Electronic Journal of Differential Equations, Vol. 2009(2009), No. 28, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LIAPUNOV-TYPE INTEGRAL INEQUALITIES FOR CERTAIN HIGHER-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we obtain Liapunov-type integral inequalities for certain nonlinear, nonhomogeneous differential equations of higher order with without any restriction on the zeros of their higher-order derivatives of the solutions by using elementary analysis. As an applications of our results, we show that oscillatory solutions of the equation converge to zero as $t \to \infty$. Using these inequalities, it is also shown that $(t_{m+k} - t_m) \to \infty$ as $m \to \infty$, where $1 \le k \le n-1$ and $\langle t_m \rangle$ is an increasing sequence of zeros of an oscillatory solution of $D^n y + y f(t,y)|y|^{p-2} = 0$, $t \ge 0$, provided that $W(.,\lambda) \in L^\sigma([0,\infty), \mathbb{R}^+)$, $1 \le \sigma \le \infty$ and for all $\lambda > 0$. A criterion for disconjugacy of nonlinear homogeneous equation is obtained in an interval $[a,b]$.

1. Introduction

The Russian mathematician A. M. Liapunov [15] proved the following remarkable inequality: If $y(t)$ is a nontrivial solution of

$$y'' + p(t)y = 0, \tag{1.1}$$

with $y(a) = 0 = y(b)$ $(a < b)$ and $y(t) \neq 0$ for $t \in (a,b)$, then

$$\frac{4}{b-a} < \int_a^b |p(t)|dt, \tag{1.2}$$

where $p \in L^1_{\text{loc}}$. This inequality provides a lower bound for the distance between consecutive zeros of $y(t)$. If $p(t) = p > 0$, then (1.2) yields

$$(b-a) > 2/\sqrt{p}.$$

In [12], the inequality (1.2) is strengthened to

$$\frac{4}{b-a} < \int_a^b p_+(t)dt, \tag{1.3}$$

where $p_+(t) = \max\{p(t), 0\}$. The inequality (1.3) is the best possible in the sense that if the constant 4 in (1.3) is replaced by any larger constant, then there exists

2000 *Mathematics Subject Classification.* 34C10.
Key words and phrases. Liapunov-type inequality; oscillatory solution; disconjugacy; higher order differential equations.

Submitted October 19, 2008. Published February 5, 2009.
Supported by National Board of Higher Mathematics, Department of Atomic Energy, India.

an example of (1.1) for which (1.3) no longer holds (see [12, p. 345], [13]). However, stronger results were obtained in [2, 13]. In [13] it is shown that

$$\int_a^c p_+(t)dt > \frac{1}{c-a} \quad \text{and} \quad \int_c^b p_+(t)dt > \frac{1}{b-c},$$

where $c \in (a,b)$ such that $y'(c) = 0$. Hence

$$\int_a^b p_+(t)dt > \frac{1}{c-a} + \frac{1}{b-c} = \frac{(b-a)}{(c-a)(b-c)} \geq \frac{4}{b-a}.$$

In [2, Corollary 4.1], the authors obtained

$$\frac{4}{b-a} < \Big| \int_a^b p(t)dt \Big|$$

from which (1.2) can be obtained. The inequality finds applications in the study of boundary value problems. It may be used to provide a lower bound on the first positive proper value of the Sturm-Liouville problems

$$\begin{gathered} y''(t) + \lambda q(t)y = 0 \\ y(c) = 0 = y(d) \quad (c < d) \end{gathered}$$

and

$$\begin{gathered} y''(t) + (\lambda + q(t))y = 0 \\ y(c) = 0 = y(d) \quad (c < d) \end{gathered}$$

by letting $p(t)$ to denote $\lambda q(t)$ and $\lambda + q(t)$ respectively in (1.2). The disconjugacy of (1.1) also depends on (1.2). Indeed, equation (1.1) is said to be disconjugate if

$$\int_a^b |p(t)|dt \leq 4/(b-a).$$

Equation (1.1) is said to be disconjugate on $[a,b]$ if no non-trivial solution of (1.1) has more than one zero. Thus (1.2) may be regarded as a necessary condition for conjugacy of (1.1). Inequality (1.2) has lots of applications in eigenvalue problems, stability, etc. A number of proofs are known and generalizations and improvements have also been given (see [12, 14, 22, 24, 25]. Inequality (1.3) was generalized to the condition

$$\int_a^b (t-a)(b-t)p_+(t)dt > (b-a) \tag{1.4}$$

by Hartman and Wintner [11]. An alternate proof of the inequality (1.4), due to Nihari [17], is given in [12, Theorem 5.1 Ch XI]. For the equation

$$y''(t) + q(t)y' + p(t)y = 0, \tag{1.5}$$

where $p, q \in C([0,\infty), R)$, Hartman and Wintner [11] established the inequality

$$\int_a^b (t-a)(b-t)p_+(t)dt + \max\Big\{ \int_a^b (t-a)|q(t)|, \int_a^b (b-t)|q(t)|dt \Big\} > (b-a) \tag{1.6}$$

which reduces to (1.4) when $q(t) = 0$. In particular, (1.6) implies the *de la vallee Poussin inequality* [23]. In [10], Galbraith has shown that if a and b are successive zeros of (1.1) with $p(t) \geq 0$ a linear function, then

$$(b-a)\int_a^b p(t)dt \leq \pi^2.$$

This inequality provides an upper bound for two successive zeros of an oscillatory solution of (1.1). Indeed, if $p(t) = p > 0$, then $(b-a) \leq \pi/(p)^{1/2}$. Fink [8], obtained both upper and lower bounds of $(b-a)\int_a^b p(t)dt$, where $p(t) \geq 0$ is linear. Indeed, he showed that

$$\frac{9}{8}\lambda_0^2 \leq (b-a)\int_a^b p(t)dt \leq \pi^2$$

and that these are the best possible bounds, where λ_0 is the first positive zero of $J_{1/3}$ and J_n is the Bessel function. The constant $\frac{9}{8}\lambda_0^2 = 9.478132\dots$ and $\pi^2 = 9.869604\dots$, so that it gives a delicate test for the spacing of the zeros for linear p. Fink [9] investigated the behaviour of the functional $(b-a)\int_a^b p(t)dt$, where p is in a certain class of sub or supper functions. Eliason [4, 5] obtained upper and lower bounds of the functional $(b-a)\int_a^b p(t)dt$, where $p(t)$ is concave or convex. St Marry and Eliason [16] considered the same problem for (1.5). Bailey and Waltman [1] applied different techniques to obtain both upper and lower bounds for the distance between two successive zeros of solution of (1.5). They also considered nonlinear equations. In a recent paper, Brown and Hinton [2] used Opial's inequality to obtain lower bounds for the spacing of the zeros of a solution of (1.1) and lower bounds of the spacing $\beta - \alpha$, where $y(t)$ is a solution of (1.1) satisfying $y(\alpha) = 0 = y'(\beta)$ and $y'(\alpha) = 0 = y(\beta)(\alpha < \beta)$.

Inequality (1.2) is generalized to second order nonlinear differential equation by Eliason [5], to delay differential equations of second order in [6, 7] and by Dahiya and Singh [3], and to higher order differential equation by Pachpatte [18]. In a recent work [20], the authors have obtained a Liapunov-type inequality for third order differential equations of the form

$$y''' + p(t)y = 0, \tag{1.7}$$

where $p \in L^1_{\rm loc}$. The inequality is used to study many interesting properties of the zeros of an oscillatory solution of (1.7) (see [20, Theorems 5, 6]). Indeed, Pachpatte derived Liapunov-type inequalities for the equation of the form

$$\begin{gathered}
D^n[r(t)D^{n-1}(p(t)g(y'(t)))] + y(t)f(t,y(t)) = Q(t),\\
D^n[r(t)D^{n-1}(p(t)h(y(t))y'(t))] + y(t)f(t,y(t)) = Q(t),\\
D^n[r(t)D^{n-1}(p(t)h(y(t))g(y'(t)))] + y(t)f(t,y(t)) = Q(t),
\end{gathered} \tag{1.8}$$

under appropriate conditions, where $n \geq 2$ is an integer and $D = d^n/dt^n$. It is clear that the results in [18] are not applicable to odd order equations. Furthermore, he has taken the restriction on the zeros of higher order derivatives [18, Theorem 1]. We may observe that in [18, p.530, Example], $y'''(3\pi/4) \neq 0$ because $y'''(t) = 2e^{-t}(\cos t - \sin t)$. On the other hand, $y'''(\pi/4) = 0$ but $\pi/4 \notin (\pi/2, 3\pi/2)$ and $y'''(5\pi/4) = 0$ but $5\pi/4 < \pi$. Although this example does not illustrate [18, Theorem 1], it has motivated us to remove the restriction on the zeros of higher order derivatives of the solution of (1.5).

The objective of this paper is to obtain Liapunov-type integral inequality for the nth-order differential equation

$$\Big(\frac{1}{r_{n-1}(t)}\dots\Big(\frac{1}{r_2(t)}\Big(\frac{1}{r_1(t)}|y'(t)|^{p-2}y'(t)\Big)'\Big)'\dots\Big)' + |y(t)|^{p-2}f(t,y(t))y = Q(t), \tag{1.9}$$

under appropriate assumptions on $r_i(t)$, $1 \le i \le n-1, f$ and Q. Here $n \ge 2, p > 1$ are even and odd integers. In this work we remove this restriction on the zeros of higher order derivatives. Further, we show that every oscillatory solution of (1.9) converges to zero as $t \to \infty$ with the help of Liapunov-type inequality. We also generalize a theorem of Patula [22, Theorem 2] to higher order equations. A criteria for diconjugacy of nonlinear homgeneous equation is obtained in an interval $[a, b]$ by the help of the inequality.

2. Main results

Equation (1.9) may be written as

$$D^n y + y f(t, y)|y(t)|^{p-2} = Q(t), \tag{2.1}$$

where $n \ge 2$ is an integer,

$$Dy = \frac{1}{r_1(t)}|y'(t)|^{p-2}y'(t), \quad D^i y = \frac{1}{r_i(t)}(D^{i-1}y)',$$

$2 \le i \le n$, and $r_n(t) \equiv 1$. We assume that

- (C1) $r_i : I \to \mathbb{R}$ is continuous and $r_i(t) > 0, 1 \le i \le n-1$ and $Q : I \to \mathbb{R}$ is continuous, where I is a real interval.
- (C2) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous such that $|f(t, y)| \le W(t, |y|)$, where $W : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $W(t, u) \le W(t, v)$ for $0 \le u \le v$ and $\mathbb{R}^+ = [0, \infty]$.

We define

$$\begin{aligned}
&E(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); z(s_{n-1})) \\
&= r_2(t) \int_{\alpha_1}^{t} r_3(s_2) \int_{\alpha_2}^{s_2} r_4(s_3) \dots \\
&\quad \int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \int_{\alpha_{n-2}}^{s_{n-2}} z(s_{n-1}) ds_{n-1} ds_{n-2} \dots ds_2,
\end{aligned}$$

where $z(t)$ is a real valued continuous function defined on $[a, b] \subset I (a < b)$ and $\alpha_1, \alpha_2, \dots, \alpha_{n-2}$ are suitable points in $[a, b]$, and

$$\begin{aligned}
&\overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); z(s_{n-1})) \\
&= r_2(t) \Big| \int_{\alpha_1}^{t} r_3(s_2) \Big| \int_{\alpha_2}^{s_2} r_4(s_3) \dots \Big| \int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \Big| \int_{\alpha_{n-2}}^{s_{n-2}} z(s_{n-1}) ds_{n-1} \Big| ds_{n-2} \Big| \\
&\quad \dots \Big| ds_2 \Big|.
\end{aligned}$$

Theorem 2.1. *Suppose that* (C1)-(C2) *hold. Let $\alpha_1, \alpha_2, \dots, \alpha_{n-2} \in [a, b]$, where $\alpha_1, \alpha_2, \dots, \alpha_{n-2}$ are the zeros of $D^2y(t), D^3y(t), \dots, D^{n-2}y(t), D^{n-1}y(t)$ respectively, $[a, b] \subset I (a < b)$ and $y(t)$ is a nontrivial solution of* (2.1) *with $y(a) = 0 = y(b)$. If c is a point in (a, b) where $|y(t)|$ attains maximum and $M = \max\{|y(t)| : t \in [a, b]\} = |y(c)|$, then*

$$\begin{aligned}
1 < (\frac{1}{2})^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1 \Big)^{p-1} \Big(\int_a^b \big[\overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); \\
W(s_{n-1}, M)) + \frac{1}{M^{p-1}} \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|) \big] ds_1 \Big),
\end{aligned} \tag{2.2}$$

for $n \geq 3$ and

$$1 < \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(t))^{1/(p-1)} dt\Big)^{p-1} \Big[\int_a^b W(t,M)dt + \frac{1}{M^{p-1}} \int_a^b |Q(t)|dt\Big], \quad (2.3)$$

for $n = 2$.

Proof. Let $n \geq 3$. Integrating (2.1) from α_{n-2} to $t \in [a,b]$, we obtain

$$\begin{aligned} &D^{n-1}y(t) + \int_{\alpha_{n-2}}^t y(s_{n-1}) f(s_{n-1}, y(s_{n-1}))|y(s_{n-1})|^{p-2} ds_{n-1} \\ &= \int_{\alpha_{n-2}}^t Q(s_{n-1}) ds_{n-1}; \end{aligned}$$

that is,

$$\begin{aligned} &(D^{n-2}y(t))' + r_{n-1}(t) \int_{\alpha_{n-2}}^t y(s_{n-1}) f(s_{n-1}, y(s_{n-1}))|y(s_{n-1})|^{p-2} ds_{n-1} \\ &= r_{n-1}(t) \int_{\alpha_{n-2}}^t Q(s_{n-1}) ds_{n-1}. \end{aligned}$$

Further integration from α_{n-3} to $t \in [a,b]$ yields

$$\begin{aligned} &D^{n-2}y(t) \\ &+ \int_{\alpha_{n-3}}^t r_{n-1}(s_{n-2}) \Big(\int_{\alpha_{n-2}}^{s_{n-2}} y(s_{n-1}) f(s_{n-1}, y(s_{n-1}))|y(s_{n-1})|^{p-2} ds_{n-1}\Big) ds_{n-2} \\ &= \int_{\alpha_{n-3}}^t r_{n-1}(s_{n-2}) \Big(\int_{\alpha_{n-2}}^{s_{n-2}} Q(s_{n-1}) ds_{n-1}\Big) ds_{n-2}. \end{aligned}$$

Proceeding as above we obtain

$$\begin{aligned} &D^2y(t) + \int_{\alpha_1}^t r_3(s_2) \int_{\alpha_2}^{s_2} r_4(s_3) \dots \\ &\int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \int_{\alpha_{n-2}}^{s_{n-2}} y(s_{n-1}) f(s_{n-1}, y(s_{n-1}))|y(s_{n-1})|^{p-2} ds_{n-1} ds_{n-2} \dots ds_2, \\ &= \int_{\alpha_1}^t r_3(s_2) \int_{\alpha_2}^{s_2} r_4(s_3) \dots \int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \int_{\alpha_{n-2}}^{s_{n-2}} Q(s_{n-1}) ds_{n-1} ds_{n-2} \dots ds_2; \end{aligned}$$

that is,

$$\begin{aligned} &(Dy(t))' + E(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); y(s_{n-1}) f(s_{n-1}, y(s_{n-1}))|y(s_{n-1})|^{p-2}) \\ &= E(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); Q(s_{n-1})). \end{aligned}$$

Hence

$$\begin{aligned} |(Dy(t))'| &\leq M^{p-1} \overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)) \\ &\quad + \overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|). \end{aligned} \quad (2.4)$$

Since

$$\begin{aligned} M = |y(c)| &= \Big|\int_a^c y'(s_1) ds_1\Big| \leq \int_a^c |y'(s_1)| ds_1, \\ M = |y(c)| &= \Big|\int_c^b y'(s_1) ds_1\Big| \leq \int_c^b |y'(s_1)| ds_1, \end{aligned}$$

it follows that

$$2M \le \int_a^b |y'(s_1)| ds_1.$$

First, using Hölders inequality with indices p and $p/(p-1)$ and then integrating by parts we obtain

$$\begin{aligned}
M^p &\le \big(\frac{1}{2}\big)^p \Big(\int_a^b |y'(s_1)| ds_1\Big)^p \\
&= \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/p}(r_1(s_1))^{-1/p}|y'(s_1)| ds_1\Big)^p \\
&\le \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1\Big)^{p-1} \Big(\int_a^b (r_1(s_1))^{-1}|y'(s_1)|^p ds_1\Big) \\
&= \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1\Big)^{p-1} \Big([(r_1(s_1))^{-1}|y'(s_1)|^{p-2}y'(s_1)y(s_1)]_a^b \qquad (2.5)\\
&\quad - \int_a^b [(r_1(s_1))^{-1}|y'(s_1)|^{p-2}y'(s_1)]'y(s_1) ds_1\Big) \\
&= -\big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1\Big)^{p-1} \int_a^b (Dy)'(s_1)y(s_1) ds_1 \\
&\le \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1\Big)^{p-1} \int_a^b |(Dy)'(s_1)||y(s_1)| ds_1.
\end{aligned}$$

Using (2.4),

$$\begin{aligned}
M^p &< \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1\Big)^{p-1} \\
&\quad \times \Big[M^p \int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)) ds_1 \\
&\quad + M \int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|) ds_1\Big];
\end{aligned}$$

that is,

$$\begin{aligned}
1 &< \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1\Big)^{p-1} \\
&\quad \times \Big[\int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)) ds_1 \\
&\quad + \frac{1}{M^{p-1}} \int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|) ds_1\Big].
\end{aligned}$$

When $n = 2$, (2.1) has the form

$$(Dy)'(t) + y(t)f(t, y(t))|y(t)|^{p-2} = Q(t).$$

Hence (2.5) yields

$$M^p < \big(\frac{1}{2}\big)^p \Big(\int_a^b (r_1(s_1))^{1/(p-1)} ds_1\Big)^{p-1} \Big[\int_a^b |y(t)|^p |f(t, y(t))| dt + \int_a^b |y(t)||Q(t)| dt\Big];$$

that is,

$$1 < \Big(\frac{1}{2}\Big)^p \Big(\int_a^b (r_1(t))^{1/(p-1)} dt \Big)^{p-1} \Big[\int_a^b W(t,M) dt + \frac{1}{M^{p-1}} \int_a^b |Q(t)| dt \Big].$$

Thus the proof is complete. □

Remarks. If $r_i(t) = 1; i = 1, 2, \dots, n-1$; $p = 2$; $f(t, y) = p(t)$ and $n = 2, 3$; then inequalities (2.3) and (2.2) reduce respectively, to the inequalities (1.2) and

$$\int_a^b |p(t)| dt > 4/(b-a)^2.$$

This inequality provides a lower bound of the distance between consecutive zeros of the solution $y(t)$. For the various applications of this inequality one can see [20].

Liapunov-type integral inequalities for (1.8) can be obtained under suitable assumptions on g and h.

If $r_i(t) = 1$; $i = 1, 2, \dots, n-1$; $n = 3$, $p = 2$, $f(t, y) = q(t)|y(t)|^{\beta-1}$ and $Q(t) = 0$, then (2.1) reduces to

$$y'''(t) + q(t)|y(t)|^{\beta-1} y = 0, \quad t \ge 0, \tag{2.6}$$

where β is a positive constant and $q : [0, \infty) \to [0, \infty)$ is a continuous function is called an *Emden-Fowler* equations of third order. If $y(t)$ is a solution of (2.6) with $y(a) = 0 = y(b)$, $(a < b)$ and $y(t) \neq 0$ for $t \in (a, b)$, then the spacing between zeros of solutions of (2.6) may be computed by using (2.2).

Example 2.2. Consider

$$y'''(t) + y^2(t) = \sin^2 t - \cos t, \quad t \ge 0. \tag{2.7}$$

Clearly, $y(t) = \sin t$ is a solution of (2.7) with $y(0) = 0 = y(\pi)$, $y''(0) = 0 = y''(\pi)$. $M = \max_{t\in[0,\pi]} |\sin t| = 1$. From Theorem 2.1 it follows that

$$1 < \frac{\pi}{4} \int_0^\pi [\overline{E}(s_1, r_2(s_1), W(s_2, M)) + \frac{1}{M} \overline{E}(s_1, r_2(s_1), |Q(s_2)|)] ds_1,$$

where

$$\overline{E}(s_1, r_2(s_1), W(s_2, M)) = \Big| \int_0^{s_1} M ds_2 \Big| = \begin{cases} s_1, & s_1 > 0, \\ -s_1, & s_1 < 0, \end{cases}$$

$$\overline{E}(s_1, r_2(s_1), |Q(s_2)|) = \Big| \int_0^{s_1} \Big| \sin^2 s_2 - \cos s_2 \Big| ds_2 \Big| = \begin{cases} 2s_1, & s_1 > 0, \\ -2s_1, & s_1 < 0. \end{cases}$$

Hence

$$\int_0^\pi \overline{E}(s_1, r_2(s_1), W(s_2, M)) ds_1 = \begin{cases} \pi^2/2, & s_1 > 0, \\ -\pi^2/2, & s_1 < 0, \end{cases}$$

$$\int_0^\pi \overline{E}(s_1, r_2(s_1), |Q(s_2)|) ds_1 = \begin{cases} \pi^2, & s_1 > 0, \\ -\pi^2, & s_1 < 0. \end{cases}$$

As $\overline{E} > 0$, then $s_1 > 0$ and

$$\int_0^\pi \overline{E}(s_1, r_2(s_1), W(s_2, M))ds_1 = \pi^2/2,$$
$$\int_0^\pi \overline{E}(s_1, r_2(s_1), |Q(s_2)|)ds_1 = \pi^2.$$

Thus by Theorem 2.1, $1 < 3\pi^3/8$ or $3\pi^3 > 8$, which is obviously true.

Theorem 2.3. *Suppose that* (C1)-(C2) *hold. Let* $\alpha_1, \alpha_2, \dots, \alpha_{n-3}, \alpha_{n-2}$ *be the zeros of* $D^2y(t), D^3y(t), \dots, D^{n-2}y(t), D^{n-1}y(t)$ *respectively, in* $[a, b] \subset I(a < b)$, *where* $y(t)$ *is a nontrivial solution of*

$$D^n y + yf(t, y)|y(t)|^{p-2} = 0$$

with $y(a) = 0 = y(b)$. *If* c *is a point in* (a, b), *where* $|y(t)|$ *attains a maximum, then the point 'c' cannot be very close to 'a' as well as 'b'.*

Proof. Let $M = \max\{|y(t)| : t \in [a, b]\} = |y(c)|$. Then $y'(c) = 0$. Since

$$y(c) = \int_a^c y'(t)dt,$$

using Hölders inequality with indices p and $p/(p-1)$ and then integrating by parts we obtain

$$\begin{aligned}
&M^p \\
&\le (\frac{1}{2})^p \Big(\int_a^c |y'(t)|dt\Big)^p \\
&= (\frac{1}{2})^p \Big(\int_a^c r_1(t)^{1/p} r_1(t)^{-1/p}|y'(t)|dt\Big)^p \\
&\le (\frac{1}{2})^p \Big(\int_a^c r_1(t)^{1/(p-1)}\Big)^{p-1} \Big(\int_a^c r_1(t)^{-1}|y'(t)|^p dt\Big) \\
&= (\frac{1}{2})^p \Big(\int_a^c r_1(t)^{1/(p-1)}\Big)^{p-1} \Big(\Big[r_1(t)^{-1}|y'(t)|^{p-2}y'(t)y(t)\Big]_a^c - \int_a^c (Dy)'(t)y(t)dt\Big) \\
&\le (\frac{1}{2})^p \Big(\int_a^c r_1(t)^{1/(p-1)}\Big)^{p-1} \Big(\int_a^c |(Dy)'(t)||y(t)|dt\Big).
\end{aligned}$$

Proceeding as Theorem 2.1 we obtain

$$|(Dy)'(t)| \le M^{p-1}\overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)).$$

Hence

$$\begin{aligned}
1 < (\frac{1}{2})^p &\Big(\int_a^c r_1(t)^{1/(p-1)}\Big)^{p-1} \\
&\times \Big(\int_a^c \overline{E}(t, r_2(t), r_3(s_3), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M))dt\Big);
\end{aligned}$$

that is,

$$\begin{aligned}
&\Big[\Big(\int_a^c r_1(t)^{1/(p-1)}\Big)^{p-1}\Big]^{-1} \\
&< (\frac{1}{2})^p \Big(\int_a^c \overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M))dt\Big) < \infty.
\end{aligned} \tag{2.8}$$

Thus 'c' cannot be very close to 'a' because

$$\lim_{c\to a+}\Big[\Big(\int_a^c r_1(t)^{1/(p-1)}\Big)^{p-1}\Big]^{-1}=\infty.$$

Next we have to show that 'c' cannot be very close to 'b'. Since

$$|y(c)|=\Big|\int_a^c y'(t)dt\Big|,$$

then proceeding as above to obtain

$$\begin{aligned}
M^p &\le (\frac{1}{2})^p\Big(\int_c^b |y'(t)|dt\Big)^p\\
&= (\frac{1}{2})^p\Big(\int_c^b r_1(t)^{1/(p-1)}\Big)^{p-1}\Big(\Big[\int_c^b r_1(t)^{p-1}|y'(t)|^{p-2}y'(t)y(t)\Big]_c^b\\
&\quad -\int_c^b (Dy)'(t)y(t)dt\Big)\\
&\le (\frac{1}{2})^p\Big(\int_c^b r_1(t)^{1/(p-1)}\Big)^{p-1}\int_c^b |(Dy)'(t)||y(t)|dt\\
&< M^p(\frac{1}{2})^p\Big(\int_c^b r_1(t)^{1/(p-1)}\Big)^{p-1}\\
&\quad\times\Big(\int_c^b \overline{E}(t,r_2(t),r_3(s_2),\dots,r_{n-1}(s_{n-2});W(s_{n-1},M))dt\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big[\Big(\int_c^b r_1(t)^{1/(p-1)}\Big)^{p-1}\Big]^{-1}\\
&< (\frac{1}{2})^p\Big(\int_c^b \overline{E}(t,r_2(t),r_3(s_2),\dots,r_{n-1}(s_{n-2});W(s_{n-1},M))dt\Big)<\infty.
\end{aligned}$$

Thus 'c' cannot be very close to 'b' because

$$\lim_{c\to b-}\Big[\Big(\int_c^b r_1(t)^{1/(p-1)}\Big)^{p-1}\Big]^{-1}=\infty.$$

This completes the proof of the theorem. □

We remark that Theorem 2.3 need not hold if $\alpha_i\notin[a,b]$ for some $i\in\{1,2,\dots,n-2\}$.

3. Applications

In this section we present some of the applications of the Liapunov-type inequality obtained in Theorem 2.1 to study the asymptotic behaviour of oscillatory solution of (2.1).

Definition. A solution $y(t)$ of (2.1) is said to be *oscillatory* if there exists a sequence $<t_m>\subset[0,\infty)$ such that $y(t_m)=0$, $m\ge 1$ and $t_m\to\infty$ as $m\to\infty$.

Theorem 3.1. *Suppose that* (C1)-(C2) *hold. Let* $W(t,\lambda)\in L^\sigma([0,\infty),\mathbb{R}^+)$ *for all* $\lambda>0$, *where* $1\le\sigma<\infty$. *Let* $r_i(t)\le K$ *for* $t\ge 0$ *and* $1\le i\le n-1$, *where* $K>0$ *is a constant. If* $<t_m>$ *is an increasing sequence of zeros of an oscillatory solution* $y(t)$ *of*

$$D^n y+yf(t,y)|y(t)|^{p-2}=0\quad t\ge 0,$$

such that $\alpha_1, \alpha_2, \dots, \alpha_{n-2} \in (t_m, t_{m+k})$, $1 \le k \le n-1$, for every large m, then $(t_{m+k} - t_m) \to \infty$, as $m \to \infty$, where $\alpha_1, \dots, \alpha_{n-2}$ are the zeros of $D^2y(t)$, $D^3y(t)$, $\dots$, $D^{n-2}y(t)$, $D^{n-1}y(t)$, respectively.

Proof. If possible, let there exist a subsequence $\langle t_{m_i} \rangle$ of $\langle t_m \rangle$ such that $(t_{m_i+k} - t_{m_i}) \le M$ for every i, where $M > 0$ is a constant. Let $M_{m_i} = \max\{|y(t)| : t \in [t_{m_i}, t_{m_i+k}]\} = |y(s_{m_i})|$, where $s_{m_i} \in (t_{m_i}, t_{m_i+k})$. Since $W(t,\lambda) \in L^\sigma([0,\infty), \mathbb{R}^+)$ for all $\lambda > 0$, then

$$\int_0^\infty W^\sigma(t,\lambda)dt < \infty, \quad \text{for all } \lambda > 0.$$

Hence

$$\int_t^\infty W^\sigma(t,\lambda)dt \to 0 \quad \text{as } t \to \infty.$$

Thus, for $1 < \sigma < \infty$, we may have

$$\int_{t_{m_i}}^\infty W^\sigma(t,\lambda)dt < [K^{n-1}M^{n-1+\frac{1}{\mu}}]^{-1}$$

for large i, where $\frac{1}{\mu} + \frac{1}{\sigma} = 1$. From (2.8) we obtain

$$\Big[\int_{t_{m_i}}^{s_i} ((r_1(t)^{1/(p-1)})^{p-1}\Big]^{-1} < \big(\frac{1}{2}\big)^p K^{n-2}\big(t_{m_i+k} - t_{m_i}\big)^{n-2} \int_{t_{m_i}}^{t_{m_i+k}} W(t, M_{m_i})dt;$$

that is,

$$1 < \big(\frac{1}{2}\big)^p K^{n-1}\big(t_{m_i+k} - t_{m_i}\big)^{n-1} \int_{t_{m_i}}^{t_{m_i+k}} W(t, M_{m_i})dt.$$

The use of Hölder's inequality yields

$$\begin{aligned} 1 &< \big(\frac{1}{2}\big)^p K^{n-1}\big(t_{m_i+k} - t_{m_i}\big)^{n-1}\big(t_{m_i+k} - t_{m_i}\big)^{1/\mu}\Big[\int_{t_{m_i}}^{t_{m_i+k}} W^\sigma(t, M_{m_i})dt\Big]^{1/\sigma} \\ &\le \big(\frac{1}{2}\big)^p K^{n-1}\big(t_{m_i+k} - t_{m_i}\big)^{n-1+\frac{1}{\mu}}\Big[\int_{t_{m_i}}^{\infty} W(t, M_{m_i})dt\Big]^{1/\sigma} \\ &< \big(\frac{1}{2}\big)^p K^{n-1}M^{n-1+\frac{1}{\mu}}\Big[K^{n-1}M^{n-1+\frac{1}{\mu}}\Big]^{-1} = \frac{1}{2^p},. \end{aligned}$$

a contradiction. For $\sigma = 1$, we can choose i large enough such that

$$\int_{t_{m_i}}^\infty W(t, M_{m_i}) < [K^{n-1}M^{n-1}]^{-1}$$

and

$$\begin{aligned} 1 &< \big(\frac{1}{2}\big)^p K^{n-1}(t_{m_i+k} - t_{m_i})^{n-1} \int_{t_{m_i}}^{t_{m_i+k}} W(t, M_{m_i})dt \\ &< \big(\frac{1}{2}\big)^p K^{n-1}M^{n-1}[K^{n-1}M^{n-1}]^{-1} = \frac{1}{2^p}, \end{aligned}$$

a contradiction. Hence the Theorem is proved. □

Theorem 3.2. *Suppose that* (C1)-(C2) *hold with $I = [0,\infty)$. Let there exist a continuous function $H : I \to \mathbb{R}^+$ such that $W(t,L) \le H(t)$ for every constant $L > 0$. Let*

$$\int_0^\infty r_1(t)^{1/(p-1)}ds_1 < \infty.$$

If

$$\int_0^\infty \overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|)dt < \infty,$$
$$\int_0^\infty \overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); H(s_{n-1}))dt < \infty,$$

for $n \geq 3$, *and*

$$\int_0^\infty H(t)dt < \infty, \quad \int_0^\infty |Q(t)|dt < \infty$$

for $n = 2$; *then every oscillatory solution of* (2.1) *converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be an oscillatory solution of (2.1) on $[T_y, \infty), T_y \geq 0$. Hence $\lim inf_{t\to\infty}|y(t)| = 0$. To complete the proof of the theorem it is sufficient to show that $limsup_{t\to\infty}|y(t)| = 0$. If possible, let $limsup_{t\to\infty}|y(t)| = \lambda > 0$. Choose $0 < d < \lambda/2$. From the given assumptions it follows that it is possible to choose a large $T_0 > 0$ such that, for $t \geq T_0$,

$$\int_t^\infty r_1(s_1)^{1/(p-1)}ds_1 < 2^{p/(p-1)},$$
$$\int_t^\infty \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|)ds_1 < d^{p-1},$$
$$\int_t^\infty \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); H(s_{n-1}))ds_1 < 1$$

for $n \geq 3$, and

$$\int_t^\infty H(s)ds < d^{p-1}, \quad \int_t^\infty |Q(s)|ds < d$$

for $n = 2$. Since $y(t)$ is oscillatory, we can find a $t_1 > T_0$ such that $y(t_1) = 0$. Let $T_0^* > t_1$ be such that $\alpha_1, \alpha_2, \dots, \alpha_{n-3}, \alpha_{n-2} \in [t_1, T_0^*]$, where $\alpha_1, \alpha_2, \dots, \alpha_{n-3}, \alpha_{n-2}$ are the zeros, respectively, of $D^2y(t), \dots, D^{n-2}y(t)$. Further, $\limsup_{t\to\infty}|y(t)| > 2d$ implies that we can find a $T^{**} > t_1$ such that $\sup\{|y(t) : t \in [t_1, T_0^{**}]\} > d$. Let $T_1 = max\{T_0^*, T_0^{**}\}$. Let $t_2 > T_1$ such that $y(t_2) = 0$. Let $M = \max\{|y(t)| : t \in [t_1, t_2]\}$, then $M > d$. From Theorem 2.1 we obtain (2.2) for $n \geq 3$ and (2.3) for $n = 2$, with $a = t_1$ and $b = t_2$. Hence, For $n \geq 3$,

$$\begin{aligned}
1 &< \big(\frac{1}{2}\big)^p \Big(\int_{t_1}^\infty ((r_1(s_1))^{1/(p-1)}ds_1\Big)^{p-1} \\
&\quad \times \int_{t_1}^\infty \Big[\overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); H(s_{n-1})) \\
&\quad + \frac{1}{M^{p-1}}\overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|)\Big]ds_1 \\
&< \big(\frac{1}{2}\big)^p \big(2^{p/(p-1)}\big)^{p-1}\big[1 + \big(\frac{d}{M}\big)^{p-1}\big] < 2,
\end{aligned}$$

a contradiction. Hence $\limsup_{t\to\infty}|y(t)| = 0$. Thus the proof of the theorem is complete. □

Example 3.3. Consider

$$(e^t(e^ty^2y')')' + y^3 = e^{-4t}(8cos^3t + 13sin^3t + 10\cos t - 6\sin t) + e^{-6t}\sin^3 t, \quad (3.1)$$

where $t \geq 0$. Thus $r_1(t) = e^{-t}$, $r_2(t) = e^{-t}$, $f(t,y) = 1$, and hence $H(t) = 1$. Clearly, $y(t) = e^{-2t}\sin t$ is a solution of (3.1) with $y(0) = 0$ and $(e^t y^2(t)y'(t))' = 0$ for $t = 0, \pi$. Hence $\alpha_1 = 0, \pi$. Let $\alpha_1 = 0$. Since

$$\overline{E}(s_1, r_2(s_1); H(s_2)) = s_1 e^{-s_1} \quad \text{for } s_1 > 0,$$
$$\overline{E}(s_1, r_2(s_1); |Q(s_2)|) \leq 38 s_1 e^{-s_1} \quad \text{for } s_1 > 0,$$

it follows that

$$\int_0^\infty \overline{E}(s_1, r_2(s_1); H(s_2)) ds_1 = 1,$$
$$\int_0^\infty \overline{E}(s_1, r_2(s_1); |Q(s_2)|) ds_1 \leq 38.$$

Again taking $\alpha_1 = \pi$, we obtain

$$\overline{E}(s_1, r_2(s_1); H(s_2)) = (s_1 - \pi) e^{-s_1} \quad \text{for } s_1 > \pi,$$
$$\overline{E}(s_1, r_2(s_1); |Q(s_2)|) \leq 38 (s_1 - \pi) e^{-s_1} \quad \text{for } s_1 . > \pi,$$

Then

$$\int_\pi^\infty \overline{E}(s_1, r_2(s_1); H(s_2)) ds_1 = e^{-\pi},$$
$$\int_\pi^\infty \overline{E}(s_1, r_2(s_1); |Q(s_2)|) ds_1 \leq 38 e^{-\pi}.$$

From Theorem 3.2 it follows that every oscillatory solution of (3.1) tends to zero as t tends to infinity.

Theorem 3.4. *If*

$$\begin{aligned} &\left(\frac{1}{2}\right)^p \Big(\int_a^b r_1(s_1)^{1/(p-1)} ds_1 \Big)^{p-1} \\ &\times \int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |p(s_{n-1})| ds_1 \leq 1, \end{aligned} \tag{3.2}$$

then

$$D^n y + p(t) y |y|^{p-2} = 0 \tag{3.3}$$

is disconjugate on $[a,b]$*, where* $p(t)$ *is a real-valued continuous function on* $[a,b]$*.*

Definition. Equation (3.3) is said to be disconjugate in $[a,b]$ if no non-trivial solution of (3.3) has more than $n-1$ zeros (counting multiplicities).

Proof of Theorem 3.4. Indeed, if (3.3) is not disconjugate on $[a,b]$, then it admits a nontrivial solution $y(t)$ has n zeros in $[a,b]$. Let these zeros be given by $a \leq a_1 < a_2 < \dots < a_{n-1} < a_n \leq b$. Then $D^2 y(t), D^3 y(t), \dots, D^{n-1} y(t)$ have zeros in

$[a_1, a_n]$. From Theorem 2.1, it follows that

$$\begin{aligned} 1 &< \big(\frac{1}{2}\big)^p \Big(\int_{a_1}^{a_n} r_1(s_1)^{1/(p-1)} ds_1\Big)^{p-1} \\ &\quad \times \int_{a_1}^{a_n} \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |p(s_{n-1}|) ds_1 \\ &\le \big(\frac{1}{2}\big)^p \Big(\int_a^b r_1(s_1)^{1/(p-1)} ds_1\Big)^{p-1} \\ &\quad \times \int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |p(s_{n-1}|) ds_1, \end{aligned}$$

a contradiction. Hence (3.3) is disconjugate on $[a, b]$. □

Remark. If $r_i(t) = 1; i = 1, 2, \dots, n-1;\ p = 2, n = 3$, then (3.2) reduces to

$$\int_a^b |p(t)| dt \le 4/(b-a)^2.$$

Thus the above inequality may be regarded as a sufficiency condition for the disconjugacy of the equation (1.7).

As a final remark, we note that the results obtained in this paper generalize the results by Pachpatte [19].

Acknowledgements. The author would like to thank the anonymous referee for his/her valuable suggestions.

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