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LATERAL ESTIMATES FOR ITERATED ELLIPTIC OPERATORS AND ANALYTICITY

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Abstract. Analyticity of functions satisfying the lateral estimates for iterated elliptic operators is shown.

1. Introduction

Bernstein [1] showed that a function $f(x)$ satisfying the inequalities

$$\frac{d^k}{dx^k} f(x) \le 0 \quad \text{on } (a,b) \text{ for any integer } k \ge 0$$

is real analytic on (a,b). According to [2], to obtain the analyticity, it is sufficient to have the above inequalities only for an increasing sequence k_j satisfying $k_{j+1} \le Ak_j$ with some $A > 0$.

Lelong [7] showed as an extension to a multidimensional case, that the inequalities for the iterated Laplacian Δ^k: for any $k = 0, 1, 2, \dots,$

$$\Delta^k u(x) \le 0 \quad \text{on a domain } D \text{ in } \mathbb{R}^n$$

imply the analyticity of $u(x)$ on D. Novickii [8] showed the above assertion is still valid if the Laplacian Δ is replaced by a second order strongly elliptic operator L with real-valued and real analytic coefficients, as a corollary of his representation theorem for L-superharmonic functions.

On the other hand, Kotake and Narasimhan [6] showed that the analyticity of $u(x)$ on D follows from the estimates: For any $k = 0, 1, 2, \dots$

$$\|P^k u\|_{L^2(D)} \le C_0 C^{mk} (mk)!^{mk}, \tag{1.1}$$

for an ellipitc operator of order m with real analytic coefficients. Bolley, Camus and Metivier [3] (see also [4]) showed the above assertion is still valid if we have the estimates (1.1) for an increasing sequence of natural numbers k_j satisfying $k_{j+1} \le Ak_j$ with some $A > 0$. We note that they showed in [3] that the conclusion holds even if P is a principal type and hypoelliptic operator with real analytic coefficients.

In this short note, we show that in the case where P is an elliptic operator with real-valued and real analytic coefficients, the above assertion is still valid if the estimates (1.1) are replaced by lateral estimates.

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Theorem 1.1. *Let D be an open set in $\mathbb{R}^n$. Let P be an elliptic operator of order m with real valued and real analytic coefficients. Assume that the inequalities*

$$P^{k_j}u(x) \le C_0 C^{mk_j}(mk_j)!^{mk_j} \quad \text{on } D \tag{1.2}$$

hold for an increasing sequence of natural numbers k_j satisfying $k_{j+1} \le Ak_j$ with some $A > 0$. Then the function $u(x)$ is real analytic on D.

2. Proof of Theorem

Proof. Indeed the theorem follows from simple integration by parts and Bolley-Camus-Metivier's theorem mentioned above.

Since the argument is local, we may consider the case where D is an open ball with center at the origin, and it is sufficient to show that $u(x)$ is real analytic near the origin. Then we assume that $D = B(r)$ where $B(r) = \{x \in \mathbb{R}^n \mid |x| < r\}$ with $r > 0$. First of all, we remark that $u(x)$ is C^∞ even if the inequalities (1.2) are satisfied in distribution sense. Indeed since (1.2) implies that $P^{k_j}u$ is a measure and P^{k_j} is a mk_j-th order elliptic operator, we see that $u(x)$ belongs to the Sobolev space $H_{loc}^{mk_j-(n+1)/2}(D)$.

We use cut-off functions $\chi_k(x)$. Let $\chi_k(x)$ $(k = 1, 2, 3, \dots)$ be non-negative smooth functions satisfying the following conditions:

(P-1) $1 \ge \chi_k(x) \ge 0$, $\chi_k(x) = 1$ for $|x| \le r/2$ and $\chi_k(x) = 0$ for $|x| \le 2r/3$

(P-2) For any α with $|\alpha| \le k$, we have

$$\Big|\frac{d^\alpha}{dx^\alpha}\chi_k(x)\Big| \le C_0 C_1^{|\alpha|} k^{|\alpha|} \quad \text{on } D. \tag{2.1}$$

where the constants C_0, C_1 are independent of k and α. (See [5])

Then, noting that $P^{k_j}u(x) - C_0 C_1^{mk_j}(mk_j)!^{mk_j} \le 0$ and (P-1), we have

$$\begin{aligned} &\int_D \chi_{mk_j}(x)\Big(P^{k_j}u(x) - C_0 C_1^{mk_j}(mk_j)!^{mk_j}\Big)\,dx \\ &\le \int_{|x|\le r/2}\Big(P^{k_j}u(x) - C_0 C_1^{mk_j}(mk_j)!^{mk_j}\Big)\,dx \le 0. \end{aligned} \tag{2.2}$$

Through the integration by parts, we see that the left hand side is equal to

$$\int_D \Big(({}^tP)^{k_j}\chi_{mk_j}(x)\Big)u(x)\,dx - CC_0 C_1^{mk_j}(mk_j)!^{mk_j}$$

where ${}^tP$ is the transposed operator of P. Since the coefficients of P are real analytic, it follows from (2.1) that

$$\big|({}^tP)^{k_j}\chi_{mk_j}(x)\big| \le K_0 K_1^{mk_j}(mk_j)^{mk_j},$$

with some constants K_0, K_1, see for example [5, Lemma 8.6.3]. Then we see that the absolute value of the left hand side of (2.2) is not greater than

$$K_0 K_1^{mk_j}(mk_j)^{mk_j}|D|(\|u(x)\|_{L^\infty(D)} + 1).$$

Here we replace the constants K_0, K_1 by larger constants, if necessary.

While $P^{k_j}u(x) - C_0C_1^{mk_j}(mk_j)!^{mk_j} \le 0$ implies

$$\begin{aligned}
&\int_{|x|\le r/2} |P^{k_j}u(x)|\,dx \\
&\le (-1)\int_{|x|\le r/2}\Big(P^{k_j}u(x) - C_0C_1^{mk_j}(mk_j)!^{mk_j}\Big)\,dx + C_rC_0C_1^{mk_j}(mk_j)!^{mk_j},
\end{aligned}$$

where the first term of the right hand side is not greater than

$$K_0K_1^{mk_j}(mk_j)^{mk_j}|D|(\|u(x)\|_{L^\infty(D)}+1).$$

Hence we have

$$\int_{|x|\le r/2} |P^{k_j}u(x)|\,dx \le K_0K_1^{mk_j}(mk_j)^{mk_j}|D|(\|u(x)\|_{L^\infty(D)}+1).$$

with some positive constants K_0, K_1. From the above L^1-estimates, we see that $u(x)$ is real analytic on a neighborhood of the origin thanks to Bolley-Camus-Metivier's theorem [3]. Indeed, according to [4, Theorem 1.2], we see that [3, Proposition 3.3] is still valid using L^1 estimates for $P^n u$. Then we have the desired conclusion. The proof is complete. □

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