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AN INCORRECTLY POSED PROBLEM FOR NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We study properties of solutions to non-linear elliptic problems involving the Laplace operator on the unit sphere. In particular, we show that solutions do not depend continuously on the initial data.

1. Introduction

In this paper we study properties of solutions to the initial-value problem

$$u_{rr}+\frac{n-1}{r}u_r+\frac{1}{r^2}\Delta_S u=f(r,u),\quad r\geq r_0,\tag{1.1}$$

$$u|_{r=r_0}=u_0\in X,\quad u_r|_{r=r_0}=u_1\in Y,\tag{1.2}$$

where $n\geq 2$, $r_0\geq 1$ is suitable chosen and fixed number, X and Y are Banach spaces, $f\in\mathcal{C}([r_0,\infty))\times\mathcal{C}^1(\mathbb{R}^1)$, $f(r,0)=0$ for every $r\geq r_0$, $a|u|\leq f'_u(r,u)\leq b|u|$ for every $r\geq r_0$, $u\in\mathbb{R}^1$, a and b are positive constants, Δ_S is the Laplace operator on the unit sphere S^{n-1}. More precisely we prove that the initial-value problem (1.1)-(1.2) is incorrectly posed in the following sense.

When we say that (1.1)-(1.2) is incorrectly posed when the following happens: (1.1)-(1.2) has exactly one solution $u(r)\in X$ for each $u_0\in X$, $u_1\in Y$; there exists $\epsilon>0$ such that for every $\delta>0$, we have: $\|u_0-u'_0\|_X<\delta$, $\|u_1-u'_1\|_Y<\delta$ and $\|u-u'\|_X\geq\epsilon$, where u is a solution with initial data u_0,u_1, and u' is a solution with initial data u'_0,u'_1.

In this article, we obtain the following results using the same approach as in [3, 4, 5, 6],

Theorem 1.1. *Let $n\geq 2$, $r_0\geq 1$, $f\in\mathcal{C}([r_0,\infty))\times\mathcal{C}^1(\mathbb{R}^1)$ $f(r,0)=0$ for every $r\geq r_0$, and $X=Y=L^2(S^{n-1})$. Assume that there are positive constants, $a\leq b$, such that $a|u|\leq f'_u(r,u)\leq b|u|$ for every $r\geq r_0$ and every $u\in\mathbb{R}$. Then* (1.1)-(1.2) *is incorrectly posed.*

Theorem 1.2. *Let $n\geq 2$, $r_0\geq 1$, $f\in\mathcal{C}([r_0,\infty))\times\mathcal{C}^1(\mathbb{R}^1)$, $f(r,0)=0$ for every $r\geq r_0$, $X=\mathcal{C}^2(S^{n-1})$ and $Y=\mathcal{C}^1(S^{n-1})$. Assume that there are positive constants, $a\leq b$, such that $a|u|\leq f'_u(r,u)\leq b|u|$ for every $r\geq r_0$ and every $u\in\mathbb{R}$. Then* (1.1)-(1.2) *is incorrectly posed.*

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This paper is organized as follows. In section 2 we prove our main results. In the appendix we prove results needed for the proof of Theorems 1.1 and 1.2.

2. Proof of Main Results

Here and bellow we will assume that $r_0 \geq 1$ and $n \geq 2$. First we will consider the initial-value problem

$$u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2}\Delta_S u = f(u), \quad r \geq r_0, \tag{2.1}$$

$$u(r)|_{r=r_0} = u_0 \in L^2(S^{n-1}), u_r(r)|_{r=r_0} = u_1 \in L^2(S^{n-1}), \tag{2.2}$$

where Δ_S is the Laplace operator on the unit sphere S^{n-1}, $f \in \mathcal{C}^1(\mathbb{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$ for every $u \in \mathbb{R}^1$, $a \leq b$ are fixed positive constants.

For fixed positive constants $n \geq 2$, $r_0 \geq 1$, a, b, $a \leq b$, we suppose that the constants A, B, c_1, d_1 satisfy the following conditions

$$\begin{gathered} r_0 \leq c_1 \leq d_1, \\ A \geq B > 0, \\ \frac{a}{2A}\frac{d_1^n}{(d_1+1)^n} \geq 1. \end{gathered} \tag{2.3}$$

Example. Let $n \geq 1$, $r_0 \gg 1$, $A = 2$, $B = 1$, $a = r_0^{10n}$,$b = 2r_0^{10n}$, $c_1 = r_0 + 1$, $d_1 = r_0 + 2$.

Let N be the set

$$\begin{aligned} N = \Big\{ & u(r) : u(r) \in \mathcal{C}^2([r_0, \infty)),\ u(\infty) = u_r(\infty) = 0, \\ & r^\alpha |\partial_r^\beta u(r)| \leq 1\ \forall r \geq r_0,\ \forall \alpha \in \mathbb{N} \cup \{0\}, \beta = 0, 1, \\ & u(r) \geq 0\ \forall r \geq r_0, u(r) \leq \frac{1}{B}\ \forall r \geq r_0, \\ & u(r) \geq \frac{1}{A}\ \forall r \in [c_1, d_1],\ u(r) \in L^2([r_0, \infty)) \Big\}. \end{aligned}$$

For $n \geq 1$, $f(u) \in \mathcal{C}^1(\mathbb{R}^1)$, $a|u| \leq f'(u) \leq b|u|$, where $a \geq b$ are positive constants, and $u \in N$ we define the operator and the initial values

$$P(u) = \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n f(u) d\tau ds,$$

$$u_0 = \int_{r_0}^\infty \frac{1}{s^n} \int_s^\infty \tau^n f(u) d\tau ds, \quad u_1 = -\frac{1}{r_0^n} \int_{r_0}^\infty \tau^n f(u) d\tau.$$

Theorem 2.1. *Let $n \geq 2$, $r_0 \geq 1$, $f \in \mathcal{C}^1(\mathbb{R}^1)$, and $f(0) = 0$. Assume that there exist positive constants $a \leq b$ such that $a|u| \leq f'(u) \leq b|u|$. Then* (2.1)-(2.2) *has exactly one solution $u \in N$.*

Proof. First we prove that $P : N \to N$. Let $u \in N$ be fixed. Then
(1) Since $f \in \mathcal{C}^1([r_0, \infty))$, $u \in \mathcal{C}^2([r_0, \infty))$ we have that $P(u) \in \mathcal{C}^2([r_0, \infty))$. Also

we have

$$P(u)_{|r=\infty}=0,$$
$$\frac{\partial P(u)}{\partial r}=-\frac{1}{r^n}\int_r^\infty \tau^n f(u)d\tau,$$
$$\frac{\partial P(u)}{\partial r}_{|r=\infty}=0.$$

(2) Let $\alpha\in\mathbb{N}\cup\{0\}$. We choose $k\in\mathbb{N}$ such that $k\geq\alpha+3$ and $\frac{b}{2B(k-1)}<1$. Then

$$r^\alpha P(u)=r^\alpha\int_r^\infty\frac{1}{s^n}\int_s^\infty\tau^n f(u)d\tau ds\,.$$

Now we use that for $u\in N$, we have $u\geq 0$ for every $r\geq r_0$, $f(0)=0$, $f'(u)\leq bu$, from here $f(u)\leq\frac{b}{2}u^2$, since $u\leq\frac{1}{B}$ for every $r\geq r_0$ we get $f(u)\leq\frac{b}{2B}u$. Then

$$\begin{aligned}
r^\alpha P(u)&\leq\frac{b}{2B}r^\alpha\int_r^\infty\frac{1}{s^n}\int_s^\infty\tau^n ud\tau ds\\
&=r^\alpha\frac{b}{2B}\int_r^\infty\frac{1}{s^n}\int_s^\infty\tau^{n+k}\frac{1}{\tau^k}ud\tau ds\quad(\text{use that }\tau^{n+k}u\leq 1)\\
&\leq\frac{b}{2B}r^\alpha\int_r^\infty\frac{1}{s^n}\int_s^\infty\frac{1}{\tau^k}d\tau ds\\
&\leq\frac{b}{2B}\frac{1}{(k-1)(n+k-2)}\frac{1}{r_0^{n+k-\alpha-2}}\leq 1.
\end{aligned}$$

In the above inequality we use our choice of the constant k. Also,

$$\begin{aligned}
\big|r^\alpha\frac{\partial P(u)}{\partial r}\big|&\leq\frac{b}{2B}r^\alpha\frac{1}{r^n}\int_r^\infty\tau^n ud\tau\\
&=r^\alpha\frac{b}{2B}\frac{1}{r^n}\int_r^\infty\tau^{n+k}\frac{1}{\tau^k}ud\tau\quad(\text{use }\tau^{n+k}u\leq 1)\\
&\leq r^\alpha\frac{b}{2B}\int_r^\infty\frac{1}{s^n}\int_s^\infty\frac{1}{\tau^k}d\tau ds\\
&\leq\frac{b}{2B}\frac{1}{(k-1)}\frac{1}{r_0^{n+k-\alpha-1}}\leq 1.
\end{aligned}$$

In the above inequality we use our choice of the constant k.
(3) First we note that for $u\in N$ we have $f(u)\geq au^2/2$. Therefore for every $r\geq r_0$ we have

$$P(u)\geq\frac{a}{2}\int_r^\infty\frac{1}{s^n}\int_s^\infty\tau^n u^2 d\tau ds\geq 0.$$

(4) Let $r\in[c_1,d_1]$. Then

$$P'(u)=\int_r^\infty\frac{1}{s^n}\int_s^\infty\tau^n f'(u)d\tau ds\geq a\int_r^\infty\frac{1}{s^n}\int_s^\infty\tau^n ud\tau ds\geq 0.$$

Therefore, for $u \in N$ the function $P(u)$ is increase function of u. Since for every $r \in [c_1, d_1]$ we have that $u \geq 1/A$ we get

$$\begin{aligned} P(u) \geq P(\frac{1}{A}) &= \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n f\big(\frac{1}{A}\big) d\tau ds \\ &\geq \frac{a}{2A^2} \int_{d_1}^{d_1+1} \frac{1}{s^n} \int_{d_1}^{d_1+1} \tau^n d\tau ds \\ &\geq \frac{a}{2A^2} \frac{d_1^n}{(d_1+1)^n} \geq \frac{1}{A}, \end{aligned}$$

in the above inequality we use (2.3).
(5) Choose $k \in \mathbb{N}$ such that

$$k > 3, \quad \frac{b}{2(k-1)(n+k-2)} < 1.$$

Then

$$\begin{aligned} P(u) &\leq \frac{b}{2B} \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n u d\tau ds \\ &\leq \frac{b}{2B} \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^{k+n} \frac{1}{\tau^k} u d\tau ds \\ &\leq \frac{b}{2B} \int_r^\infty \frac{1}{s^n} \int_s^\infty \frac{1}{\tau^k} d\tau ds \\ &= \frac{b}{2B(k-1)(n+k-2) r_0^{n+k-2}} \leq \frac{1}{B}. \end{aligned}$$

(6) Now we prove that $P(u) \in L^2([r_0, \infty))$. Indeed,

$$\begin{aligned} \|P(u)\|^2_{L^2([r_0,\infty))} &= \int_{r_0}^\infty \Big(\int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n f(u) d\tau ds \Big)^2 dr \\ &\leq \frac{b^2}{4} \int_{r_0}^\infty \Big(\int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n u^2 d\tau ds \Big)^2 dr \\ &\leq \frac{b^2}{4} \int_{r_0}^\infty \Big(\int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^{k+n} u \frac{u}{\tau^k} d\tau ds \Big)^2 dr \quad (\text{use that } \tau^{k+n} u \leq 1) \\ &\leq \frac{b^2}{4} \int_{r_0}^\infty \Big(\int_r^\infty \frac{1}{s^n} \int_s^\infty \frac{u}{\tau^k} d\tau ds \Big)^2 dr \leq \quad (\text{use Hölder's inequality}) \\ &\leq \frac{b^2}{4} \int_{r_0}^\infty \Big(\int_r^\infty \frac{1}{s^n} \Big(\int_s^\infty \frac{1}{\tau^{2k}} d\tau \Big)^{1/2} \Big(\int_s^\infty u^2 d\tau \Big)^{1/2} ds \Big)^2 dr \\ &\leq \frac{b^2}{4(2k-1)(n+k-\frac{3}{2})^2 (2n+2k-4) r_0^{2n+2k-4}} \|u\|^2_{L^2([r_0,\infty))} < \infty, \end{aligned}$$

because $u \in L^2([r_0, \infty))$. From (1)–(6) we conclude that $P : N \to N$.

Now we prove that the operator P has exactly one fixed point in N. Let $u_1, u_2 \in N$ are fixed and $\alpha = \|u_1 - u_2\|_{L^2([r_0,\infty))}$. We choose the constant $k \in \mathbb{N}$ large so that $Q_1/\alpha < 1$, where

$$Q_1 = \frac{2b^2}{B^2 (\frac{4}{3}k - 1)^{\frac{3}{2}} (n+k-\frac{7}{4})^2 (2n+2k-\frac{9}{2}) r_0^{2n+2k-\frac{9}{2}}}.$$

Then

$$
\begin{aligned}
&\|P(u_1)-P(u_2)\|^2_{L^2([r_0,\infty))}\\
&=\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n(f(u_1)-f(u_2))d\tau ds\Big)^2 dr \quad \text{(mean value theorem)}\\
&=\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n f'(\xi)(u_1-u_2)d\tau ds\Big)^2 dr\\
&\le\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n |f'(\xi)||u_1-u_2|d\tau ds\Big)^2 dr\\
&\quad \text{(use that } |f'(\xi)|\le b|\xi|\le\frac{b}{B},\ |\xi|\le\max\{|u_1|,|u_2|\})\\
&\le\frac{b^2}{B^2}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n|u_1-u_2|d\tau ds\Big)^2 dr\\
&=\frac{b^2}{B^2}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\sqrt{\tau^{2k+2n}|u_1-u_2|}\frac{1}{\tau^k}\sqrt{|u_1-u_2|}d\tau ds\Big)^2 dr\\
&\quad \text{(use that } \sqrt{\tau^{2k+2n}|u_1-u_2|}\le\sqrt{2})\\
&\le\frac{2b^2}{B^2}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\sqrt{|u_1-u_2|}\frac{1}{\tau^k}d\tau ds\Big)^2 dr \quad \text{(Hölder's inequality)}\\
&\le\frac{2b^2}{B^2}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\Big(\int_s^{\infty}\frac{1}{\tau^{\frac{4k}{3}}}d\tau\Big)^{3/4}\Big(|u_1-u_2|^2 d\tau\Big)^{1/4}ds\Big)^2 dr\\
&\le Q_1\|u_1-u_2\|_{L^2([r_0,\infty))};
\end{aligned}
$$

i.e.,

$$\|P(u_1)-P(u_2)\|^2_{L^2([r_0,\infty))}\le Q_1\|u_1-u_2\|_{L^2([r_0,\infty))}.$$

From this,

$$\|P(u_1)-P(u_2)\|^2_{L^2([r_0,\infty))}\le\frac{Q_1}{\alpha}\alpha\|u_1-u_2\|_{L^2([r_0,\infty))}\le\frac{Q_1}{\alpha}\|u_1-u_2\|^2_{L^2([r_0,\infty))}.$$

For our next step we need the theorem in [8, page 294]:

> Let B be the complete metric space for which $AB\subset B$ and for the operator A satisfies the condition
>
> $$\rho(Ax,Ay)\le L(\alpha,\beta)\rho(x,y),\quad x,y\in B,\alpha\le\rho(x,y)\le\beta,$$
>
> where $L(\alpha,\beta)<1$ for $0<\alpha\le\beta<\infty$. Then the operator A has exactly one fixed point in B.

From the above result and our choice of k we conclude that the operator P has exactly one fixed point $u\in N$. Consequently u is a solution to the problem (2.1)-(2.2). In the appendix we will prove that the set N is closed subset of the space $L^2([r_0,\infty))$. We have that $u_0\in L^2(S^{n-1})$, $u_1\in L^2(S^{n-1})$. □

Theorem 2.2. *Let $n\ge 2$, $r_0\ge 1$, $f\in\mathcal{C}^1(\mathbb{R}^1)$, and $f(0)=0$. Assume that there exists positive constants, $a\le b$, such that $a|u|\le f'(u)\le b|u|$. Then* (2.1)-(2.2) *is incorrectly posed.*

Proof. On the contrary, suppose that (2.1)-(2.2) is correctly posed. Let u is the solution from Theorem 2.1. We choose ϵ such that $0 < \epsilon < 1/Q_2$, where

$$Q_2 = \frac{b^2}{4(4k-1)^{1/2}(n+k-\frac{5}{4})^2(2n+2k-\frac{7}{2})r_0^{2n+2k-\frac{7}{2}}}.$$

Then there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|u_0\|_{L^2(S^{n-1})} < \delta, \quad \|u_1\|_{L^2(S^{n-1})} < \delta$$

imply

$$\|u\|_{L^2([r_0,\infty))} < \epsilon.$$

From the definition of u, we have

$$\begin{aligned}
\|u\|^2_{L^2([r_0,\infty))} &= \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n f(u)d\tau ds\Big)^2 dr \\
&\le \frac{b^2}{4}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n u^2 d\tau ds\Big)^2 dr \\
&= \frac{b^2}{4}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\sqrt{\tau^{2k+2n}u}u^{\frac{3}{2}}\frac{1}{\tau^k}d\tau ds\Big)^2 dr \\
&\quad (\text{use that } \sqrt{\tau^{2k+2n}u} \le 1) \\
&\le \frac{b^2}{4}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}u^{\frac{3}{2}}\frac{1}{\tau^k}d\tau ds\Big)^2 dr \quad (\text{Hölder's inequality}) \\
&\le \frac{b^2}{4}\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\Big(\int_s^{\infty}u^2 d\tau\Big)^{3/4}\Big(\frac{1}{\tau^{4k}}d\tau\Big)^{1/4}ds\Big)^2 dr \\
&\le Q_2\|u\|^3_{L^2([r_0,\infty))};
\end{aligned}$$

i.e.,

$$\|u\|^2_{L^2([r_0,\infty))} \le Q_2\|u\|^3_{L^2([r_0,\infty))}.$$

From this,,

$$\|u\|_{L^2([r_0,\infty))} \ge \frac{1}{Q_2} > \epsilon$$

which is a contradiction. Consequently the problem (2.1)-(2.2) is incorrectly posed. □

Theorem 2.3. *Let $n \ge 2$, $r_0 \ge 1$, $f \in \mathcal{C}^1(\mathbb{R}^1)$, and $f(0) = 0$. Assume that there are positive constants $a \le b$ such that $a|u| \le f'(u) \le b|u|$. Then the problem*

$$u_{rr} + \frac{n-1}{r}u_r + \frac{1}{r^2}\Delta_S u = f(u), \quad r \ge r_0, \tag{2.4}$$

$$u(r)_{|r=r_0} = u_0 \in \mathcal{C}^2(S^{n-1}), \quad u_r(r)_{|r=r_0} = u_1 \in \mathcal{C}^1(S^{n-1}), \tag{2.5}$$

is incorrectly posed.

Proof. Let us suppose that (2.4)-(2.5) is correctly posed, and let

$$Q_3 = \frac{b}{2(k-1)(n+k-2)r_0^{n+k-2}}.$$

Then for $0 < \epsilon < 1/Q_3^2$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|u_0\|_{\mathcal{C}^2(S^{n-1})} < \delta, \quad \|u_1\|_{\mathcal{C}^1(S^{n-1})} < \delta,$$

imply

$$\max_{r\in[r_0,\infty)}|u|<\epsilon,\quad \max_{r\in[r_0,\infty)}|u_r|<\epsilon,\quad \max_{r\in[r_0,\infty)}|u_{rr}|<\epsilon,$$

where u is the solution from the Theorem 2.1. From the definition of u, and $k\in\mathbb{N}$, we have

$$\begin{aligned}
u(r) &\le \frac{b}{2}\int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n u^2 d\tau ds\\
&= \frac{b}{2}\int_r^\infty \frac{1}{s^n}\int_s^\infty \sqrt{\tau^{2k+2n}u}u^{\frac{3}{2}}\frac{1}{\tau^k} d\tau ds\\
&\le \frac{b}{2}\int_r^\infty \frac{1}{s^n}\int_s^\infty u^{\frac{3}{2}}\frac{1}{\tau^k} d\tau ds\\
&\le \frac{b}{2}(\max_{r\in[r_0,\infty)} u)^{\frac{3}{2}}\int_r^\infty \frac{1}{s^n}\int_s^\infty \frac{1}{\tau^k} d\tau ds\\
&\le Q_3(\max_{r\in[r_0,\infty)} u)^{\frac{3}{2}}.
\end{aligned}$$

From this it follows that

$$Q_3(\max_{r\in[r_0,\infty)} u)^{1/2}\ge 1,\quad \text{or}\quad \max_{r\in[r_0,\infty)} u > \frac{1}{Q_3^2}>\epsilon,$$

which is a contradiction with our assumption. Consequently (2.4)-(2.5) is incorrectly posed. □

The proofs of Theorems 1.1 and 1.2 follow from the method used in the proof of Theorems 2.2 and 2.3.

3. Appendix

Lemma 3.1. *The set N is a closed subset of $L^2([r_0,\infty))$.*

Proof. Let $\{u_n\}$ is a sequence of elements in N for which

$$\lim_{n\to\infty}\|u_n-\tilde{u}\|_{L^2([r_0,\infty))}=0,$$

where $\tilde{u}\in L^2([r_0,\infty))$. Since $P(u)$ is a continuous differentiable function of u, for $r\in[r_0,c_1]$ and $u\in N$ we have

$$\begin{aligned}
P'(u) &= \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n f'(u) d\tau ds\\
&\ge a\int_{c_1}^{d_1}\frac{1}{s^n}\int_{c_1}^{d_1}\tau^n u d\tau ds\\
&\ge \frac{a}{A}\frac{c_1^n}{d_1^n}(d_1-c_1)^2.
\end{aligned}$$

From this, it follows that for every $u\in N$ there exists

$$L=\min_{r\in[r_0,c_1]}|P'(u)(r)|>0.$$

Let

$$M_1=\max_{r\in[r_0,c_1]}\Big|\frac{\partial}{\partial r}P'(u)(r)\Big|.$$

Now we prove that for every $\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that from $|x-y|<\delta$ we have

$$|u_m(x)-u_m(y)|<\epsilon\quad \forall m\in\mathbb{N}.$$

We suppose that there exists $\tilde{\epsilon} > 0$ such that for every $\delta > 0$ there exist natural number m and $x, y \in [r_0, \infty)$, $|x - y| < \delta$ for which $|u_m(x) - u_m(y)| \geq \tilde{\epsilon}$. We choose $\tilde{\tilde{\epsilon}}$ such that $0 < \tilde{\tilde{\epsilon}} < L\tilde{\epsilon}$. We note that $P(u_m)(x)$ is uniformly continuous for $x \in [r_0, \infty)$. For $u \in N$ $P(u)(r)$ is uniformly continuous function for $r \in [r_0, \infty)$ because $P(u)(r) \in \mathcal{C}([r_0, \infty))$ and as in the proof of the Theorem 2.1 we have that there exists positive constant C such that $\left|\frac{\partial}{\partial r} P(u)(r)\right| \leq C$. Then there exists $\delta_1 = \delta_1(\tilde{\tilde{\epsilon}}) > 0$ such that for every natural m we have

$$|P(u_m)(x) - P(u_m)(y)| < \tilde{\tilde{\epsilon}}, \quad \forall x, y \in [r_0, \infty) : |x - y| < \delta_1.$$

Consequently we can choose

$$0 < \delta < \min\{c_1 - r_0, \delta_1, \frac{(L\tilde{\epsilon} - \tilde{\tilde{\epsilon}})B}{M_1}\}$$

such that there exist natural number m and $x_1, x_2 \in [r_0, \infty)$ for which

$$|x_1 - x_2| < \delta, \quad |u_m(x_1 - x_2 + r_0) - u_m(r_0)| \geq \tilde{\epsilon}.$$

In particular,

$$|P(u_m)(x_1 - x_2 + r_0) - P(u_m)(r_0)| < \tilde{\tilde{\epsilon}}. \tag{3.1}$$

Let us suppose for convenience that $x_1 - x_2 > 0$. Then $x_1 - x_2 < c_1 - r_0$ and for every $u \in N$ we have $P'(u)(x_1 - x_2 + r_0) \geq L$. Then from the middle point theorem we have $P(0) = 0$, $P(u_m)(x_1 - x_2 + r_0) = P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0)$, $P(u_m)(r_0) = P'(\xi)(r_0)u_m(r_0)$,

$$\begin{aligned}
&|P(u_m)(x_1 - x_2 + r_0) - P(u_m)(r_0)| \\
&= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(r_0)u_m(r_0)| \\
&= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) \\
&\quad + P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) - P'(\xi)(r_0)u_m(r_0)| \\
&\geq |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0)| \\
&\quad - |P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) - P'(\xi)(r_0)u_m(r_0)| \\
&= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0)| \\
&\quad - \Big|\frac{\partial}{\partial r} P'(\xi)\Big| |x_1 - x_2| |u_m(r_0)| \\
&\geq L\tilde{\epsilon} - M_1 \delta \frac{1}{B} \geq \tilde{\tilde{\epsilon}},
\end{aligned}$$

which is a contradiction with (3.1). Therefore, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that from $|x - y| < \delta$ follows

$$|u_m(x) - u_m(y)| < \epsilon \quad \forall m \in \mathbb{N}. \tag{3.2}$$

On the other hand from the definition of the set N we have that for every natural number m

$$u_m(r) \leq \frac{1}{B} \quad \forall r \geq r_0. \tag{3.3}$$

From this inequality and (3.2) it follows that the set $\{u_m\}$ is a compact subset of the space $\mathcal{C}([r_0, \infty))$. Therefore there is a subsequence $\{u_{n_k}\}$ and function $u \in \mathcal{C}([r_0, \infty))$ for which

$$|u_{n_k}(x) - u(x)| < \epsilon \quad \forall x \in [r_0, \infty).$$

Now we suppose that that $u \neq \tilde{u}$ a.e. in $[r_0, \infty)$. Then there exist $\epsilon_1 > 0$ and subinterval $\Delta \subset [r_0, \infty)$ such that $\mu(\Delta) > 0$ and

$$|u - \tilde{u}| > \epsilon_1 \quad \text{for } r \in \Delta.$$

Let $\epsilon > 0$ is chosen such that

$$\epsilon < \frac{\epsilon_1 (\mu(\Delta))^{1/2}}{\mu(\Delta)^{1/2} + 1}. \tag{3.4}$$

Then, for every $n_k \in \mathbb{N}$ sufficiently large, we have $\|u_{n_k} - \tilde{u}\|_{L^2([r_0,\infty))} < \epsilon$,

$$\begin{aligned}
\epsilon\mu(\Delta) &= \epsilon \int_\Delta dx \\
&> \int_\Delta |u_{n_k} - u| dx = \int_\Delta |u_{n_k} - \tilde{u} + \tilde{u} - u| dx \\
&\geq \int_\Delta |\tilde{u} - u| dx - \int_\Delta |u_{n_k} - \tilde{u}| dx \\
&\geq \epsilon_1 \mu(\Delta) - \Big(\int_\Delta |u_{n_k} - \tilde{u}|^2 dx \Big)^{1/2} \big(\mu(\Delta)\big)^{1/2} \\
&\geq \epsilon_1 \mu(\Delta) - \|u_{n_k} - \tilde{u}\|_{L^2([r_0,\infty))} \big(\mu(\Delta)\big)^{1/2} \\
&> \epsilon_1 \mu(\Delta) - \epsilon \big(\mu(\Delta)\big)^{1/2},
\end{aligned}$$

which is a contradiction with (3.4). From this, $u = \tilde{u}$ a.e. in $[r_0, \infty)$, $|u_n - u|^2 = |\tilde{u} - u_n|^2$ a.e. in $[r_0, \infty)$, $\|u_n - u\|_{L^2([r_0,\infty))} = \|u_n - \tilde{u}\|_{L^2([r_0,\infty))}$. Consequently, for every sequence $\{u_n\}$ from elements of the set N, which is convergent in $L^2([r_0, \infty))$, there exists a function $u \in \mathcal{C}([r_0, \infty))$, $u \in L^2([r_0, \infty))$ for which

$$\lim_{n \to \infty} \|u_n - u\|_{L^2([r_0,\infty))} = 0.$$

Bellow we will suppose that $\{u_n\}$ is a sequence from elements of the set N, which is convergent in $L^2([r_0, \infty))$. Then there exists a function $u \in \mathcal{C}([r_0, \infty))$, $u \in L^2([r_0, \infty))$ for which

$$\lim_{n \to \infty} \|u_n - u\|_{L^2([r_0,\infty))} = 0.$$

Now we suppose that $u(\infty) \neq 0$. Then there exist sufficiently large $Q > 0$, a large natural number m and $\epsilon_2 > 0$ for which

$$u_m(r) = 0, \quad u(r) > \epsilon_2, \quad \forall r \geq Q.$$

We choose

$$0 < \epsilon_3 < \epsilon_2. \tag{3.5}$$

Then, for every $n \in \mathbb{N}$ sufficiently large, we have $|u_n(r) - u(r)| < \epsilon_3$ and

$$\begin{aligned}
\epsilon_3 &> \int_Q^{Q+1} |u_n(r) - u(r)| dr \\
&\geq \int_Q^{Q+1} (|u(r)| - |u_n(r)|) dr \\
&= \int_Q^{Q+1} |u(r)| dr > \epsilon_2,
\end{aligned}$$

which is a contradiction with (3.5). Therefore, $u(\infty) = 0$.

Now we prove that $\frac{\partial}{\partial r}u(r)$ exists for every $r \geq r_0$. Let us suppose that there exists $r_1 \in [r_0, \infty)$ such that $\frac{\partial}{\partial r}u(r_1)$ does not exists. Then for every $h > 0$, which is enough small, exists $\epsilon_4 > 0$ such that

$$\Big|\frac{u(r_1+h)-u(r_1)}{h}\Big| > \epsilon_4,$$

and

$$0 < \epsilon_5 < \frac{h}{2}\epsilon_4, \tag{3.6}$$

such that $|u_n(r_1+h) - u(r_1)| < \epsilon_5$. From this,

$$\begin{aligned}
\epsilon_5 &> |u_n(r_1+h) - u(r_1+h)| \\
&= |u_n(r_1+h) - u(r_1) + u(r_1) - u(r_1+h)| \\
&\geq |u(r_1) - u(r_1+h)|\frac{1}{h}h - |u_n(r_1+h) - u(r_1)| \\
&\geq \epsilon_4 h - \epsilon_5,
\end{aligned}$$

which is a contradiction of our choice of ϵ_5. Therefore $\frac{\partial}{\partial r}u(r)$ exists for every $r \in [r_0, \infty)$. As in above we can see that $u(r) \in \mathcal{C}^2([r_0,\infty))$ $u_r(\infty) = 0$.

Now we suppose that there exists interval $\Delta_2 \subset [r_0, \infty)$ such that

$$u(r) \geq \frac{1}{B} + \epsilon_7 \quad \text{for } r \in \Delta_2.$$

Let $n \in \mathbb{N}$ be large and $\epsilon_8 > 0$ chosen such that

$$|u_n(r) - u(r)| < \epsilon_8 \quad \text{for } r \in \Delta_2, 0 < \epsilon_8 < \epsilon_7. \tag{3.7}$$

From this, for $r \in \Delta_2$, we have

$$\epsilon_8 > |u_n(r) - u(r)| \geq |u(r)| - |u_n(r)| \geq \frac{1}{B} + \epsilon_7 - \frac{1}{B} = \epsilon_7,$$

which is a contradiction with (3.7). Therefore, $u(r) \leq \frac{1}{B}$ for every $r \geq r_0$.

Now we suppose that there exists interval $\Delta_3 \subset [c_1, d_1]$ for which $u(r) < \frac{1}{A}$ for every $r \in \Delta_3$. From this, there exists $\epsilon_9 > 0$ such that $u(r) \leq \frac{1}{A} - \epsilon_9$ for $r \in \Delta_3$. Also, let

$$0 < \epsilon_{10} < \epsilon_9 \tag{3.8}$$

and $n \in \mathbb{N}$ is enough large such that $\epsilon_{10} > |u_n(r) - u(r)|$ for $r \in \Delta_3$. Then for $r \in \Delta_3$ we have

$$\epsilon_{10} > |u_n(r) - u(r)| \geq |u_n(r)| - |u(r)| \geq \frac{1}{A} - \frac{1}{A} + \epsilon_9,$$

which is a contradiction with (3.8). Consequently, for every $r \in [c_1, d_1]$ we have $u(r) \geq \frac{1}{A}$.

Now we suppose that there exist $\alpha \in \mathbb{N} \cup \{0\}$, interval $\Delta_4 \subset [r_0, \infty)$ and $\epsilon_{11} > 0$ such that

$$|r^\alpha u(r)| > 1 + \epsilon_{11} \quad \text{for } r \in \Delta_4.$$

Let $\epsilon_{12} > 0$ and $n \in \mathbb{N}$ be chosen such that

$$|r^\alpha(u_n(r) - u(r))| < \epsilon_{12} \quad \text{for } r \in \Delta_4,\ 0 < \epsilon_{12} < \epsilon_{11}. \tag{3.9}$$

From this,

$$\epsilon_{12} > |r^\alpha(u_n(r) - u(r))| \geq |r^\alpha u(r)| - r^\alpha |u_n(r)| \geq \epsilon_{11},$$

which is a contradiction with (3.9). Therefore for every $\alpha \in \mathbb{N} \cup \{0\}$ and for every $r \in [r_0, \infty)$ we have $r^\alpha u(r) \le 1$. After we use the same arguments we can see that for every $\alpha \in \mathbb{N} \cup \{0\}$ and for every $r \in [r_0, \infty)$ we have $r^\alpha |u_r(r)| \le 1$.

Now we suppose that there exist interval $\Delta_5 \subset [r_0, \infty)$ and $\epsilon_{13} > 0$ such that for $r \in \Delta_5$ we have $u(r) < -\epsilon_{13}$. Let $n \in \mathbb{N}$ is enough large and $\epsilon_{14} > 0$ are fixed for which

$$|u_n(r) - u(r)| < \epsilon_{14} \quad \text{for } r \in \Delta_5, \quad 0 < \epsilon_{14} < \epsilon_{13}. \tag{3.10}$$

Then for $r \in \Delta_5$ we have

$$\epsilon_{14} > u_n(r) - u(r) > \epsilon_{13}$$

which is a contradiction with (3.10). □

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