2.1, we prove the operator F admits a fixed point. Define the norm $\|\cdot\|_b$ in $C(I,X)$, for $u\in C(I,X)$, by

$$\|u\|_b=\max\{\frac{1}{b(t)}\|u(t)\|:t\in I\}.$$

Then the norms $\|\cdot\|_C$ and $\|\cdot\|_b$ are equivalent. Let $P=\sup_{x\in X}\|g(x)\|$,

$$Q=\frac{1}{1-\delta}(\|x_0\|+P+\frac{C}{\Gamma(q+1)}+\frac{L_2KB(q,1-p_2)}{\Gamma(q)}),$$

and $B_Q=\{u\in C(I,X):\|u\|_b\le Q\}$. For $u\in B_Q$, noting the assumption (F2'), we have

$$\|[\theta u](t)\|\le\int_0^t\|k(t,r,u(r))\|dr\le\|h\|_{L^1}\sup_{r\in[0,t]}\|x(r)\|+K.$$

By the assumption (F1') and Lemma 4.1, for $u\in B_Q$, we obtain

$$\begin{aligned}
\|Fu(t)\| &\le \|x_0\|+P+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|f(s,u(s),[\theta u](s))\|ds\\
&\le \|x_0\|+P+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(L_1s^{-p_1}\|u(s)\|+L_2s^{-p_2}\|\theta u(s)\|+C)ds\\
&\le \|x_0\|+P+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(L_1s^{-p_1}+L_2\|h\|_{L^1}s^{-p_2})\sup_{r\in[0,s]}\|u(s)\|ds\\
&\quad+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(L_2Ks^{-p_2}+C)ds\\
&\le \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(L_1s^{-p_1}+L_2\|h\|_{L^1}s^{-p_2})b(s)ds\|u\|_b\\
&\quad+\|x_0\|+P+\frac{C}{\Gamma(q+1)}+\frac{L_2KB(q,1-p_2)}{\Gamma(q)}\\
&\le \delta b(t)\|u\|_b+\|x_0\|+P+\frac{C}{\Gamma(q+1)}+\frac{L_2KB(q,1-p_2)}{\Gamma(q)}.
\end{aligned}$$

Thus

$$\|Fu\|_b\le\delta Q+\|x_0\|+P+\frac{C}{\Gamma(q+1)}+\frac{L_2KB(q,1-p_2)}{\Gamma(q)}=Q.$$

This implies $F(B_Q)\subset B_Q$.

Similar arguments as in the proof of Theorem 2.1 show that A is completely continuous and B is contraction mapping. Thus, by Lemma 3.1, we conclude that F has a fixed point in B_Q. Thus there exists a solution of Cauchy problem (1.1)-(1.2). The proof is complete. □

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