Electronic Journal of Differential Equations, Vol. 2009(2009), No. 32, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF MULTIPLE SOLUTIONS FOR A NONLINEARLY PERTURBED ELLIPTIC PARABOLIC SYSTEM IN $\mathbb{R}^2$

MICHINORI ISHIWATA, TAKAYOSHI OGAWA, FUTOSHI TAKAHASHI

ABSTRACT. We consider the following nonlinearly perturbed version of the elliptic-parabolic system of Keller-Segel type:

$$\begin{gathered}\partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + v - v^p = u, \quad t > 0,\ x \in \mathbb{R}^2, \\ u(0, x) = u_0(x) \geq 0, \quad x \in \mathbb{R}^2,\end{gathered}$$

where $1 < p < \infty$. It has already been shown that the system admits a positive solution for a small nonnegative initial data in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ which corresponds to the local minimum of the associated energy functional to the elliptic part of the system. In this paper, we show that for a radially symmetric nonnegative initial data, there exists *another* positive solution which corresponds to the critical point of mountain-pass type. The v-component of the solution bifurcates from the unique positive radially symmetric solution of $-\Delta w + w = w^p$ in $\mathbb{R}^2$.

1. INTRODUCTION

In this paper, we consider the nonlinearly perturbed version of the elliptic-parabolic system modeling chemotaxis:

$$\begin{gathered}\partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + v - v^p = u, \quad t > 0,\ x \in \mathbb{R}^2, \\ u(0, x) = u_0(x) \geq 0, \quad x \in \mathbb{R}^2.\end{gathered} \tag{1.1}$$

In the context of mathematical biology, Keller and Segel [13] introduced a parabolic system, called the Keller-Segel system, as a mathematical model of chemotactic collapse (see also Herrero-Velázquez [11], [12], Nagai [18], [19], Biler [1], Nagai-Senba-Yoshida [21], Nagai-Senba-Suzuki [20] and Senba-Suzuki [24]). When the diffusion of the chemical substance is much slower than that of chemotaxis ameba, then the dynamics of chemotaxis is described by the following simplified

---

2000 *Mathematics Subject Classification.* 35K15, 35K55, 35Q60, 78A35.
Key words and phrases. Multiple existence; elliptic-parabolic system; unconditional uniqueness.

Submitted August 22, 2008. Published February 16, 2009.

system:

$$\begin{aligned} \partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + v = u, \quad t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = u_0(x) \geq 0, \quad x \in \mathbb{R}^2. \end{aligned} \tag{1.2}$$

It is well known that the existence of the finite time blow up of the solution for (1.2) which corresponds to the concentration of ameba (Herrero-Velázquez [11], [12], Nagai [18]).

Chen-Zhong [5] introduced a perturbed system of (1.2): For $p > 1$,

$$\begin{aligned} \partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + v + v^p = u, \quad t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = u_0(x) \geq 0, \quad x \in \mathbb{R}^2. \end{aligned} \tag{1.3}$$

This system is considered as a model of the chemotaxis with a nonlinear diffusion for the chemical substance. It has been proved that the solution of (1.3) has a similar behavior to the original system (1.2). In fact, one can show the local existence theory and finite time blow up with mass concentration phenomena as is shown for (1.2), see Chen-Zhong [5] and Kurokiba-Suzuki [16].

Note that the nonlinear term v^p in the second equation in (1.1) has a different sign compared to (1.3). According to this difference, the behavior of the solution for (1.1) is much different from the one for (1.3). Indeed, the nonhomogeneous elliptic problem corresponding to the second equation of (1.1):

$$-\Delta v + v - v^p = f, \quad x \in \mathbb{R}^2 \tag{1.4}$$

admits at least two positive solutions when f is a sufficiently small nonnegative nontrivial function in $H^{-1}(\mathbb{R}^2)$, while

$$-\Delta v + v + v^p = f, \quad x \in \mathbb{R}^2$$

has only one solution. Moreover, it is also known that if the external force f is large in H^{-1} sense, then there is no positive solution for the equation (1.4). Hence it is an interesting question whether the finite time blow up of the solution may occur in the case (1.1), or more primitively, whether the time local solution exists properly and the system is well posed in some sense or not. In this point, the structure of the time dependent positive solutions of (1.1) seems to be very much different from that of the original system (1.2) or the perturbed system (1.3).

In this paper, we shall consider solutions of (1.1) in the following sense:

$$\begin{aligned} &u \in C([0,T); L^2(\mathbb{R}^2)) \cap C^1((0,T); L^2(\mathbb{R}^2)) \cap C((0,T); \dot{H}^2(\mathbb{R}^2)), \\ &v \in C((0,T); H^1(\mathbb{R}^2)) \cap C((0,T); W^{2,2}(\mathbb{R}^2)) \end{aligned}$$

for some $T > 0$.

Recently Kurokiba-Ogawa-Takahashi [15] proved that, for a small nonnegative initial data, there exists a solution for (1.1) which is, in a sense, "small" one. On the other hand, as is mentioned above, the perturbed nonlinear elliptic equation (1.4) admits at least two positive solutions for small and nonnegative $f \not\equiv 0$. Therefore it is natural to ask whether the time dependent equation (1.1) also has a second positive solution. The main issue of this paper is to show the existence of *two positive time dependent solutions of* (1.1) under the radially symmetric setting.

Theorem 1.1 (Multiple existence). *Let $1 < p < \infty$. Then there exists a constant $C_{**} > 0$ such that, if the radially symmetric nonnegative initial data $u_0 \in L^1 \cap L^2(\mathbb{R}^2)$ satisfies*

$$\|u_0\|_2 \leq C_{**},$$

then there exist two positive radial pair of solutions $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ for (1.1)*. One of them is different from the solution obtained in* [15].

Note that the solution obtained in [15] exists globally in time if in addition $\|u_0\|_1$ is sufficiently small.

The main idea to construct the time dependent solutions heavily relies on the variational structure of the elliptic part of the system. The v-component of the solution obtained in [15] corresponds to the solution of (1.4) bifurcating from the trivial solution with $f = 0$. On the other hand, it has been known that the problem (1.4) with $f = 0$ has a unique positive solution w (see Berestycki-Lions [2], Gidas-Ni-Nirenberg [10] and Kwong [14]). This solution is obtained as a mountain pass critical point of the energy functional

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |v|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |v|^{p+1} dx.$$

If the second variation of I_0 at w is nondegenerate and if f is small, then we may construct the solution v of (1.4) bifurcating from the mountain pass solution w. This is not always possible, since the kernel of the Hessian of I_0 at w is nontrivial. If we restrict the class of initial data, however, there is a possibility of constructing the second local-in-time solution of (1.1). In this paper, we shall show that this is indeed possible under the radially symmetric setting.

Also it should be noted that our problem is related to the unconditional uniqueness problem in the general nonlinear evolution equations. Let X be a Banach space. If an initial value problem admits the unique solution in the class $C([0, T); X)$ with initial data in X, then we call the *unconditional uniqueness* holds for this problem. If the class of the solution is reasonably restricted, the unconditional uniqueness is expected to hold for the well-posed problem. For our problem (1.1), however, there is no possibility to have the unconditional uniqueness by restricting the regularity. Namely, no matter how the class of the solution is restricted from the regularity point of view, at least two solutions for (1.1) do exist. Only the variational characterization of the second component v distinguishes two solutions and the uniqueness class is not definable by means of function spaces. In this sense, the unconditional uniqueness never holds for (1.1). This kind of phenomena may occur for a general nonlinear problem. In our particular setting, there exists at least two time dependent solutions and are uniquely continued in time each other under the variational restriction.

We use the following notation. The Lebesgue space $L^p(\mathbb{R}^2)$ is denoted by L^p with $1 \leq p \leq \infty$ with the norm $\|\cdot\|_p$. For $k = 1, 2, \cdots$ and for $1 \leq p \leq \infty$, let $W^{k,p} = W^{k,p}(\mathbb{R}^2)$ be the Sobolev space with the norm $\|f\|_p + \|\nabla f\|_p$. We frequently use $H^1 = W^{1,2}(\mathbb{R}^2)$, and L^2_r and H^1_r denote the radially symmetric subspaces of L^2 and H^1, respectively. $(H^1_r)^*$ denotes the dual space of H^1_r. For a Banach space X, $B_{\delta,X}$ stands for the open ball in X with the radius $\delta > 0$ and the center 0. The constant C may vary from line to line.

2. Variational Structure of the Lagrangian Functional

The existence of multiple positive solutions for the semilinear elliptic equation

$$-\Delta v + v = v^p + f, \quad x \in \mathbb{R}^2 \tag{2.1}$$

is known for small nonnegative external forces $f \not\equiv 0$ in H^{-1}, see e.g. Zhu [25] and Cao-Zhou [4]. According to their results, there exists a solution of (2.1) for small f (in the H^{-1} sense) which is not a local minimizer of the functional I_f defined by

$$I_f(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |v|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^2} |v|^{p+1} - \int_{\mathbb{R}^2} fv dx, \quad v \in H^1(\mathbb{R}^2).$$

In this section, we give some analysis on the dependence of this non-minimal solution with respect to f, namely, we show some refined results compared to those of Zhu [25] and Cao-Zhou [4] from a bifurcation theoretical point of view.

As is mentioned in the introduction, the nonlinear elliptic problem (2.1) with $f \equiv 0$,

$$-\Delta v + v = v^p, \quad x \in \mathbb{R}^2, \tag{2.2}$$

has a radially symmetric positive unique solution w [2, 10, 14]. This solution is obtained as a critical point of the variational functional I_0 by the well known mountain pass lemma in H^1. Note that the Hessian operator of I_0 at $u \in H^1$ is realized by $L_u := -\Delta + 1 - p|u|^{p-1}$, which is an operator from H^1 to H^{-1}. As for the kernel of the linearized operator L_w at w, the following is well-known (see e.g. [6, 7, 8, 22]).

Proposition 2.1 (Kernel of the linearized operator). *For the radially symmetric positive unique solution w to* (2.2), $\ker L_w$, *the kernel of the operator $L_w = -\Delta + 1 - p|w|^{p-1}$, is spanned by $\partial_{x_1} w$ and $\partial_{x_2} w$. In particular,* $\ker L_w \cap H^1_r = \{0\}$.

According to Proposition 2.1, we may construct a solution branch of the non-minimal solution of (2.1) with the aid of the implicit function theorem if we restrict our problem to the class of radially symmetric functions.

Proposition 2.2. *There exists $\delta > 0$ and $h \in C(B_{\delta,(H^1_r)^*}; H^1_r)$ such that $h(f)$ is a critical point of I_f which is not a local minimum for $f \in B_{\delta,(H^1_r)^*}$ with $h(0) = w$. Moreover, h is a Lipschitz continuous mapping in $B_{\delta,(H^1_r)^*}$, namely, there exists $C > 0$ such that*

$$\|h(f_1) - h(f_2)\|_{H^1} < C\|f_1 - f_2\|_{(H^1_r)^*}, \quad \forall f_1,\ f_2 \in B_{\delta,(H^1_r)^*}. \tag{2.3}$$

If $f \geq 0$, then $h(f) \geq 0$ holds.

Proof. We employ the implicit function theorem for

$$g : (H^1_r)^* \times H^1_r \ni (f, u) \mapsto g(f, u) := (dI_f)_u \in (H^1_r)^*$$

around $(0, w) \in (H^1_r)^* \times H^1_r$. Hereafter the functional derivatives of g with respect to $f \in (H^1_r)^*$ and $u \in H^1_r$ are denoted by $D_1 g$ and $D_2 g$, respectively.

Let $\varphi \in H^1_r$, $\eta \in (H^1_r)^*$ and $v \in H^1_r$. Then it is easy to see that

$$(D_1 g)_{(f,u)}(\eta)\varphi = \frac{d}{dt} g(f + t\eta, u)\varphi\Big|_{t=0} = \int_{\mathbb{R}^2} \eta\varphi.$$

Hence $(D_1 g)_{(f,u)}$: $(H^1_r)^* \to (H^1_r)^*$ is an identity mapping. Therefore

$$D_1 g : (H^1_r)^* \times H^1_r \ni (f, u) \mapsto (D_1 g)_{(f,u)} \in L((H^1_r)^*; (H^1_r)^*)$$

is a constant map, especially continuous, where $L(X,Y)$ denotes the space of bounded linear operators between Banach spaces X and Y. Similarly,

$$(D_2 g)_{(f,u)}(v)\varphi = \frac{d}{dt} g(f, u+tv)\varphi\Big|_{t=0} = \int_{\mathbb{R}^2} L_u v\varphi,$$

i.e., $(D_2 g)_{(f,u)} : H_r^1 \to (H_r^1)^*$ is given by

$$(D_2 g)_{(f,u)}(v) = L_u v, \quad v \in H_r^1. \tag{2.4}$$

In particular,

$$D_2 g : (H_r^1)^* \times H_r^1 \ni (f,u) \mapsto (D_2 g)_{(f,u)} \in L(H_r^1; (H_r^1)^*)$$

is also a continuous map; thus $g \in C^1((H_r^1)^* \times H_r^1; (H_r^1)^*)$. Note that (2.4) implies $(D_2 g)_{(0,w)} = L_w$ and L_w restricted to H_r^1 should have a trivial kernel by virtue of Proposition 2.1. Therefore by the implicit function theorem (see e.g. ([17, Theorem 5.9])), there exist $\delta > 0$ and h: $B_{\delta,(H_r^1)^*} \to H_r^1$ such that $g(f, h(f)) = 0$ in $(H_r^1)^*$ for any $f \in B_{\delta,(H_r^1)^*}$ and $h(0) = w$. The latter implies that $h(f)$ is a critical point which is not a local minimum of I_f. Moreover, by the symmetric criticality principle of Palais [23], $h(f)$ is a critical point of I_f not only on H_r^1 but also on H^1. Therefore the first part of the proposition follows.

Also by the implicit function theorem, we have $h \in C^1(B_{\delta,(H_r^1)^*}; H_r^1)$, thus there exists a constant $C > 0$ such that

$$\|(dh)_f\|_{L((H_r^1)^*; H_r^1)} < C \quad \text{for } f \in B_{\delta,(H_r^1)^*}$$

if $\delta > 0$ is sufficiently small. Then for $f_1,\ f_2 \in B_{\delta,(H_r^1)^*}$,

$$\begin{aligned}
\|h(f_1) - h(f_2)\|_{H^1} &\le \Big\| \int_0^1 dt \Big(\frac{d}{dt} h(tf_2 + (1-t)f_1) \Big) \Big\|_{H^1} \\
&\le \int_0^1 dt \|(dh)_{tf_2+(1-t)f_1}\|_{L((H_r^1)^*; H_r^1)} \|f_2 - f_1\|_{(H_r^1)^*} \\
&\le C\|f_2 - f_1\|_{(H_r^1)^*},
\end{aligned}$$

hence (2.3) follows. The nonnegativity of $h(f)$ for $f \ge 0$ follows from the standard argument as in e.g. [4]. $\square$

The following corollary follows immediately from Proposition 2.2.

Corollary 2.3. *There exists $\rho > 0$ such that the conclusion of Proposition 2.2 holds when $B_{\delta;(H_r^1)^*}$ and $(H_r^1)^*$ are replaced by B_{ρ,L_r^2} and L_r^2, respectively.*

3. Proof of Main Theorem

In this section, we give the proof of Theorem 1.1. Let $1 < p < \infty$. We choose M with $M < \rho$ where ρ is the number obtained in Corollary 2.3. We shall construct a solution of (1.1) in the complete metric space

$$X_{T,M} = \big\{\phi \in C([0,T]; L_r^2) \cap L^2(0,T; \dot{H}_r^1);\ \phi \ge 0,\ \|\phi\|_X \le M\big\}$$

with the metric $d(\phi,\psi) \equiv \sup_{t\in[0,T]} \|\phi - \psi\|_X$, where

$$\|\phi\|_X \equiv \Big(\sup_{\tau\in[0,T)} \|\phi(\tau)\|_2^2 + \int_0^T \|\nabla\phi(\tau)\|_2^2 d\tau \Big)^{1/2}$$

and $T > 0$ is chosen to be small later.

For a nonnegative function $a \in L^2_r$, we define a map

$$\Phi_a : X_{T,M} \ni f \mapsto u \in X_{T,M},$$

where u is the solution of the following system:

$$\begin{gathered} \partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ -\Delta v + v = v^p + f, \quad t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = a(x), \quad x \in \mathbb{R}^2. \end{gathered} \tag{3.1}$$

Here we choose the solution $v(t)$ of the elliptic part of the above system as $h(f(t))$ where h is a map constructed in Corollary 2.3. Note that Φ_a is well defined by virtue of Corollary 2.3, since $\sup_{\tau\in[0,T)} \|f(\tau)\|_2 \le \|\!|f|\!\|_X \le M < \rho$.

It is also easy to see that Corollary 2.3 yields

$$\begin{aligned} \sup_{\tau\in[0,T)} \|h(f(\tau))\|_{H^1} \le & \sup_{\tau\in[0,T)} \|h(f(\tau)) - h(0)\|_{H^1} + \|h(0)\|_{H^1} \\ \le & C \sup_{\tau\in[0,T)} \|f(\tau) - 0\|_2 + \|w\|_{H^1} \\ \le & CM + \|w\|_{H^1} =: \sigma, \end{aligned} \tag{3.2}$$

where w is the unique, radially symmetric positive function satisfying $-\Delta w + w = w^p$ in $\mathbb{R}^2$. Hereafter for f and $\bar{f} \in X_{T,M}$, we denote $h(f(\tau))$ and $h(\bar{f}(\tau))$ by $v(\tau)$ and $\bar{v}(\tau)$ (or simply v and $\bar{v}$), respectively.

Our first lemma is as follows.

Lemma 3.1. *For any $q \ge 2$, there exists a constant $C_q > 0$ such that*

$$\|v(\tau)\|_{W^{1,q}} < C_q, \tag{3.3}$$

$$\|v(\tau) - \bar{v}(\tau)\|_{W^{1,q}} < C_q \|f(\tau) - \bar{f}(\tau)\|_2 \tag{3.4}$$

holds for any $f,\ \bar{f} \in X_{T,M}$ and for any $\tau \in [0,T)$.

Proof. Recall that v satisfies $-\Delta v + v = v^p + f$ in $\mathbb{R}^2$. Now for a given $g \in L^2$, the unique solution of $-\Delta \tilde{v} + \tilde{v} = g$ in $\mathbb{R}^2$ satisfies

$$\|\tilde{v}\|_{W^{1,q}} \le A_q \|g\|_2 \tag{3.5}$$

for some constant $A_q > 0$ when $q \ge 2$. Thus the Sobolev embedding $H^1 \hookrightarrow L^{2p}$ and (3.2) yields

$$\|v\|_{W^{1,q}} \le A_q \|v^p + f\|_2 \le A_q (C\|v\|_{H^1}^p + \|f\|_2) \le A_q (C\sigma^p + M) =: C_q,$$

hence (3.3).

Since v and $\bar{v}$ satisfy

$$-\Delta(v - \bar{v}) + (v - \bar{v}) = v^p - \bar{v}^p + f - \bar{f},$$

we have again from (3.5),

$$\|v - \bar{v}\|_{W^{1,q}} \le A_q \big(\|v^p - \bar{v}^p\|_2 + \|f - \bar{f}\|_2\big). \tag{3.6}$$

Here we note that, by the Sobolev embedding $H^1 \hookrightarrow L^{2p}$, (3.2) and Corollary 2.3,

$$\|v^p - \bar{v}^p\|_2^2 \le C(\|v\|_{2p}^{2(p-1)} + \|\bar{v}\|_{2p}^{2(p-1)})\|v - \bar{v}\|_{2p}^2 \le C\|v - \bar{v}\|_{H^1}^2 \le C\|f - \bar{f}\|_2^2$$

holds for suitable $C > 0$. Hence this fact together with (3.6) yields (3.4). ∎

Lemma 3.2. *There exists $C > 0$ such that*

$$\|\nabla v(\tau)\|_\infty^2 \le C(1 + \|\nabla f(\tau)\|_2)$$

for $\tau \in [0, T)$.

Proof. The second equation of (3.1), the Sobolev embedding $H^1 \hookrightarrow L^{2p}$ and (3.2) lead

$$\|\Delta v\|_2 \le \|v\|_2 + \|v^p\|_2 + \|f\|_2 < C$$

for some $C > 0$. Hence by using a version of the Brezis-Gallouet inequality [3]:

$$\|h\|_\infty^2 \le C\left(\|h\|_{H^1}^2(1 + \|\Delta h\|_2^{1/2}) + \|\Delta h\|_2\right)$$

for all $h \in H^2(\mathbb{R}^2)$, we have

$$\begin{aligned} \|\nabla v\|_\infty^2 &\le C((\|\Delta v\|_2^2 + \|\nabla v\|_2^2)(1 + \|\nabla \Delta v\|_2^{1/2}) + \|\nabla \Delta v\|_2) \\ &\le C(1 + \|\nabla \Delta v\|_2). \end{aligned} \tag{3.7}$$

Note that by Lemma 3.1, the Sobolev embedding $H^1 \hookrightarrow L^{2p}$ and (3.2),

$$\|\nabla v^p\|_2^2 = p^2 \int_{\mathbb{R}^2} |v|^{2(p-1)} |\nabla v|^2 \le p^2 \|v\|_{2p}^{2(p-1)} \|\nabla v\|_{2p}^2 < C \tag{3.8}$$

holds. Then the second equation of (3.1) together with (3.2) and (3.8) yields

$$\|\nabla \Delta v\|_2 \le \|\nabla v\|_2 + \|\nabla v^p\|_2 + \|\nabla f\|_2 \le C(1 + \|\nabla f\|_2).$$

Hence combining this relation with (3.7), we have the conclusion. $\square$

Using the estimate for v obtained above, we can verify the following key proposition for the verification of Theorem 1.1.

Proposition 3.3. *Let a, $\overline{a} \in L_r^2$ be smooth nonnegative radial functions. Then for some $C > 0$, we have for the solution operator Φ_a defined by (3),*

$$\big(1 - CT^{1/2}(T^{1/2} + M)\big) \|\!|\Phi_a(f)|\!\|_X^2 \le \|a\|_2^2, \tag{3.9}$$

$$\begin{aligned} &\big(1 - CT^{1/2}(T^{1/2} + M)\big) \|\!|\Phi_a(f) - \Phi_{\overline{a}}(\bar f)|\!\|_X^2 \\ &\qquad \le \|a - \overline{a}\|_2^2 + C \|\!|\Phi_a(f)|\!\|_X^2 T^{1/2} \|\!|f - \bar f|\!\|_X^2 \end{aligned} \tag{3.10}$$

for f, $\bar f \in X_{T,M}$.

Proof. The existence of a smooth solution for the system (3.1) with a smooth initial data follows from the standard theory of evolution equations. Under the assumption of the proposition, we denote solutions $\Phi_a(f(\tau))$ and $\Phi_{\overline{a}}(\bar f(\tau))$ of (3.1) by $u(\tau)$ and $\bar u(\tau)$ (or simply u and $\bar u$), respectively. We also denote $h(f(\tau))$ and $h(\bar f(\tau))$ by $v(\tau)$ and $\bar v(\tau)$ (or simply v and $\bar v$), respectively. Now multiplying the first equation of (3.1) by $u = u(\tau)$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{d\tau}\|u(\tau)\|_2^2 + \|\nabla u(\tau)\|_2^2 \le \frac{1}{2}\|u(\tau)\|_2^2 \|\nabla v(\tau)\|_\infty^2 + \frac{1}{2}\|\nabla u(\tau)\|_2^2. \tag{3.11}$$

Then, the integration of (3.11) from 0 to t in τ leads

$$
\begin{aligned}
\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 d\tau &\le \|a\|_2^2 + \int_0^t \|u(\tau)\|_2^2 \|\nabla v(\tau)\|_\infty^2 d\tau \\
&\le \|a\|_2^2 + \sup_{\tau\in[0,T)} \|u(\tau)\|_2^2 C \int_0^t (1 + \|\nabla f(\tau)\|_2) d\tau \\
&\le \|a\|_2^2 + \sup_{\tau\in[0,T)} \|u(\tau)\|_2^2 C T^{1/2}(T^{1/2} + M)
\end{aligned}
$$

for $t \in [0,T)$, here we have used Lemma 3.2 and $\sqrt{\int_0^t \|\nabla f\|_2^2 d\tau} \le M$, thus (3.9).

Next, we consider equations

$$
\begin{gathered}
\partial_t u - \Delta u + \nabla\cdot(u\nabla v) = 0, \quad u(0) = a, \\
\partial_t \bar{u} - \Delta \bar{u} + \nabla\cdot(\bar{u}\nabla \bar{v}) = 0, \quad \bar{u}(0) = \bar{a}.
\end{gathered}
$$

Multiplying $u - \bar{u}$ to the difference of these equations and integrating by parts, we see

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|u(\tau) - \bar{u}(\tau)\|_2^2 + \|\nabla(u(\tau) - \bar{u}(\tau))\|_2^2 \\
&= \int_{\mathbb{R}^2} u\nabla(v - \bar{v})\cdot\nabla(u - \bar{u})dx + \int_{\mathbb{R}^2} (u - \bar{u})\nabla\bar{v}\cdot\nabla(u - \bar{u})dx.
\end{aligned}
\tag{3.12}
$$

Then

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^2} u\nabla(v - \bar{v})\nabla(u - \bar{u})dx\Big| &\le \|u\|_4 \|\nabla(v - \overline{v})\|_4 \|\nabla(u - \overline{u})\|_2 \\
&\le \|u\|_4^2 \|\nabla(v - \overline{v})\|_4^2 + \frac{1}{4}\|\nabla(u - \overline{u})\|_2^2 \\
&\le \|u\|_4^2 C \sup_{\tau\in[0,T)} \|f(\tau) - \overline{f}(\tau)\|_2^2 + \frac{1}{4}\|\nabla(u - \overline{u})\|_2^2,
\end{aligned}
\tag{3.13}
$$

where we have used Lemma 3.1. Also Lemma 3.2 gives

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2} (u - \bar{u})\nabla\bar{v}\cdot\nabla(u - \bar{u})dx\Big| \\
&\le \|u - \bar{u}\|_2 \|\nabla\bar{v}\|_\infty \|\nabla(u - \bar{u})\|_2 \\
&\le \|u - \bar{u}\|_2^2 \|\nabla\bar{v}\|_\infty^2 + \frac{1}{4}\|\nabla(u - \bar{u})\|_2^2 \\
&\le C(1 + \|\nabla f\|_2) \sup_{\tau\in[0,T)} \|u(\tau) - \bar{u}(\tau)\|_2^2 + \frac{1}{4}\|\nabla(u - \bar{u})\|_2^2.
\end{aligned}
\tag{3.14}
$$

Then plugging (3.13) and (3.14) into (3.12) and integrating from 0 to t in τ, we have

$$
\begin{aligned}
&\|u(t) - \bar{u}(t)\|_2^2 + \int_0^t \|\nabla(u(\tau) - \bar{u}(\tau))\|_2^2 d\tau \\
&\le \|a - \bar{a}\|_2^2 + 2C\int_0^T \|u(\tau)\|_4^2 d\tau \sup_{\tau\in[0,T)} \|f(\tau) - \overline{f}(\tau)\|_2^2 \\
&\quad + 2C \sup_{\tau\in[0,T)} \|u(\tau) - \bar{u}(\tau)\|_2^2 \int_0^T (1 + \|\nabla f(\tau)\|_2) d\tau.
\end{aligned}
\tag{3.15}
$$

Here, we recall the Ladyzhenskaya inequality (see e.g., [9]):

$$\Big(\int_0^T \|\varphi(\tau)\|_4^4 d\tau\Big)^{1/2} \le C\Big(\sup_{\tau\in[0,T)} \|\varphi(\tau)\|_2^2 + \int_0^T \|\nabla\varphi(\tau)\|_2^2 d\tau\Big)$$

for $\varphi \in C([0,T);L^2) \cap L^2(0,T;\dot{H}^1)$. Then we obtain

$$\int_0^T \|u(\tau)\|_4^2 d\tau \le \Big(\int_0^T \|u(\tau)\|_4^4 d\tau\Big)^{1/2} T^{1/2} \le C\|\!|u|\!\|_X^2 T^{1/2}. \tag{3.16}$$

Hence plugging (3.16) into (3.15) and noting $\int_0^T (1+\|\nabla f(\tau)\|_2)d\tau \le T^{1/2}(T^{1/2}+M)$ again, we have

$$\begin{aligned}\Big(1 - CT^{1/2}(T^{1/2}+M)\Big) &\sup_{\tau\in[0,T)} \|u(\tau)-\bar{u}(\tau)\|_2^2 + \int_0^t \|\nabla(u(\tau)-\bar{u}(\tau))\|_2^2 d\tau \\ &\le \|a-\bar{a}\|_2^2 + C\|\!|u|\!\|_X^2 T^{1/2} \sup_{\tau\in[0,T)} \|f(\tau)-\bar{f}(\tau)\|_2^2\end{aligned}$$

and (3.10) follows. □

Now we are in the position to give the proof of Theorem 1.1.

Proof. Take any $M < \rho$ where ρ is the number which is obtained in Corollary 2.3. Then choose $T > 0$ so small that $\frac{1}{2} \le 1 - CT^{1/2}(T^{1/2}+M)$ and $CM^2T^{1/2} \le \frac{1}{4}$ hold, where C is the constant in Proposition 3.3. Let $u_0 \in L_r^2(\mathbb{R}^2)$ be a nonnegative initial data with $\|u_0\|_2^2 < M^2/2 =: C_{**}^2$. Then by using the approximation of u_0 by a sequence of smooth functions and Proposition 3.3, we can easily verify that Φ_{u_0} is a contraction mapping from $X_{T,M}$ to $X_{T,M}$. Therefore, the Banach fixed point theorem implies that there exists a unique solution of $u = \Phi_{u_0}(u)$. It is obvious that $(u,v) = (u,h(u))$ gives a solution of (1.1). The standard parabolic regularity argument gives that the solution becomes regular immediately after $t > 0$. The continuous dependence of the solution on the initial data also follows from (3.10). □

Acknowledgments. Work of M. Ishiwata is partially supported by Grant-in-Aid for Young Scientists (B) #19740081, The Ministry of Education, Culture, Sports, Science and Technology, Japan. Work of T. Ogawa is partially supported by JSPS Grant-in-Aid for Scientific Research (A) #20244009. Work of F. Takahashi is partially supported by JSPS Grant-in aid for Scientific Research (C) #20540216.@

References

[1] P. Biler, *Local and global solvability of some parabolic systems modeling chemotaxis*, Adv. Math. Sci. Appl. , **8** (1998), 715–743.
[2] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state.* Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345.
[3] H. Brezis, T. Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. T.M.A., **4** (1980), 677-681.
[4] D.-M. Cao, H.-S. Zhou, *Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^n$,* Proc. Roy. Soc. Edinburgh Ser A, **126** (1996), 443-463.
[5] H. Chen, X.- H. Zhong, *Global existence and blow-up for the solutions to the nonlinear parabolic-elliptic system modeling chemotaxis,* IMA J. Appl. Math., **70** (2005), 221-240.
[6] C. Cortazar, M. del Pino, M. Elgueta, *The problem of uniqueness of the limit in a semilinear heat equation.* Comm. Partial Differential Equations **24** (1999), no. 11-12, 2147–2172.
[7] E. N. Dancer, *On the uniqueness of the positive solution of a singularly perturbed problem.* Rocky Mountain J. Math. **25** (1995), no. 3, 957–975.

[8] E. Feireisl, H. Petzeltova, *Convergence to a ground state as a threshold phenomenon in nonlinear parabolic equations.* Differential Integral Equations **10** (1997), no. 1, 181–196.
[9] C. Foias, O. Manley, R. Rosa, R. Temam, *Navier-Stokes equations and Tuebulance.* Cambridge University Press, 2001.
[10] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.
[11] M. A. Herrero, J.J.L. Velázquez, *Chemotaxis collapse for the Keller-Segel model.* J. Math. Biol., **35**(1996), 177-194.
[12] M. A. Herrero, J.J.L. Velázquez, *Singularity patterns in a chemotaxis model.* Math. Ann., **306** (1996), 583-623.
[13] E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability,* J. Theor. Biol., **26** (1970), 399-415.
[14] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* R^n*,* Arch. Rational Mech. Anal. **105** (1989), no. 3, 243–266.
[15] M. Kurokiba, T. Ogawa, F. Takahashi, *Global existence of solutions for a nonlinearly perturbed elliptic parabolic system in* $\mathbb{R}^2$, Z. Angew. Math. Phys., to appear.
[16] M. Kurokiba, T. Suzuki, *On a perturbed system of chemotaxis,* Int. J. Math. Anal. (Ruse) 1 (2007), no. 25-28, pp. 1261-1271.
[17] S. Lang, *Differential and Riemannian Manifold.* Springer (1995).
[18] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl., **5** (1995), 581-601.
[19] T. Nagai, *Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains*, J. Inequal. Appl. 6 (2001), no.1, pp.37-55.
[20] T. Nagai, T. Senba, T. Suzuki, *Chemotactic collapse in a parabolic system of mathematical biology*, Hiroshima Math. J., **30** (2000), 463-497
[21] T. Nagai, T. Senba, K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. **40** (1997), no. 3, 411–433.
[22] W. M. Ni, I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem.* Duke Math. J. **70** (1993), no. 2, 247–281.
[23] R. S. Palais, *The principle of symmetric criticality,* Comm. Math. Phys. **69** (1979), 19-30.
[24] T. Senba, T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology.* Adv. Differential Equations, **6** (2001), 21-50.
[25] X. P. Zhu, *A perturbation result on positive entire solutions of a semilinear elliptic equation.* J. Differential Equations **92** (1991), no. 2, 163–178.

Michinori Ishiwata
Common Subject Division, Muroran Institute of Technology, Muroran 050-8585, Japan
E-mail address: `ishiwata@mmm.muroran-it.ac.jp`

Takayoshi Ogawa
Mathematical Institute, Tohoku University, Sendai 980-8578, Japan
E-mail address: `ogawa@math.tohoku.ac.jp`

Futoshi Takahashi
Graduate School of Science, Osaka City University, Osaka 558-8585, Japan
E-mail address: `futoshi@sci.osaka-cu.ac.jp`