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SECOND ORDER TANGENCY CONDITIONS AND DIFFERENTIAL INCLUSIONS: A COUNTEREXAMPLE AND A REMEDY

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Abstract. In this paper we show that second order tangency conditions are superfluous not to say useless while discussing the existence condition for certain second order differential inclusions. In this regard, a counterexample is provided even in the simpler setting of second order differential equations, where a substitute condition is propound. In the setting of differential inclusions, the corresponding substitute condition allows for us to prove existence of sufficiently many approximate solutions without the use of any convexity, measurability, or upper semicontinuity assumption. Accordingly, some proofs in the related literature are greatly simplified.

1. A second order differential equation: the theory

Consider the second order differential equation

$$X''(t) = g(t, X(t), X'(t)) \tag{1.1}$$

where $g : [a, b) \times D \to \mathbb{R}^n$ is a function, $D \subseteq \mathbb{R}^{2n}$ is a nonempty set, and $[a, b) \subseteq \mathbb{R}$ is a nonempty, possibly unbounded interval. The existence condition for the equation (1.1) states that

> for every $(x, y) \in D$ and for every $\tau \in [a, b)$ there exist a subinterval $[\tau, \upsilon)$ of $[\tau, b)$ and a solution $X : [\tau, \upsilon) \to \mathbb{R}^n$ to the differential equation (1.1) such that $X(\tau) = x$ and $X'(\tau) = y$. (1.2)

By a solution to the equation (1.1) we mean a Carathéodory solution, that is, a locally absolutely continuous function $X : [\tau, \upsilon) \to \mathbb{R}^n$ such that also $X' : [\tau, \upsilon) \to \mathbb{R}^n$ is locally absolutely continuous, such that $(X(t), X'(t)) \in D$ for all $t \in [\tau, \upsilon)$, and such that $(t, X(t), X'(t), X''(t))$ renders true the equality (1.1) for almost all $t \in [\tau, \upsilon)$. Throughout this paper, by a solution to a differential equation, inclusion, and so on we mean a Carathéodory solution.

A characterization of the existence condition (1.1) can be given by using a tangency concept which springs from two papers published, in 1931, in the same issue of the journal "Annales de la Société Polonaise de Mathématique." The authors of these papers are Bouligand (see [6]) and Severi (see [14]).

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For every subset S and for every point p_0 of a Hausdorff topological vector space E, we denote by $\mathcal{T}_S(p_0)$ the set of all points $p_1 \in E$ with the property that

for every neighborhood Q of the origin in E and for every $H > 0$ there exist $h \in (0, H)$ and $q \in Q$ such that $p_0 + h(p_1 + q) \in S$.

Obviously, $\mathcal{T}_S(p_0) \neq \emptyset$ if and only if $p_0 \in \text{closure}(S)$, in which case $\mathcal{T}_S(p_0)$ is a closed cone.

The existence condition (1.2) can be characterized through the first order tangency condition which involves the first order tangency relation

$$(y, g(t, x, y)) \in \mathcal{T}_D(x, y) \tag{1.3}$$

and which states that

there exists a set $\mathcal{N} \subseteq [a, b)$ of null Lebesgue measure such that the tangency relation (1.3) holds for all $(x, y) \in D$ and for all $t \in [a, b) \setminus \mathcal{N}$. (1.4)

Such a characterization does hold if the set D is locally closed, whereas the function g is a Carathéodory function, i.e.

(i) the functions $(x, y) \to g(t, x, y)$ are continuous on D for almost all $t \in [a, b)$;
(ii) the functions $t \to g(t, x, y)$ are measurable on $[a, b)$ for all $(x, y) \in D$;
(iii) for every $(x, y) \in D$ and for every $\tau \in [a, b)$ there exist a neighborhood W of (x, y), a subinterval $[\tau, \upsilon)$ of $[a, b)$, and a locally integrable function $m : [\tau, \upsilon) \to \mathbb{R}$ such that $\sup_{(u,v) \in W \cap D} \|g(t, u, v)\| \leq m(t)$ for almost all $t \in [\tau, \upsilon)$.

Here, $\|\cdot\|$ stands for a norm, e.g. the Euclidean norm, on $\mathbb{R}^n$.

Theorem 1.1. *Let D be locally closed and let the function g be a Carathéodory function. Then the existence condition* (1.2) *is equivalent to the tangency condition* (1.4).

The conclusion follows from a result in [16, p. 484, Theorem] (see also [15, pp. 5-6, Theorem]), for the second order differential equation (1.1) is equivalent to the first order differential system

$$X'(t) = Y(t),$$
$$Y'(t) = g(t, X(t), Y(t)).$$

Note that condition (1.4) is superfluous if the set D is open because $\mathcal{T}_D(x, y) = \mathbb{R}^{2n}$ for all $(x, y) \in D$.

Suppose further $D = K \times L$ where $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n$ are nonempty sets. In this case, D is locally closed if and only if so are both K and L. No matter whether L is locally closed and no matter whether g is a Carathéodory function, it is possible to characterize the tangency condition (1.4) through a couple of simpler conditions. The first condition of the couple involves a "confluence" of closed cones,

$$\mathcal{L}(K) = \cap_{x \in K} \mathcal{T}_K(x),$$

and states that

$$L \subseteq \mathcal{L}(K). \tag{1.5}$$

The second condition of the couple involves the x-"collection" of first order tangency relations

$$g(t, x, y) \in \mathcal{T}_L(y), \tag{1.6}$$

and states that

$$\text{there exists a set } \mathcal{N} \subseteq [a,b) \text{ of null Lebesgue measure such that the tangency relation (1.6) holds for all } x \in K, \text{ for all } y \in L, \text{ and for all } t \in [a,b) \setminus \mathcal{N}. \tag{1.7}$$

Note that condition (1.5) is superfluous if the set K is open because $\mathcal{L}(K) = \mathbb{R}^n$, whereas condition (1.7) is superfluous if the set L is open because $\mathcal{T}_L(y) = \mathbb{R}^n$ for all $y \in L$.

Theorem 1.2. *Let $D = K \times L$. Then the existence condition* (1.2) *implies the confluence tangency condition* (1.5)*, whereas the tangency condition* (1.4) *implies both the confluence tangency conditions* (1.5) *and the collective tangency condition* (1.7)*.*

Let, in addition, the set K be locally closed. Then the tangency condition (1.4) *is equivalent to the couple of tangency conditions* (1.5) *and* (1.7)*.*

The fact that condition (1.2) implies condition (1.5) follows from lemma below.

Lemma 1.3. *Let $x : [\tau, \upsilon) \to \mathbb{R}^n$ be an absolutely continuous function such that also $x' : [\tau, \upsilon) \to \mathbb{R}^n$ is absolutely continuous and such that $x(t) \in K$ for all $t \in [\tau, \upsilon)$. Then $x'(\tau) \in \mathcal{T}_K(x(\tau))$.*

The conclusion of the lemma follows from the fact that, if $h \in (0, \upsilon - \tau)$, then

$$X(\tau) + h\Big(X'(\tau) + \frac{1}{h}\int_\tau^{\tau+h}\Big(\int_\tau^t X''(s)ds\Big)dt\Big) = X(\tau + h) \in K.$$

Further, the fact that condition (1.4) implies the couple of conditions (1.5) and (1.7) follows from the inclusion $\mathcal{T}_{K\times L}(x,y) \subseteq \mathcal{T}_K(x) \times \mathcal{T}_L(y)$.

Finally, note that, if $x \in K$, $y \in L$, and there exists $H > 0$ such that $x + hy \in K$ for all $h \in (0, H)$, then $\{y\} \times \mathcal{T}_L(y) \subseteq \mathcal{T}_{K\times L}(x,y)$.

Now, the fact that the couple of conditions (1.5) and (1.7) implies condition (1.4) follows from the inclusion $\mathcal{L}(K) \times \mathcal{T}_L(y) \subseteq \mathcal{T}_{K\times L}(x,y)$, a consequence of lemma below.

Lemma 1.4. *Let the set K be locally closed. Then $y \in \mathcal{L}(K)$ if and only if for every $x \in K$ there exists $H > 0$ such that $x + hy \in K$ for all $h \in (0, H)$.*

Moreover, for every compact subset P of K and for every bounded subset Q of $\mathcal{L}(K)$ there exists $H > 0$ such that $P + hQ \subset K$ for all $h \in (0, H)$.

The "if" part of the lemma is obvious. Now, let $y \in \mathcal{L}(K)$. Since $y \in \mathcal{T}_K(x)$ for all $x \in K$, it follows from a result of Nagumo (see [12, p. 552]) that for every $x \in K$ there exist $T > 0$ and a classical solution $X : [0, T) \to \mathbb{R}^n$ to the restricted Cauchy problem

$$X'(t) = y, \quad X(t) \in K,$$
$$X(0) = x,$$

But $X(t) = x + ty$ for all $t \in [0, T)$, and the "only if" part of the lemma follows.

Further, for every $x \in K$ and for every $y \in \mathcal{L}(K)$, let $H(x,y)$ be the supremum of all $H > 0$ such that $x + hy \in K$ for all $h \in (0, H)$. Obviously, either $H(x,y) = +\infty$ or both $H(x,y) < +\infty$ and $x + H(x,y)y \notin K$.

Finally, let $P \subseteq K$ be compact and let $Q \subseteq \mathcal{L}(K)$ be bounded. We have to show that $0 < \inf_{x\in P, y\in Q} H(x,y)$. Suppose, by contradiction, there exists a sequence

$(x_j, y_j) \in P \times Q$ such that $H(x_j, y_j)$ converges to 0. Since P is compact, we can suppose, taking a subsequence if necessary, that x_j converges to a point $x \in P$. Since K is locally closed, it follows $\tilde{U} \cap K$ is closed for some neighborhood $\tilde{U}$ of x. Since Q is bounded, it follows $U + [0, H]Q \subset \tilde{U}$ for some neighborhood U of x and for some $H > 0$. Further, $x_j \in U$ and $H(x_j, y_j) \leq H$ for some j. Since $x_j + hy_j \in \tilde{U} \cap K$ for all $h \in (0, H(x_j, y_j))$, it follows $x_j + H(x_j, y_j)y_j \in K$, a contradiction, and the lemma is proved.

In view of lemma above, if the set K is closed, then $\mathcal{L}(K)$ equals the asymptotic cone of K, that is, $y \in \mathcal{T}(K)$ if and only if $x + hy \in K$ for all $x \in K$ and for all $h > 0$. Accordingly, the confluence tangency condition (1.5) is equivalent to the condition that $K + hL \subseteq K$ for all $h > 0$.

To close this section we rephrase Theorem 1.1 with a statement which does not explicitly involve any tangency condition except (1.5). Consider the x-"collection" of first order differential equations

$$Y'(t) = g(t, x, Y(t)). \tag{1.8}$$

The collective existence condition for the differential equation (1.8) states that

> for every $x \in K$, for every $y \in L$, and for every $\tau \in [a, b)$ there exist a subinterval $[\tau, \upsilon)$ of $[\tau, b)$ and a solution $Y : [\tau, \upsilon) \to \mathbb{R}^n$ to the differential equation (1.8) such that $Y(\tau) = y$. (1.9)

In view of the cited result in [16], if the set L is locally closed and if the function $(t, y) \in [a, b) \times L \to g(t, x, y) \in \mathbb{R}^n$ is a Carathéodory function for each $x \in K$, then the collective existence condition (1.9) is equivalent to the collective tangency condition (1.7).

Theorem 1.5. *Let $D = K \times L$, let the sets K and L be locally closed, and let the function g be a Carathéodory function. Then the existence condition* (1.2) *is equivalent to the couple made up of the confluence tangency condition* (1.5) *and the collective existence condition* (1.9).

2. A second order tangency condition: the counterexample

In this section we try to characterize the couple of tangency conditions (1.5) and (1.7) by using the second order version of the tangency concept $\mathcal{T}$ (see [5]).

For every subset S and for every couple of points p_0 and p_1 of a Hausdorff topological vector space E, we denote by $\mathcal{T}_S^{(2)}(p_0, p_1)$ the set of all points $p_2 \in E$ with the property that

> for every neighborhood Q of the origin in E and for every $H > 0$ there exist $h \in (0, H)$ and $q \in Q$ such that $p_0 + hp_1 + \frac{h^2}{2}(p_2 + q) \in S$.

Clearly, $\mathcal{T}_S^{(2)}(p_0, 0) = \mathcal{T}_S(p_0)$. Moreover, if $\mathcal{T}_S^{(2)}(p_0, p_1) \neq \emptyset$, then $p_1 \in \mathcal{T}_S(p_0)$, but the converse may fail. For example, if $p_0 \in E$, $p_1 \in E$, $p_2 \in E$, and

$$S = \{p_0 + hp_1 + \frac{h\sqrt{h}}{2} p_2; h \geq 0\},$$

then $p_1 \in \mathcal{T}_S(p_0)$, but $\mathcal{T}_S^{(2)}(p_0, p_1)$ is empty if the vectors p_1 and p_2 are linearly independent.

Now, consider the second order tangency condition which involves the second order tangency relation

$$g(t,x,y) \in \mathcal{T}_K^{(2)}(x,y) \tag{2.1}$$

and which states that

> there exists a set $\mathcal{N} \subseteq [a,b)$ of null Lebesgue measure such that the tangency relation (2.1) holds for all $x \in K$, for all $y \in L$, and for all $t \in [a,b) \setminus \mathcal{N}$. (2.2)

Theorem 2.1. *Let $D = K \times L$ and let the set K be locally closed. Then the couple of tangency conditions* (1.5) *and* (1.7) *implies the second order tangency condition* (2.2). *Conversely, condition* (2.2) *implies the confluence tangency condition* (1.5), *but may fail to imply the collective tangency condition* (1.7). *Let, in addition, $L \subseteq \operatorname{interior}(\mathcal{L}(K))$. Then condition* (2.2) *is superfluous because $\mathcal{T}_K^{(2)}(x,y) = \mathbb{R}^n$ for all $x \in K$ and for all $y \in L$, and condition* (1.7) *still may fail to hold.*

The fact that (1.5) and (1.7) taken together imply (2.2) follows from the inclusion $\mathcal{T}_L(y) \subseteq \mathcal{T}_{\mathcal{L}(K)}(y)$ and from lemma below.

Lemma 2.2. *Let the set K be locally closed. Then $\mathcal{T}_{\mathcal{L}(K)}(y) \subseteq \bigcap_{x\in K} \mathcal{T}_K^{(2)}(x,y)$ for all $y \in \mathcal{L}(K)$. Let, in addition, $\mathcal{L}(K)$ have a nonempty interior. Then $\mathcal{T}_K^{(2)}(x,y) = \mathbb{R}^n$ for all $x \in K$ and for all $y \in \operatorname{interior}(\mathcal{L}(K))$.*

To prove the first part of the lemma, let $y \in \mathcal{L}(K)$, let $z \in \mathcal{T}_{\mathcal{L}(K)}(y)$, and let $x \in K$. We have to show that $z \in \mathcal{T}_K^{(2)}(x,y)$. According to the definition of the tangency concept $\mathcal{T}$, there exist a sequence $h_i > 0$ which converges to 0 and a sequence $q_i \in \mathbb{R}^n$ which converges to 0 such that $y + (h_i/2)(z+q_i) \in \mathcal{L}(K)$ for all i. According to Lemma 1.4, there exists $H > 0$ such that $x + h(y + (h_i/2)(z+q_i)) \in K$ for all $h \in (0,H)$ and for all i. We can suppose, taking a subsequence if necessary, that $h_i \in (0,H)$ for all i. Since $x + h_i(y + (h_i/2)(z+q_i)) \in K$ for all i, it follows $z \in \mathcal{T}_K^{(2)}(x,y)$, and the first part of the lemma is proved.

The inclusion we have just obtained can not be improved to the corresponding equality. Let $K = \{x \in \mathbb{R}^2; x_2 \geq (x_1)^2\}$ and $y = (0,1)$, so that $\mathcal{L}(K) = \{x \in \mathbb{R}^2; x_1 = 0, x_2 \geq 0\}$ and $y \in \mathcal{L}(K)$. On the one hand $\mathcal{T}_{\mathcal{L}(K)}(y) = \{z \in \mathbb{R}^2; z_1 = 0, z_2 \in R\}$. On the other hand, $\mathcal{T}_K^{(2)}(x,y) = \mathbb{R}^2$ for all $x \in K$.

The additional part of the lemma follows from the fact that $\mathcal{T}_{\mathcal{L}(K)}(y) = \mathbb{R}^n$ for all $y \in \operatorname{interior}(\mathcal{L}(K))$.

The fact that (2.2) implies (1.5) follows from lemma below.

Lemma 2.3. *Let the set K be locally closed. Then*

$$\mathcal{L}(K) = \{y \in \mathbb{R}^n; \forall x \in K, \mathcal{T}_K^{(2)}(x,y) \neq \emptyset\}.$$

To prove the lemma, denote by S the right hand side of the equality above. Since $\mathcal{T}_K^{(2)}(x,y) \neq \emptyset$ implies $y \in \mathcal{T}_K(x)$, it follows from Lemma 1.4 that $S \subseteq \mathcal{L}(K)$. To prove the converse inclusion, let $y \in \mathcal{L}(K)$ and let $x \in K$. We have to show that $\mathcal{T}_K^{(2)}(x,y) \neq \emptyset$. According to the definition of the set $\mathcal{L}(K)$, there exists $\tilde{H} > 0$ such that $x + hy \in K$ for all $h \in (0,\tilde{H})$. Let $H > 0$ such that $H + H^2/2 = \tilde{H}$, and let $h \in (0,H)$. Then $x + hy + (h^2/2)y \in K$, hence $y \in \mathcal{T}_K^{(2)}(x,y)$, and the lemma is proved.

The fact that (2.2) may not imply (1.7) is illustrated through the simplest counterexample given by $[a,b)=[0,+\infty)$, $K=[0,+\infty)$, $L=\{1\}$, and $g(t,x,y)=y$. Clearly, condition (2.2) holds because $\mathcal{T}^2_K(x,y)=\mathbb{R}$ for all $x\in K$ and $y\in L$, but condition (1.2) does not hold, that is, the restricted differential system $X''(t)=X'(t), X(t)\in K, X'(t)\in L$ has no solution. Indeed, if $x\in K$, $y\in L$, and X is a solution to the system $X''(t)=X'(t)$ such that $X(0)=x$ and $X'(0)=y$, then $X(t)=x+(\exp(t)-1)y$ for all $t>0$. Now, $X(t)\in K$ for all $t>0$, but $X'(t)\notin L$ for any $t>0$. Note also $\mathcal{L}(K)=K$, hence $L\subseteq \text{interior}(\mathcal{L}(K))$.

To conclude, if $D=K\times L$, then the second order tangency condition (2.2) is useless not to say superfluous while discussing the existence condition (1.2). *Nevertheless*, condition (2.2) may be useful (cf. [13, p. 38, Theorem 2.4]) while discussing some adjacent existence conditions, e.g.

> there exist $x\in K$, $y\in L$, $\tau\in[a,b)$, a subinterval $[\tau,\upsilon)$ of $[\tau,b)$, and a solution $X:[\tau,\upsilon)\to\mathbb{R}^n$ to the second order differential equation (1.1) such that $X(\tau)=x$ and $X'(\tau)=y$.

3. A second order differential inclusion: the corresponding results

Consider the second order differential inclusion

$$X''(t)\in G(t,X(t),X'(t)) \tag{3.1}$$

where $G:[a,b)\times D\to\mathbb{R}^n$ is a multifunction with nonempty values, $D\subseteq\mathbb{R}^{2n}$ is a nonempty set, and $[a,b)\subseteq\mathbb{R}$ is a nonempty, possibly unbounded interval. The existence condition for the inclusion (3.1) states that

> for every $(x,y)\in D$ and for every $\tau\in[a,b)$ there exist a subinterval $[\tau,\upsilon)$ of $[\tau,b)$ and a solution $X:[\tau,\upsilon)\to\mathbb{R}^n$ to the second order differential inclusion (3.1) such that $X(\tau)=x$ and $X'(\tau)=y$. (3.2)

Parallel results to the ones in Section 1 do hold if the set D is locally closed, whereas the multifunction G has compact, convex values and is a Carathéodory multifunction, i.e.

- (I) the multifunctions $(x,y)\to G(t,x,y)$ are continuous on D for almost all $t\in[a,b)$;
- (II) the multifunctions $t\to G(t,x,y)$ are measurable on $[a,b)$ for all $(x,y)\in D$;
- (III) for every $(x,y)\in D$ and for every $\tau\in[a,b)$ there exist a neighborhood W of (x,y), a subinterval $[\tau,\upsilon)$ of $[a,b)$, and a locally integrable function $m:[\tau,\upsilon)\to\mathbb{R}$ such that $\sup_{(u,v)\in W\cap D}\|G(t,u,v)\|\le m(t)$ for almost all $t\in[\tau,\upsilon)$.

Here, $\|G(t,u,v)\|$ stands for the supremum of all $\|p\|$ with $p\in G(t,u,v)$.

If *continuity* is replaced with *upper semicontinuity* in condition (I) above, we say that the multifunction G is an *upper* Carathéodory multifunction.

The existence condition (3.1) can be characterized through the first order tangency condition which involves the first order tangency relation

$$(\{y\}\times G(t,x,y))\cap\mathcal{T}_D(x,y)\neq\emptyset \tag{3.3}$$

and which states that

> there exists a set $\mathcal{N}\subseteq[a,b)$ of null Lebesgue measure such that the tangency relation (3.3) holds for all $(x,y)\in D$ and for all $t\in[a,b)\setminus\mathcal{N}$. (3.4)

Theorem 3.1. *Let the set D be locally closed, and let the multifunction G have compact, convex values and be an upper Carathéodory multifunction. Then the existence condition (3.2) implies the tangency condition (3.4). Let, in addition, the multifunction G be a Carathéodory multifunction. Then the existence condition (3.2) is equivalent to the tangency condition* (3.4)*.*

The result above is a particular form of a more general result derived in [10, pp. 279, 280 Theorems 4.2 and 4.3].

Suppose further $D = K \times L$ where $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n$ are nonempty sets. In this case, no matter whether L is locally closed and no matter whether G is a Carathéodory multifunction, it is possible to characterize the tangency condition (3.4) through a couple of simpler conditions. The first condition of the couple is just the confluence tangency condition (1.5). The second condition of the couple involves the x-"collection" of first order tangency relations

$$G(t, x, y) \cap \mathcal{T}_L(y) \neq \emptyset, \tag{3.5}$$

and states that

> there exists a set $\mathcal{N} \subseteq [a, b)$ of null Lebesgue measure such that the tangency condition (3.5) holds for all $x \in K$, $y \in L$, and $t \in [a, b) \setminus \mathcal{N}$. (3.6)

Recall condition (1.5) is superfluous if the set K is open, whereas condition (3.6) is superfluous if the set L is open.

Theorem 3.2. *Let $D = K \times L$. Then the existence condition (3.2) implies the confluence tangency condition* (1.5)*, whereas the tangency condition* (3.4) *implies both the confluence tangency conditions* (1.5) *and the collective tangency condition* (3.6)*.*

Let, in addition, the set K be locally closed. Then tangency condition (3.4) *is equivalent to the couple of tangency conditions* (1.5) *and* (3.6)*.*

Now, consider the second order tangency condition which involves x-"collection" of second order tangency relations

$$G(t, x, y) \cap \mathcal{T}_K^{(2)}(x, y) \neq \emptyset \tag{3.7}$$

and which states that

> there exists a set $\mathcal{N} \subseteq [a, b)$ of null Lebesgue measure such that the tangency relation (3.7) holds for all $x \in K$, for all $y \in L$, and for all $t \in [a, b) \setminus \mathcal{N}$. (3.8)

Theorem 3.3. *Let $D = K \times L$ and let the set K be locally closed. Then the couple of tangency conditions* (1.5) *and* (3.6) *implies the second order tangency condition* (3.8)*. Conversely, condition* (3.8) *implies the confluence tangency condition* (1.5)*, but may fail to imply the collective tangency condition* (3.6)*. Let, in addition, $L \subseteq \operatorname{interior}(\mathcal{L}(K))$. Then condition* (3.8) *is superfluous because $\mathcal{T}_K^{(2)}(x, y) = \mathbb{R}^n$ for all $x \in K$ and for all $y \in L$, but condition* (3.6) *still may fail to hold.*

To conclude, if $D = K \times L$, then the second order tangency condition (3.8) is useless not to say superfluous while discussing the existence condition (3.2). *Nevertheless*, condition (3.8) may be useful (cf. [4, p. 214, Theorem 4.1]) while discussing some adjacent existence conditions, e.g.

> there exist $x \in K$, $y \in L$, $\tau \in [a,b)$, a subinterval $[\tau, \upsilon)$ of $[\tau, b)$, and a solution $X : [\tau, \upsilon) \to \mathbb{R}^n$ to the second differential inclusion (3.1) such that $X(\tau) = x$ and $X'(\tau) = y$.

To close this section we rephrase Theorem 3.1 with a statement which does not explicitly involve any tangency condition except (1.5). Consider the x-"collection" of first order differential inclusions

$$Y'(t) \in G(t, x, Y(t)). \tag{3.9}$$

The collective existence condition for the differential inclusion (3.9) states that

> for every $x \in K$, for every $y \in L$, and for every $\tau \in [a,b)$ there exist a subinterval $[\tau, \upsilon)$ of $[\tau, b)$ and a solution $Y : [\tau, \upsilon) \to \mathbb{R}^n$ to the differential inclusion (3.9) such that $Y(\tau) = y$. (3.10)

In view of the cited result in [10], if the set L is locally closed and if the multi-function $(t, y) \in [a,b) \times L \to G(t, x, y) \in \mathbb{R}^n$ has compact, convex values and is a Carathéodory multifunction for each $x \in K$, then the collective existence condition (3.10) is equivalent to the collective tangency condition (3.6).

Theorem 3.4. *Let $D = K \times L$, let the sets K and L be locally closed, and let the multifunction G have compact, convex values and be a Carathéodory multifunction. Then the existence condition (3.2) is equivalent to the couple made up of the confluence tangency condition* (1.5) *and the collective existence condition* (3.10).

4. A second order differential inclusion: the approximate solutions

Consider the second order differential inclusion (3.1) in case $D = K \times L$. In view of Theorem 3.4, it is strongly expected for the existence condition (3.2) to be implied by the couple made up of the confluence tangency condition (1.5) and the collective existence condition (3.10) if the sets K and L are locally closed, and G is an upper Carathéodory multifunction with compact, but not necessarily convex values.

We provide such a result in the next section. In the present section, we define a new type of approximate solutions to the first order differential system

$$\begin{gathered} X'(t) = Y(t), \\ Y'(t) \in G(t, X(t), Y(t)), \end{gathered} \tag{4.1}$$

which is equivalent to the second order differential inclusion (3.1), and we prove that the couple of conditions (1.5) and (3.10) implies existence of sufficiently many approximate solutions provided that the sets K and L are locally closed, and the multifunction G enjoys only the third Carathéodory type condition (III) above.

Denote by $\Phi([\tau, \upsilon))$ the family of all functions $\phi : [\tau, \upsilon) \to \mathbb{R}$ such that there exists a finite covering of $[\tau, \upsilon)$ made up of mutually disjoint intervals $[\tilde{\tau}, \tilde{\upsilon})$ such that $\phi(t) = \tilde{\tau}$ for all $t \in [\tilde{\tau}, \tilde{\upsilon})$ (cf. [10, p.280, (iii)], where the covering may be infinite). Obviously, $\tau \le \phi(t) \le t$ for all $t \in [\tau, \upsilon)$.

Let $[\tau, \upsilon)$ be a subinterval of $[a,b)$, let $\phi \in \Phi([\tau, \upsilon))$, and consider the ϕ-differential system

$$\begin{gathered} X'(t) = Y(t), \\ Y'(t) \in G(t, X(\phi(t)), Y(t)). \end{gathered} \tag{4.2}$$

Definition 4.1. A function $(X,Y):[\tau,\upsilon)\to\mathbb{R}^{2n}$ is said to be a ϕ-approximate solution to the differential system (4.1) if (X,Y) is a solution to the ϕ-differential system (4.2).

Note that, if $Y:[\tau,\upsilon)\to\mathbb{R}^n$ is a solution to the differential inclusion (3.9), if $\phi(t)=\tau$ for all $t\in[\tau,\upsilon)$, and if $X(t)=x+\int_\tau^t Y(s)ds$ for all $t\in[\tau,\upsilon)$, then $\phi\in\Phi([\tau,\upsilon))$ and (X,Y) is a ϕ-approximate solution to the differential system (4.1).

In view of this remark, condition (3.10) can be rephrased as follows:

> for every $x\in K$, for every $y\in L$, and for every $\tau\in[a,b)$ there exist a subinterval $[\tau,\upsilon)$ of $[\tau,b)$, a $\phi\in\Phi([\tau,\upsilon))$, and a ϕ-approximate solution $(X,Y):[\tau,\upsilon)\to\mathbb{R}^{2n}$ to the differential system (4.1) such that $(X,Y)(\tau)=(x,y)$.

The main result of this section shows that, under suitable hypotheses, the collective existence condition (3.10) implies the approximate existence condition which states that

> for every $\tau\in[a,b)$, for every $x\in K$, for every $y\in L$, for every neighborhood U of x, and for every neighborhood V of y there exists a subinterval $[\tau,\upsilon)$ of $[\tau,b)$ such that for every $\phi\in\Phi([\tau,\upsilon))$ there exists a ϕ-approximate solution $(X,Y):[\tau,\upsilon)\to\mathbb{R}^{2n}$ to the differential system (4.1) such that $(X,Y)(\tau)=(x,y)$, $X([\tau,\upsilon))\subseteq U\cap K$, and $Y([\tau,\upsilon))\subseteq V\cap L$. (4.3)

Theorem 4.2. *Let the sets K and L be locally closed, and let the multifunction G satisfy the Caratéodory type condition (III). Let the confluence tangency condition* (1.5) *and the collective existence condition* (3.10) *be satisfied. Then the approximate existence condition* (4.3) *is satisfied too.*

To prove the theorem, let $\tau^*\in[a,b)$, let $x^*\in K$, let $y*\in L$, let U^* be a neighborhood of x^*, and let V^* be a neighborhood of y^*.

We can suppose, taking smaller U^* and V^* if necessary, that the sets $U^*\cap K$ and $V^*\cap L$ are closed.

We can suppose, taken even smaller U^* and V^* if necessary, that there exists a subinterval $[\tau^*,\upsilon^*)$ of $[\tau^*,b)$ and a locally integrable function $m:[\tau^*,\upsilon^*)\to\mathbb{R}$ such that

$$\sup_{u\in U^*\cap K, V^*\in V\cap L}\|G(t,u,v)\|\le m(t)$$

for almost all $t\in[\tau^*,\upsilon^*)$.

We can suppose, taking a smaller υ^* if necessary, that $\upsilon^*<b$ and $\int_{\tau^*}^{\upsilon^*}m(t)dt<+\infty$, that is m is integrable.

Define $\mu:[\tau^*,\upsilon^*]\times[\tau^*,\upsilon^*]\times R\to\mathbb{R}$ through

$$\mu(t;\tau,r)=\int_\tau^t\Big(r+\int_\tau^s m(\sigma)d\sigma\Big)ds.$$

Note parenthetically that the function $t\to\mu(t;\tau,r)$ is the solution to the scalar, second order differential system

$$\begin{gathered}\mu''(t)=m(t),\\ \mu'(\tau)=r,\\ \mu(\tau)=0.\end{gathered}$$

We can suppose, taking an even smaller υ^* if necessary, that

$$\overline{B}(y^*, \mu'(\upsilon^*; \tau^*, 0)) \subseteq \text{interior}(V^*),$$
$$\overline{B}(x^*, \mu(\upsilon^*; \tau^*, \|y^*\|)) \subseteq \text{interior}(U^*).$$

Here, $\overline{B}(c, r)$ stands for the closed ball with center c and radius r.

We shall show that for every $\phi \in \Phi([\tau^*, \upsilon^*))$ there exists a ϕ-approximate solution $(X, Y) : [\tau^*, \upsilon^*) \to \mathbb{R}^{2n}$ to the differential system (4.1) such that $(X, Y)(\tau^*) = (x^*, y^*)$, $X([\tau^*, \upsilon^*)) \subseteq U^* \cap K$, and $Y([\tau^*, \upsilon^*)) \subseteq V^* \cap L$.

To this purpose, we first show that the statement above holds in the particular case that $\phi(t) = t$ for all $t \in [\tau^*, \upsilon^*)$, namely there exists a solution $Y : [\tau, \upsilon^*) \to \mathbb{R}^n$ to the differential system

$$Y'(t) \in G(t, x^*, Y(t)),$$
$$Y(\tau^*) = y^*,$$

as well as a solution $X : [\tau^*, \upsilon^*) \to \mathbb{R}^n$ to the restricted system

$$X'(t) = Y(t), \quad X(t) \in K,$$
$$X(\tau^*) = x^*.$$

In fact, we show that there holds a slightly stronger statement which involves the family of Cauchy problems

$$\begin{gathered} Y'(t) \in G(t, x, Y(t)), \\ Y(\tau) = y, \end{gathered} \tag{4.4}$$

where $\tau \in [a, b)$, $x \in K$, $y \in L$, as well as the family of restricted Cauchy problems

$$\begin{gathered} X'(t) = Y(t), \quad X(t) \in K \\ X(\tau) = x, \end{gathered} \tag{4.5}$$

where $\tau \in [a, b)$, $x \in K$, and Y is a solution to a corresponding system (4.4).

To frame the announced slightly stronger statement, we need some additional items.

For every $(\tau, x, y) \in [\tau^*, \upsilon^*) \times K \times L$, we denote by $S(\tau, x, y)$ the set of all points $(t, \zeta, \eta) \in [\tau^*, \upsilon^*) \times K \times L$ such that $\tau \le t$, $\|\zeta - x\| \le \mu(t; \tau, \|y\|)$, and $\|\eta - y\| \le \mu'(t; \tau, 0)\}$, and we note that:

- $S(\tau^*, x^*, y^*) \subseteq [\tau^*, \upsilon^*) \times \text{interior}(U^*) \times \text{interior}(V^*)$;
- $(\tau, x, y) \in S(\tau, x, y)$;
- $S(S(\tau, x, y))) \subseteq S(\tau, x, y)$.

The last property above follows from the equalities

$$\mu'(t; \tau, 0) = \mu'(t; \tilde{\tau}, 0) + \mu'(\tilde{\tau}; \tau, 0),$$
$$\mu(t; \tau, r) = \mu(t; \tilde{\tau}, r + \mu'(\tilde{\tau}; \tau, 0)) + \mu(\tilde{\tau}; \tau, r),$$

which hold whenever $r \in \mathbb{R}$ and $\tau^* \le \tau \le \tilde{\tau} \le t < \upsilon^*$.

Now, we can frame the announced statement.

Lemma 4.3. *Let the sets K and L be locally closed, and let the multifunction G satisfy the Carathéodory type condition (III). Let the confluence tangency condition* (1.5) *and the collective existence condition* (3.10) *be satisfied. Then for every $(\tau, x, y) \in S(\tau^*, x^*, y^*)$ there exist a solution $Y : [\tau, \upsilon^*) \to \mathbb{R}^n$ to the Cauchy problem* (4.4) *and a solution $X : [\tau, \upsilon^*) \to \mathbb{R}^n$ to the restricted Cauchy problem*

(4.5) *such that* $(X,Y)(\tau) = (x,y)$ *and such that* $(t, X(t), Y(t)) \in S(\tau, x, y)$ *for all* $t \in [\tau, \upsilon^*)$.

To prove this lemma we need some auxiliary results. The first three of them do not involve all of the hypotheses of Theorem 4.2. The fourth one involves all of those hypotheses through the item $[\tau^*, \upsilon^*)$.

The first auxiliary result concerns solutions to (4.4) on subintervals $[a, b)$. Recall that a solution $Y : [\tau, \upsilon) \to \mathbb{R}^n$ to the Cauchy problem (4.4) is said to be *saturated* if there does not exists any solution $\tilde{Y} : [\tau, \tilde{\upsilon}) \to \mathbb{R}^n$ to (4.4) such that both $\upsilon < \tilde{\upsilon}$ and Y equals the restriction of $\tilde{Y}$ to $[\tau, \upsilon)$. Such a definition can be given in case of the solutions of any Cauchy problem (see Section 6 below).

Lemma 4.4. *Let the existence condition* (3.10) *be satisfied. Let* $x \in K$, *let* $y \in L$, *let* $\tau \in [a, b)$, *let* $[\tau, \upsilon)$ *be a subinterval of* $[\tau, \upsilon)$, *and let* $Y : [\tau, \upsilon) \to \mathbb{R}^n$ *be a solution to the Cauchy problem* (4.4). *If* $\int_\tau^\upsilon \|Y'(t)\| dt < +\infty$, *if the set*

$$\overline{B}\Big(y, \int_\tau^\upsilon \|Y'(t)\| dt\Big) \cap L$$

is closed, and if $\upsilon < b$, *then* Y *is not a saturated solution to* (4.4).

Under the hypotheses of the lemma, $\hat{y} = \lim_{t\to\upsilon} Y(t)$ makes sense and belongs to L According to the existence condition (3.10), there exist a subinterval $[\upsilon, \tilde{\upsilon})$ of $[\upsilon, b)$ and a solution $\hat{Y} : [\upsilon, \tilde{\upsilon}) \to \mathbb{R}^n$ to (3.9) such that $\hat{Y}(\upsilon) = \hat{y}$. Let $\tilde{Y}(t)$ equal $Y(t)$ if $t \in [\tau, \upsilon)$ and let it equal $\hat{Y}(t)$ if $t \in [\upsilon, \tilde{\upsilon})$. Since $\tilde{Y}$ is a solution to (4.4), it follows Y is not saturated.

The second and third auxiliary results concern solutions to the Cauchy problems (4.4) and (4.5) on subintervals of $[a, b)$.

Lemma 4.5. *Let the set* K *be locally closed and let the confluence tangency condition* (1.5) *be satisfied. Let* $x \in K$, *let* $y \in L$, *let* $\tau \in [a, b)$, *let* $[\tau, \upsilon)$ *be a subinterval of* $[\tau, \upsilon)$, *and let* $Y : [\tau, \upsilon) \to \mathbb{R}^n$ *be a solution to the Cauchy problem* (4.4). *Then there exist a subinterval* $[\tau, \theta)$ *of* $[\tau, \upsilon)$ *and a solution* $X : [\tau, \theta) \to \mathbb{R}^n$ *to the restricted Cauchy problem* (4.5).

First of all, note $Y(t) \in \mathcal{L}(K)$ for all $t \in [\tau, \upsilon)$. Since K is locally closed and $Y(t) \in \mathcal{T}_K(x)$ for all $x \in K$ and for all $t \in [\tau, \upsilon)$, it follows from the cited result in [12] that for every $\tilde{x} \in K$ and for every $\tilde{\tau} \in [\tau, \upsilon)$ there exist a subinterval $[\tilde{\tau}, \tilde{\upsilon})$ of $[\tau, \upsilon)$ and a classical solution $X : [\tilde{\tau}, \tilde{\upsilon}) \to \mathbb{R}^n$ to the restricted Cauchy problem

$$\begin{gathered} X'(t) = Y(t), \quad X(t) \in K, \\ X(\tilde{\tau}) = \tilde{x}. \end{gathered}$$

In case $(\tilde{\tau}, \tilde{x}) = (\tau, x)$ we get the conclusion of the lemma.

Lemma 4.6. *Let the set* K *be locally closed and let the confluence tangency condition* (1.5) *be satisfied. Let* $x \in K$, *let* $y \in L$, *let* $\tau \in [a, b)$, *let* $[\tau, \upsilon)$ *be a subinterval of* $[\tau, \upsilon)$, *and let* $Y : [\tau, \upsilon) \to \mathbb{R}^n$ *be a solution to the differential system* (4.4). *Let* $[\tau, \theta)$ *be a subinterval of* $[\tau, \upsilon)$ *and let* $X : [\tau, \theta) \to \mathbb{R}^n$ *be a solution to the restricted Cauchy problem* (4.5). *If* $\int_\tau^\theta \|Y(t)\| dt < +\infty$, *if the set*

$$\overline{B}\Big(x, \int_\tau^\theta \|Y(t)\| dt\Big) \cap K$$

is closed, and if $\theta < \upsilon$, *then* X *is not a saturated solution to* (4.5).

Under the hypotheses of the lemma, $\hat{x} = \lim_{t\to\upsilon} X(t)$ makes sense and belongs to K. According to Lemma 4.5, there exist a subinterval $[\theta, \tilde{\theta})$ of $[\tau, \upsilon)$ and a solution $\hat{X} : [\theta, \tilde{\theta}) \to \mathbb{R}^n$ to the restricted Cauchy problem

$$\hat{X}'(t) = Y(t), \quad \hat{X}(t) \in K,$$
$$\hat{X}(\theta) = \hat{x}.$$

Let $\tilde{X}(t)$ equal $X(t)$ if $t \in [\tau, \theta)$ and let it equal $\hat{X}(t)$ if $t \in [\theta, \tilde{\theta})$. Since $\tilde{X}$ is a solution to (4.5), it follows X is not saturated.

The fourth auxiliary result concerns solutions to the Cauchy problems (4.4) and (4.5) on subintervals of $[\tau^*, \upsilon^*)$.

Lemma 4.7. *Let $(\tau, x, y) \in S(\tau^*, x^*, y^*)$, let $[\tau, \upsilon)$ be a subinterval of $[\tau, \upsilon^*)$, and let $Y : [\tau, \upsilon) \to \mathbb{R}^n$ be a solution to the Cauchy problem* (4.4)*. Then $\|Y(t) - y\| \le \mu'(t; \tau, 0)$ for all $t \in [\tau, \upsilon)$. Moreover, if $\upsilon < \upsilon^*$, then Y is not a saturated solution to* (4.4)*.*

Let $[\tau, \theta)$ be a subinterval of $[\tau, \upsilon)$ and let $X : [\tau, \theta) \to \mathbb{R}^n$ be a solution to the restricted Cauchy problem (4.5)*. Then $\|X(t)\| \le \mu(t; \tau, \|y\|)$ for all $t \in [\tau, \theta)$. Moreover, if $\theta < \upsilon$, then X is not a saturated solution to* (4.5)*.*

To prove the first part of the lemma, note $Y(\tau) \in \operatorname{interior}(V^*)$, let $[\tau, \lambda)$ be the greatest subinterval of $[\tau, \upsilon)$ such that $Y([\tau, \lambda)) \subseteq V$, and note that either $\lambda = \upsilon$ or both $\lambda < \upsilon$, but $Y(\lambda) \notin \operatorname{interior}(V)$. In addition, $\|Y'(t)\| \le m(t)$ for almost all $t \in [\tau, \lambda)$, hence $\|Y(t) - y\| \le \mu'(t; \tau, 0)$ for all $t \in [\tau, \lambda)$. Since

$$Y([\tau, \lambda)) \subseteq \overline{B}(y, \mu'(\upsilon; \tau, 0)) \subseteq \overline{B}(y^*, \mu'(\upsilon^*; \tau^*, 0)) \subseteq \operatorname{interior}(V^*),$$

it follows $\lambda = \upsilon$. Since the set $V^* \cap L$ is closed, so is its subset $\overline{B}(y, \mu'(\upsilon; \tau, 0)) \cap L$, and Lemma 4.4 implies Y is not saturated if $\upsilon < \upsilon^*$.

To prove the second part of the lemma, note $X(\tau) \in \operatorname{interior}(U^*)$, let $[\tau, \lambda)$ be the greatest subinterval of $[\tau, \theta)$ such that $X([\tau, \lambda)) \subseteq U$, and note that either $\lambda = \theta$ or both $\lambda < \theta$, but $X(\lambda) \notin \operatorname{interior}(U)$. In addition $\|X(t) - x\| \le \mu(t; \tau, \|y\|)$ for all $t \in [\tau, \lambda)$. Since

$$X([\tau, \lambda)) \subseteq \overline{B}(x, \mu(\theta; \tau, \|y\|)) \subseteq \overline{B}(x^*, \mu(\upsilon^*; \tau^*, \|y^*\|)) \subseteq \operatorname{interior}(U^*),$$

it follows $\lambda = \theta$. Since the set $U^* \cap K$ is closed, so is its subset $\overline{B}(x, \mu(\theta; \tau, \|y\|)) \cap K$, and Lemma 4.6 implies that X is not saturated if $\theta < \upsilon$.

Now, we are in a position to prove Lemma 4.3.

Let $(\tau, x, y) \in S(\tau^*, x^*, y^*)$. According to the collective existence condition (3.10), there exists a subinterval $[\tau, \upsilon)$ of $[\tau, b)$ and a solution $Y : [\tau, \upsilon) \subseteq [\tau, b) \to \mathbb{R}^n$ to the Cauchy problem (4.4). In view of Theorem 6.1 in Section 6 below, we can suppose Y is saturated. According to Lemma 4.5, there exists a subinterval $[\tau, \theta)$ of $[\tau, \upsilon)$ and solution $X : [\tau, \theta) \subseteq [\tau, \upsilon) \to \mathbb{R}^n$ to the restricted Cauchy problem (4.5). In view of Theorem 6.1, we can suppose X is saturated.

First, we assert that $\upsilon^* \le \upsilon$. Suppose, by contradiction, that $\upsilon < \upsilon^*$. According to Lemma 4.7, Y is not saturated, a contradiction. Second, we assert that $\upsilon^* \le \theta$. Suppose, by contradiction, that $\theta < \upsilon^*$, According to Lemma 4.7, X is not saturated, a contradiction. Now, the restrictions of X and Y to $[\tau, \upsilon^*)$ satisfy the required conclusion.

Finally, we are in a position to prove Theorem 4.2.

Let $\phi \in \Phi([\tau^*, \upsilon^*))$. Then $[\tau^*, \upsilon^*)$ equals the union of a family of j mutually disjoint intervals $[\tau_i, \tau_{i+1})$ such that $\tau_1 = \tau^*$, $\tau_{j+1} = \upsilon^*$, and $\phi(t) = \tau_i$ whenever $t \in [\tau_i, \tau_{i+1})$.

In view of Lemma 4.3, there exists a family of j functions $(X_i, Y_i) : [\tau_i, \upsilon^*) \to \mathbb{R}^{2n}$ such that:

- (X_1, Y_1) is a solution to the restricted Cauchy problem
$$\begin{gathered} X_1'(t) = Y_1(t),\ X_1(t) \in K, \\ Y_1'(t) \in G(t, x^*, Y_1(t)), \\ X_1(\tau^*) = x^*, \\ Y_1(\tau^*) = y^*, \end{gathered}$$
and moreover, $(t, X_1(t), Y_1(t)) \in S(\tau^*, x^*, y^*)$ for all $t \in [\tau^*, \upsilon^*)$;
- if $i > 1$, then (X_i, Y_i) is a solution to the restricted Cauchy problem
$$\begin{gathered} X_i'(t) = Y_i(t),\ X_i(t) \in K, \\ Y_i'(t) \in G(t, X_{i-1}(\tau_i), Y_i(t)), \\ X_i(\tau_i) = X_{i-1}(\tau_i), \\ Y_i(\tau_i) = Y_{i-1}(\tau_i), \end{gathered}$$
and moreover, $(t, X_i(t), Y_i(t)) \in S(\tau_i, X_{i-1}(\tau_i), Y_{i-1}(\tau_i))$ for all $t \in [\tau_i, \upsilon^*)$.

Now, let $X(t) = X_i(t)$ and $Y(t) = Y_i(t)$ if $t \in [\tau_i, \tau_{i+1})$. Then the function $(X, Y) : [\tau^*, \upsilon^*) \to \mathbb{R}^{2n}$ is a ϕ-approximate solution to (4.1) and $(X, Y)(\tau^*) = (x^*, y^*)$.

Since $(t, X(t), Y(t)) \in S(\tau^*, x^*, Y^*)$ for all $t \in [\tau^*, \upsilon^*)$, it follows $X([\tau^*, \upsilon^*)) \subseteq U^* \cap K$, and $Y([\tau^*, \upsilon^*)) \subseteq V^* \cap L$.

5. Relation to earlier work

The fact that the tangency condition (1.5) and the existence condition (3.10) imply together the existence condition (3.2) is implicitly dealt with in [1, 2, 9, 11]. There, the necessary tangency condition (1.5) must replace the conditions in [1, assumption (H1), p. 185] and [11, assumption (H2), p. 3] as well as in [2, condition (A5), p. 2] and [9, Hypothesis 2.3 iv), p. 177]. The use of useless or superfluous second order tangency conditions renders extremely intricate the construction of the correponding approximate solutions.

Consider the second order differential inclusion
$$X''(t) \in F(X(t), X'(t)) + f(t, X(t), X'(t)) \tag{5.1}$$
where $F : K \times L \to \mathbb{R}^n$ is a multifunction with nonempty values, $f : [a, b) \times K \times L \to \mathbb{R}^n$ is a function, $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n$ are nonempty sets, and $[a, b)$ is a nonempty, possibly unbounded interval.

The existence condition for the differential inclusion (5.1) states that

> for every $x \in K$, for every $y \in L$, and for every $\tau \in [a, b)$ there exist a subinterval $[\tau, \upsilon)$ of $[a, b)$ and a solution $X : [\tau, \upsilon) \to \mathbb{R}^n$ to the second order differential inclusion (5.1) such that $X(\tau) = x$ and $X'(\tau) = y$. (5.2)

In the literature, there are many results which establish existence of solutions to the first order differential inclusion

$$Y'(t) \in \Omega(Y(t)) + \omega(t, Y(t))$$

in case $\omega : [a, b) \times L \to \mathbb{R}^n$ is a special Carathéodory function and $\Omega : L \to \mathbb{R}^n$ is a special upper semicontinuous multifunction with compact, but not necessarily convex values. In this regard, the pioneering result is the one in [3, p. 73, Theorem], where $L = \mathbb{R}^n$. Each of these results provides a setting in which there is satisfied the collective existence condition for the x-“collection” of first order differential inclusions

$$Y'(t) \in F(x, Y(t)) + f(t, x, Y(t)). \tag{5.3}$$

This collective existence condition states that

for every $x \in K$, for every $y \in L$, and for every $\tau \in [a, b)$ there exist a subinterval $[\tau, \upsilon)$ of $[a, b)$ and a solution $Y : [\tau, \upsilon) \to \mathbb{R}^n$ to the first order differential inclusion (5.3) such that $Y(\tau) = y$. (5.4)

The result of this section shows that the confluence tangency condition (1.5) and the collective existence condition (5.4) implies the existence condition (5.2). Such a result does hold if the Carathéodory function f is a special function in that (cf. [3, p. 72, iii)])

for every $x \in K$, for every $y \in L$, and for every $\tau \in [a, b)$ there exist a neighborhood U of x, a neighborhood V of y, a subinterval $[\tau, \upsilon)$ of $[a, b)$, and a locally integrable function $m : [\tau, \upsilon) \to \mathbb{R}$ such that $\sup_{u \in U \cap K, v \in V \cap L} \|f(t, u, v)\| \leq \sqrt{m(t)}$ for almost all $t \in [\tau, \upsilon)$, (5.5)

whereas the upper semicontinuous multifunction F with compact, but not necessarily convex values is a special multifunction in that (cf. [3, p. 72, ii)])

for every $x \in K$ and for every $y \in L$ there exist a neighborhood U of x, a convex, open neighborhood V of y, and a convex function $\mathcal{V} : V \to \mathbb{R}$ such that $F(x, y) \subseteq \partial \mathcal{V}(y)$ for all for all $x \in U \cap K$ and for all $y \in V \cap L$. (5.6)

Here, $\partial \mathcal{V}(y)$ stands for the convex subdifferential of $\mathcal{V}$ at the point y. Recall

$$\partial \mathcal{V}(y) = \{z \in \mathbb{R}^n; \forall \tilde{y} \in V, \mathcal{V}(y) + \langle z, \tilde{y} - y \rangle \leq \mathcal{V}(\tilde{y})\}$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product on $\mathbb{R}^n$.

Theorem 5.1. *Let the sets K and L be locally closed, let the Carathéodory function f satisfy the condition* (5.5)*, and let the compact valued, upper semicontinuous multifunction G satisfy the condition* (5.6)*. Let the confluence tangency condition* (1.5) *and the collective existence condition* (5.4) *be satisfied. Then the existence condition* (5.2) *is satisfied too.*

To prove the theorem, we first note that the multifunction $G(t, x, y) = F(x, y) + f(t, x, y)$ satisfies the hypotheses of Theorem 4.2.

Now, let $x \in K$, $y \in L$, and $\tau \in [a, b)$.

Further, let $\tilde{U}$, $\tilde{V}$, and $\mathcal{V}$ be the items provided by the condition (5.6). Further, let U, V, $[\tau, \upsilon)$, and $m : [\tau, \upsilon)$ be the items provided by the condition (5.5). We can suppose, taking smaller U and V if necessary, that $U \subseteq \tilde{U}$ and $V \subseteq \tilde{V}$. Since

F is compact valued it follows $\|F(x,y)\| < +\infty$. Since F is upper semicontinuous at (x,y), we can suppose, taking smaller U and V if necessary, that

$$M = \sup_{u\in U\cap K, v\in V\cap L} \|F(u,v)\| < +\infty.$$

Since K and L are locally closed, we can suppose, taking smaller U and V if necessary, that $U\cap K$ and $V\cap L$ are compact. In view of Theorem 4.2, we can suppose, taking a smaller υ if necessary, that there holds the approximate existence condition (4.3).

Now, let $\epsilon_j > 0$ be a sequence which converges to 0 and, for every $j\geq 1$, let $\phi_j \in \Phi$ such that $\phi_j(t) - t \leq \epsilon_j$ for all $t\in[\tau,\upsilon)$, so that $\phi_j(t)$ converges to t uniformly on $[\tau,\upsilon)$. Further, let $(X_j,Y_j) : [\tau,\upsilon)\to\mathbb{R}^{2n}$ be a solution to the ϕ_j-differential system

$$\begin{gathered} X_j'(t) = Y_j(t),\\ Y_j'(t) \in F(X_j(\phi_j(t)), Y_j(t)) + f(t, X_j(\phi_j(t)), Y_j(t)), \end{gathered}$$

such that $(X_j,Y_j)(\tau) = (x,y)$, $X_j([\tau,\upsilon))\subseteq U\cap K$ and $Y_j([\tau,\upsilon))\subseteq V\cap L$.

Since $\|Y_j'(t)\| \leq M + \sqrt{m(t)}$ for almost all $t\in[\tau,\upsilon)$, it follows there exists a subsequence still denoted by Y_j and an absolutely continuous function $Y:[\tau,\upsilon)\to\mathbb{R}^n$ such that Y_j converges uniformly to Y, whereas Y_j' converges to Y' in the weak topology of $L^2([\tau,\upsilon))$. Let $X(t) = x + \int_\tau^t Y(s)ds$ for all $t\in[\tau,\upsilon)$, so that $X'(t) = Y(t)$ and $X(\tau) = x$. We shall show that

$$Y'(t) \in F(X(t), Y(t)) + f(t, X(t), Y(t))$$

almost everywhere. Let $Z(t) = Y'(t) - f(t, X(t), Y(t))$. We have to show that

$$Z(t) \in F(X(t), Y(t))$$

almost everywhere. Let $Z_j(t) = Y_j'(t) - f(t, X_j(\phi_j(t)), Y_j(t))$ and note

$$Z_j(t) \in F(X_j(\phi_j(t)), Y_j(t)).$$

Since $Z_j(t)\in\partial\mathcal{V}(Y_j(t))$ for all j, it follows $Z(t)\in\partial\mathcal{V}(Y(t))$. Further,

$$\begin{gathered} (\mathcal{V}\circ Y_j)'(t) = \langle Z_j(t), Y_j'(t)\rangle,\\ (\mathcal{V}\circ Y)'(t) = \langle Z(t), Y'(t)\rangle; \end{gathered}$$

therefore,

$$\begin{gathered} \mathcal{V}(Y_j(\upsilon)) - \mathcal{V}(y) = \int_\tau^\upsilon \|Y_j'(t)\|^2 ds - \int_\tau^\upsilon \langle f(t, X_j(\phi_j(t)), Y_j(t)), Y_j'(t)\rangle dt,\\ \mathcal{V}(Y(\upsilon)) - \mathcal{V}(y) = \int_\tau^\upsilon \|Y'(t)\|^2 ds - \int_\tau^\upsilon \langle f(t, X(t), Y(t)), Y'(t)\rangle dt. \end{gathered}$$

Since $f(\cdot, X_j(\phi_j(\cdot)), Y_j(\cdot))$ converges to $f(\cdot, X(\cdot), Y(\cdot))$ strongly in $L^2([\tau,\upsilon))$, it follows

$$\lim_{j\to+\infty}\int_\tau^\upsilon \langle f(t, X_j(\phi_j(t)), Y_j(t)), Y_j'(t)\rangle dt = \int_\tau^\upsilon \langle f(t, X(t), Y(t)), Y'(t)\rangle dt.$$

Further

$$\lim_{j\to+\infty}\mathcal{V}(Y_j(\upsilon)) = \mathcal{V}(Y(\upsilon)),$$

hence

$$\lim_{j\to\infty}\int_\tau^\upsilon \|Y_j'(t)\|^2 ds = \int_\tau^\upsilon \|Y'(t)\|^2 ds,$$

and Y_j' converges to Y' strongly in $L^2([\tau, \upsilon))$. Then there exists a subsequence still denoted by Y_j such that $Y_j'(t)$ converges to $Y'(t)$ almost everywhere, so that $Z_j(t)$ converges to $Z(t)$ almost everywhere. Since the restriction of F to $(U \cap K) \times (V \times K)$ is closed and since $(Z_j(t), X_j(\phi_j(t)), Y_j(t))$ belongs to graph(F) for all j, it follows also $(Z(t), X(t), Y(t))$ belongs to graph(F), and the theorem is proved.

6. Dependent choices and saturated solutions

In this final section we show that by using the axiom of dependent choices, a weaker form of the axiom of choice, it can be proved existence of saturated solutions in an abstract setting which is free of any topological feature, but is suitable for any differential equation theory (see [8, p. 382, Lemma 16], [17, p. 288], and [18, p. 76], where the source result in [7, p. 356, Corollary 1] is adapted).

Let M be an abstract space, let $[a, b) \subseteq \mathbb{R}$ be a nonempty, possibly unbounded interval, and let Ξ be a family of functions $\xi : [\tau, \upsilon) \to M$ defined on subintervals $[\tau, \upsilon)$ of $[a, b)$.

In the following we restrict the usual notion of function extension. We say that a function $\tilde{\xi} : [\tilde{\tau}, \tilde{\upsilon}) \to M$ *extends* a function $\xi : [\tau, \upsilon) \to M$ if $[\tau, \upsilon) \subseteq [\tilde{\tau}, \tilde{\upsilon})$, if ξ equals the restriction of $\tilde{\xi}$ to $[\tau, \upsilon)$, and moreover, if $\tilde{\tau} = \tau$.

Assume that for every sequence of functions $\xi_j : [\tau, \upsilon_j) \to M$ in Ξ such that each ξ_{j+1} extends ξ_j there exists a function $\xi : [\tau, \upsilon) \to M$ in Ξ such that ξ extends each ξ_j. In this case we say that Ξ is a *family of solutions* and the functions $\xi : [\tau, \upsilon) \to M$ in Ξ are *solutions*. Finally, we say that a solution is *saturated* if it is not extended by any other solution.

Theorem 6.1. *Every solution is extended by a saturated solution.*

To prove the theorem, for every solution $\xi : [\tau, \upsilon) \to M$, we consider the family of its extending solutions $\tilde{\xi} : [\tau, \tilde{\upsilon}) \to M$, we note ξ belongs to this family, and we denote by $\Upsilon(\xi)$ the supremum of the corresponding family of $\tilde{\upsilon}$'s. Clearly, $\upsilon \leq \Upsilon(\xi) \leq b$, and moreover, ξ is saturated if and only if $\upsilon = \Upsilon(\xi)$. Note parenthetically that, if a solution $\tilde{\xi} : [\tau, \tilde{\upsilon}) \to M$ extends a solution $\xi : [\tau, \upsilon) \to M$, then $\upsilon \leq \tilde{\upsilon} \leq \Upsilon(\tilde{\xi}) \leq \Upsilon(\xi)$.

Now, let $\xi : [\tau, \upsilon) \to M$ be a solution. We have to show that there exists an extending solution $\tilde{\xi} : [\tau, \tilde{\upsilon}) \to M$ such that $\tilde{\upsilon} = \Upsilon(\tilde{\xi})$.

Assume first that $\upsilon = b$. Then $\upsilon = \Upsilon(\xi)$, hence ξ is saturated, and the conclusion follows.

Assume further that both $\upsilon < b$ and $\Upsilon(\tilde{\xi}) = b$ for all solutions $\tilde{\xi}$ extending ξ. Consider a strictly increasing sequence b_j in (υ, b) which converges to b (recall b may equal $+\infty$). In view of the definition of Υ and using the axiom of dependent choices, we get a sequence of solutions $\xi_j : [\tau, \upsilon_j) \to M$ such that ξ_1 extends ξ and $b_1 < \upsilon_1$, and such that each ξ_{j+1} extends ξ_j and $b_{j+1} < \upsilon_{j+1}$. Let $\tilde{\xi} : [\tau, \tilde{\upsilon}) \to M$ be a solution extending all ξ_j. Then $\tilde{\upsilon} = b$ and $\tilde{\upsilon} = \Upsilon(\tilde{\xi})$, hence $\tilde{\xi}$ is saturated, and the conclusion follows.

Assume finally that both $\upsilon < b$ and $\Upsilon(\tilde{\xi}) < b$ for some solutions $\tilde{\xi}$ extending ξ. In view of the definition of Υ and using the axiom of dependent choices, we get a sequence of solutions $\xi_j : [\tau, \upsilon_j) \to M$ such that ξ_1 extends ξ and $\Upsilon(\xi_1) < b$, and such that each ξ_{j+1} extends ξ_j and $(1/2)(\upsilon_j + \Upsilon(\xi_j)) \leq \upsilon_{j+1} \leq \Upsilon(\xi_j)$. Since $\Upsilon(\xi_{j+1}) - \upsilon_{j+1} \leq \Upsilon(\xi_j) - \upsilon_{j+1} \leq (1/2)(\Upsilon(\xi_j) - \upsilon_j)$, it follows the increasing sequence υ_j and the decreasing sequence $\Upsilon(\xi_j)$ have the same limit. Let $\tilde{\xi} : [\tau, \tilde{\upsilon}) \to M$ be a solution extending all ξ_j. Since $\upsilon_j \le \tilde{\upsilon} \le \Upsilon(\tilde{\xi}) \le \Upsilon(\xi_j)$ for all j, it follows $\tilde{\upsilon} = \Upsilon(\tilde{\xi})$, hence $\tilde{\xi}$ is saturated, and the proof of the theorem is accomplished.

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