UNIQUENESS OF A SYMMETRIC POSITIVE SOLUTION TO
AN ODE SYSTEM

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In memory of Jack K. Hale (1928–2009)

Abstract. In this article, we prove uniqueness of symmetric positive solutions
of the variational ODE system

\[-w'' + aw - wv = 0,
- v'' + bv - \frac{w^2}{2} = 0,\]

where \(a\) and \(b\) are positive constants.

1. Introduction and Statement of the Result

In this article, we prove uniqueness of symmetric positive solutions of the vari-
tional ODE system

\[-w'' + aw - wv = 0,
- v'' + bv - \frac{w^2}{2} = 0,\]  \(1.1\)

where \(a\) and \(b\) are positive constants. The solutions under consideration are defined
for all \(x \in \mathbb{R}\) and have finite energy.

To show how \(1.1\) arises, we consider the so-called \(\chi^2\) SHG equations

\[
\begin{align*}
 i \frac{\partial w}{\partial t} + r \frac{\partial^2 w}{\partial x^2} - \theta w + w^* v &= 0, \\
 i \sigma \frac{\partial v}{\partial t} + s \frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{w^2}{2} &= 0,
\end{align*}
\]

where \(r, s, \sigma, \theta\) are positive real parameters and \(w(x)\) and \(v(x)\) are complex func-
tions. This system governs phenomena in nonlinear optics (see [5] for instance).

A solitary wave is a solution of \(1.2\) of the form

\((w(x)e^{i\gamma t}, v(x)e^{2i\gamma t})\).
Hence, \((w, v)\) satisfies
\[
-rw'' + (\theta + \gamma)w - w^* v = 0
\]
\[
-sv'' + (\alpha + 2\sigma\gamma)v - \frac{w^2}{2} = 0.
\]

The solutions of (1.3) are critical points of \(E + \gamma I\) where \(E\) and \(I\) are the following conserved quantities for (1.2)
\[
E(w, v) = \int_{-\infty}^{+\infty} \left( r|w'|^2 + s|v'|^2 + \theta|w|^2 + \alpha|v|^2 - \text{Re}(w^2v^*) \right) dx,
\]
(1.4)
\[
I(w, v) = \int_{-\infty}^{+\infty} \left( |w|^2 + 2\sigma|v|^2 \right) dx.
\]
(1.5)

If \(w\) and \(v\) are real solutions of (1.3) then it solves
\[
-rw'' + (\theta + \gamma)w - wv = 0
\]
\[
-sv'' + (\alpha + 2\sigma\gamma)v - \frac{w^2}{2} = 0.
\]
(1.6)

Replacing \((w, v)\) by \((k_1w, k_2v)\) in (1.6), with \(k_2 = r\) and \(k_1^2 = rs\), we get
\[
-w'' + \frac{(\theta + \gamma)}{r}w - wv = 0
\]
\[
-v'' + \frac{(\alpha + 2\sigma\gamma)}{s}v - \frac{w^2}{2} = 0.
\]

Therefore, we consider the real variational ODE system
\[
-w'' + aw - wv = 0
\]
(1.7)
\[
-v'' + bv - \frac{w^2}{2} = 0
\]
(1.8)

and we will be interested in solutions that have finite energy (or equivalently, tend to zero as \(|x|\) tends to infinity). The existence of positive solutions of (1.7)-(1.8) has been proved in [6]. Briefly the argument goes as follows. We define \(H = H^1(\mathbb{R}) \times H^1(\mathbb{R})\) equipped with the norm
\[
\int_{-\infty}^{+\infty} (w'^2(x) + v'^2(x) + aw^2(x) + bv^2(x)) dx.
\]

We consider the functionals
\[
E(w, v) = \int_{-\infty}^{+\infty} (w'^2(x) + v'^2(x) - w^2(x)v(x)) dx,
\]
\[
I(w, v) = \int_{-\infty}^{+\infty} (aw^2(x) + bv^2(x)) dx.
\]

Using the method of concentration-compactness ([3]), we minimize \(E(w, v)\) under \(I(w, v) = 1\) in the space \(H\). If we replace \((w(x), v(x))\) by \((|w(x)|, |v(x)|)\) then \(E\) does not increase. Therefore, any minimizer is nonnegative and solves the Euler-Lagrange system
\[
-w'' + \mu aw - wv = 0
\]
(1.9)
\[
-v'' + \mu bv - \frac{w^2}{2} = 0
\]
(1.10)
with $\mu \geq 0$ (because $(w, v) \in H$ of (1.9)-(1.10) with $\mu = 0$ is the solution identically zero. Therefore, we must have $\mu > 0$. Defining a new pair $(k_1 w(k_3 x), k_2 v(k_3 x))$ with $k_3 = 1/\mu$, we see that this new pair satisfies (1.7)-(1.8).

In [4] the symmetry of any positive solution of (1.7)-(1.8) has been proved using a result of [1]. However, as pointed out in [1], their proof works for $N \geq 2$. Since we are in dimension one, we need the following modified version given in [2].

**Theorem 1.1.** Consider the system

\[
\begin{align*}
  w'' + f(w, v) &= 0 \\
  v'' + g(w, v) &= 0
\end{align*}
\]

where $f(w, v)$ and $g(w, v)$ are $C^1$ functions satisfying the conditions:

\[
f(0, 0) = 0 = g(0, 0), \quad \frac{\partial f(w, v)}{\partial v}, \frac{\partial g(w, v)}{\partial w} \geq 0.
\]

Suppose that there exist $\epsilon > 0$ and $\delta > 0$ such that $w > 0$, $v > 0$, $w^2 + v^2 < \epsilon$ imply

\[
\frac{\partial f(w, v)}{\partial w}, \frac{\partial g(w, v)}{\partial w} < -\delta, \quad 0 < \frac{\partial f(w, v)}{\partial v}, \frac{\partial g(w, v)}{\partial w} < \delta.
\]

Then, except for translations, any positive solution of (1.11) is even and decreasing.

We conclude that, except for translations, any positive solution of (1.7)-(1.8) is symmetric and decreasing.

In [4] we have also proved the following result.

**Theorem 1.2.** The linearized operator of (1.7)-(1.8) at any positive symmetric solution has zero as a simple eigenvalue with odd eigenfunctions $(w_x, v_x)$ and it has exactly one negative eigenvalue.

The fact that zero is a simple eigenvalue of the linearized operator is not a proof of uniqueness of symmetric positive solution, but it may suggest it. Our main result is that this is indeed the case.

**Theorem 1.3.** For $a, b > 0$, the positive symmetric decreasing solution of (1.7)-(1.8) is unique.

Several interesting numerical experiments concerning system (1.7)-(1.8) are presented in [6]. They indicate uniqueness of positive solution (which is confirmed by Theorem 1.3) and that (1.7)-(1.8) may have solutions that change sign.

2. Proof of main result

First we establish the following abstract uniqueness result.

**Theorem 2.1.** Let $X$ be a Banach space and $F : X \times [0, 1] \to X$ be a continuous function with continuous Frechet derivative with respect to the first variable. Also assume that

(i) the set of the solutions $(u, \lambda)$ of $F(u, \lambda) = 0$, $u \in X, \lambda \in [0, 1]$ is precompact;

(ii) for any solution of $F(u, \lambda) = 0$, the derivative $F_u(u, \lambda)$ is invertible;

(iii) the equation $F(u, 0) = 0$ has a unique solution.

Then the equation $F(u, \lambda) = 0$ has a unique solution for $\lambda \in [0, 1]$. 

Proof. First we claim that there is a $\lambda_0 > 0$ such that the solution of $F(u, \lambda) = 0$ is unique for $0 \leq \lambda < \lambda_0$. In fact, otherwise, there is a sequence $0 < \lambda_n \to 0$ such that $F(u, \lambda_n) = 0$ has at least two distinct solution $u_n$ and $v_n$. In view of assumption (i) and passing to a subsequence if necessary, we can assume that $u_n$ converges to $u$ and $v_n$ converges to $v$. In view of (iii), we must have $u = v$. However, by (ii) and the implicit function theorem, in a neighborhood of $u$, for small $\lambda$, the solution of $F(u, \lambda) = 0$ is unique. This contradiction proves the claim. The same argument shows that the set $A$ of $\lambda$, $0 \leq \lambda \leq 1$, for which the solution of $F(u, \mu) = 0$ is unique for $0 \leq \mu \leq \lambda$ is open. Since by ii) $A$ is clearly closed, $A$ has to be the whole interval $[0, 1]$ and the theorem is proved.

□

Remark. If we take $u \in \mathbb{R}$ and $F(u, \lambda) = u(\lambda u - 1) = \lambda u^2 - u$, we have $F_u(u, \lambda) = 2\lambda - 1$. We see that, except for assumption i), all the others are satisfied but the conclusion of the theorem does not hold. This is so because there is the branch $u = 1/\lambda$ of solutions bifurcating from infinity.

Theorem 1.3 will be a consequence of Theorem 2.1. To verify all its assumptions, we start with the following result.

Lemma 2.2. The system

$$
\begin{align*}
-w'' + aw - vw &= 0 \\
n'' + av - \frac{w^2}{2} &= 0
\end{align*}
$$

(2.1)

($a = b$ in (1.7)-(1.8)) has a unique positive solution with finite energy.

Proof. Defining $z(x) = w(x) - \sqrt{2}v(x)$, multiplying the second equation by $\sqrt{2}$ and subtracting we get

$$-z'' + z + \frac{w}{\sqrt{2}}z = 0.
$$

Multiplying this last equation by $z$ and integrating we get

$$
\int_{-\infty}^{+\infty} (z'^2(x) + z^2(x) + \frac{w}{\sqrt{2}}z(x)^2) \, dx = 0
$$

and this implies $z \equiv 0$ (because $w$ is a positive). Therefore, each component of the solution of (2.1) solves a single second order equation and this implies uniqueness and the lemma is proved. □

To verify the other assumptions of Theorem 2.1, we establish a chain of estimates. Since we wish to find estimates for solutions of (1.7)-(1.8) which remain uniform for $a$ and $b$ in a certain interval, we fix two constants $0 < c_1 < c_2$ and we assume

$$
c_1 \leq a, b \leq c_2.
$$

(2.2)

In the sequel, $d_i$, $1 \leq i \leq \infty$ will indicate constants depending on $c_1$ and $c_2$ only. Let $(w(x), v(x))$ be as in Theorem 1.3. Since

$$
T(w, v, w', v') = -w'^2 - v'^2 + aw^2 + bv^2 - w^2v
$$

(2.3)

is a first integral for (1.7)-(1.8), we must have

$$
-w'^2(x) - v'^2(x) + aw^2(x) + bv^2(x) - w^2(x)v(x) = 0
$$

(2.4)

for any $x$. 

Bound for \( v(0) \). Using the fact the \((w(x), v(x))\) is symmetric, if we set \( x = 0 \) in \([2.4]\) we get
\[
  w^2(0) = \frac{bw^2(0)}{v(0) - a}.
\] (2.5)
In particular \( v(0) > a \). Moreover, \( v''(0) \leq 0 \) (because \( v(x) \) has a maximum at \( x = 0 \)) and then the second equation \([1.8]\) yields
\[
  bv(0) \leq \frac{w^2(0)}{2}.
\] (2.6)
This together with \([2.5]\) implies
\[
  bv(0) \leq \frac{1}{2} \frac{bv^2(0)}{v(0) - a}
\] (2.7)
and finally \( v(0) \leq 2a \) because \( v(0) > a \).

Bound for \( v'(x) \). Multiplying the second equation \([1.8]\) by \( v'(x) \), then for \( x \geq 0 \) we get:
\[
\frac{d}{dx}(-v'(x)^2 + bv^2(x)) = w^2(x)v'(x) \leq 0.
\]
Therefore \(-v'(x)^2 + bv^2(x)\) is decreasing and, since it vanishes at \( +\infty \), we get
\[
  -v'(x)^2 + bv^2(x) \geq 0
\]
and then
\[
  v'(x)^2 \leq bv^2(x) \leq bv^2(0) \leq 4a^2b.
\] (2.8)

Bound for \( w'(x) \). We know \( w'(x) \leq 0 \) and that \( w'(x) \) reaches its minimum when \( w''(x) = 0 \). By the first equation \([1.7]\), this occurs when \( v(x) = a \) and then, from \([2.4]\),
\[
  w'(x)^2 + v'(x)^2 = bv^2(x) \leq bv^2(0) \leq 4a^2b.
\]
We conclude
\[
  |w'(x)| = -w'(x) \leq 2a\sqrt{b}.
\] (2.9)

Bound for \( w(0) \). Suppose \( w(0) = M \) and \( w(x_0) = M/2 \) for some \( x_0 > 0 \). Since
\[
  w(0) - w(x_0) = -\int_{0}^{x_0} w'(s) \, ds,
\]
then, in view of \([2.9]\), we have \( \frac{M}{2} \leq 2a\sqrt{b}x_0 \) and this implies
\[
  x_0 \geq \frac{M}{4a\sqrt{b}}.
\] (2.10)
Moreover, the solution of the linear equation
\[
  -v''(x) + bv(x) = h(x)
\] (2.11)
is given by
\[
  v(x) = \frac{1}{2\sqrt{b}} \int_{-\infty}^{+\infty} e^{-\sqrt{b}|x-y|} h(y) \, dy,
\] (2.12)
and then, the second equation \((1.8)\) and \((2.10)\) give

\[
v(0) = \frac{1}{4\sqrt{b}} \int_{-\infty}^{+\infty} e^{-\sqrt{b}|y|} w^2(y) dy
\]

\[
= \frac{1}{2\sqrt{b}} \int_0^{+\infty} e^{-\sqrt{b}y} w^2(y) dy
\]

\[
\geq \frac{1}{2\sqrt{b}} \int_0^{x_0} e^{-\sqrt{b}y} w^2(y) dy
\]

\[
\geq \frac{M^2}{8\sqrt{b}} \int_0^{x_0} e^{-\sqrt{b}y} dy
\]

\[
= \frac{M^2}{8b}(1 - e^{-\sqrt{b}x_0})
\]

\[
\geq \frac{M^2}{8b}(1 - e^{-\frac{4}{7}}).
\]

Therefore,

\[
2a \geq v(0) \geq \frac{M^2}{8b}(1 - e^{-\frac{4}{7}})
\]

and this gives that \(M = w(0) \leq d_1\), for some constant \(d_1\). In view of \((2.5)\), this gives also that \(v(0) \geq d_2 > a\), for some constant \(d_2\), and also gives a lower bound for \(w(0) \geq d_3\).

**Bound for the length of the interval for which \(v(x) \geq a\).** By the first equation in \((1.7)\) and the previous estimates for \(v(0)\) and \(w(0)\), we have \(w''(0) \leq -d_4 < 0\) and \(|w''(x)| \leq d_5\). Defining \(X = -\frac{w''(0)}{2d_4}\) then, for \(0 \leq x \leq X\) we have

\[
w''(x) - w''(0) = \int_0^x w'''(s) ds \leq d_5X = -w''(0)/2,
\]

and then \(w''(x) \leq w''(0)/2 \leq -d_4/2\) for \(0 \leq x \leq X\). Moreover,

\[
w'(X) = w'(0) + \int_0^X w''(s) ds \leq \int_0^X w''(0) ds = X w''(0) / 2 = -w''(0)^2 / 4d_5 \leq -d_6.
\]

Since, by \((1.7)\), \(w''(x) \leq 0\) whenever \(v(x) \geq a\), we have \(w'(x) \leq -d_6\) whenever \(v(x) \geq a\) and \(x \geq X\). Furthermore,

\[
-w(0) \leq -w(X) \leq w(x) - w(X) = \int_X^x w'(s) ds \leq -d_6(x - X).
\]

Therefore, defining \(X_1 = w(0)/d_6 + X\), we see that we must have \(v(X_1) \leq a\).

**Estimate for the time \(v(x)\) stays close (and less) than \(a\).** Let \(x_0 \leq X_1\) be such that \(v(x_0) = a\) and let \(d_7 > 0\) and \(d_8 < a\) be such that

\[
(a - v)w^2 + bw^2 \geq d_7^2
\]

whenever \(d_8 \leq v \leq a, w \leq d_1\). Then, if \(d_8 \leq v(x) \leq a\) for \(x_0 \leq x \leq x_0 + X_2\), by \((2.4)\) we have \(-w'(x) - v'(x) \geq d_7\) and then

\[
w(x_0) + v(x_0) \geq -w(x) + w(x_0) - v(x) + v(x_0) \geq d_7X_2
\]

and this gives a uniform upper bound for \(X_2\).
Exponential decay for $w(x)$ and for $v(x)$. Since
\[
\frac{d}{dx} \left(-w'(x)^2 + aw^2(x) - w^2(x)v(x)\right) = -w^2(x)v'(x) \geq 0
\]
the function $-w'(x)^2 + (a - v(x))w^2(x)$ is increasing and then $-w'(x)^2 + (a - v(x))w^2(x) \leq 0$ for all $x \geq 0$ because it vanishes at infinity. Now, for $x \geq X_3 = X_1 + X_2$ we have $-w'(x)^2 + d_9 w(x)^2 \leq 0$ and then $w'(x) + d_9 w(x) \leq 0$ and finally, $w(x) \leq e^{-d_9 (x - X_3)} w(X_3)$ for $x \geq X_3$ and this implies
\[
w(x) \leq d_{10} e^{-d_{10} x}, \quad x \geq 0.
\]
From the second equation (1.8) we get
\[
v(x) = \frac{1}{4 \sqrt{b}} \int_{-\infty}^{+\infty} e^{-\sqrt{b}|x-y|} w^2(y) \, dy
\]
and this together with (2.13) and elementary calculation gives a similar exponential decay
\[
v(x) \leq d_{11} e^{-d_{11} x}, \quad x \geq 0
\]
for $v(x)$.

Proof of Theorem 1.3. Using (2.12) to invert the linear operators $-w'' + aw$ and $-v'' + bv$, we see that system (1.7)-(1.8) can be written as
\[
w(x) = \frac{1}{2 \sqrt{a}} \int_{-\infty}^{+\infty} e^{-\sqrt{a}|x-y|} w(y) v(y) \, dy
\]
\[
v(x) = \frac{1}{4 \sqrt{b}} \int_{-\infty}^{+\infty} e^{-\sqrt{b}|x-y|} w^2(y) \, dy.
\]
Defining $u$ as the pair $(w, v)$, system (2.15) can be viewed as the equation
\[
F(u, \lambda) = 0
\]
where $\lambda$, say, is $b$, with $a$ kept fixed. We denote by $H^1_{ev} \subset H^1(\mathbb{R})$ the subspace of the even functions. If we take $X = H^1_{ev} \times H^2_{ev}$, then $F : X \rightarrow X$ is a well defined very smooth function. In view of Theorem 1.2 assumption ii) of Theorem 2.1 is satisfied because $X$ consists of even functions. Uniqueness for $\lambda = a$ is given by Lemma 2.2. To verify assumption (i) of Theorem 2.1 we recall that a subset $K$ of $X$ is precompact if and only if the following conditions are satisfied:

1. for each $n$ the restriction of the functions of $K$ to the interval $[-n, n]$ is precompact;
2. for every $\epsilon > 0$, there is an $x(\epsilon) > 0$ such that for all $u \in K$ we have
\[
\int_{|x| \geq x(\epsilon)} \left( |u'|^2(x) + |u(x)|^2 \right) \, dx < \epsilon.
\]
To verify these conditions we first notice that we have obtained uniform bound for the $H^1(\mathbb{R})$ norm of the solution $(w, v)$ of (1.7)-(1.8). This implies uniform bound for the $H^2$ norm of such solutions and this verifies condition (1) for precompactness. The uniform exponential decay (2.13) and (2.14) for $w(x)$ and $v(x)$ together with (2.3) gives the uniform exponential decay also for the derivatives. This implies that condition (2) for precompactness is satisfied; therefore, Theorem 1.3 is proved. $\square$
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