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A NOTE ON NODAL NON-RADIALLY SYMMETRIC SOLUTIONS TO EMDEN-FOWLER EQUATIONS

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Abstract. We prove the existence of an unbounded sequence of sign-changing and non-radially symmetric solutions to the problem $-\Delta u = |u|^{p-1}u$ in Ω, $u = 0$ on $\partial\Omega$, $u(gx) = u(x)$, $x \in \Omega$, $g \in G$, where Ω is an annulus of $\mathbb{R}^N$ $(N \geq 3)$, $1 < p < (N+2)/(N-2)$ and G is a non-transitive closed subgroup of the orthogonal group $O(N)$.

1. Introduction

In this note we consider the sign-changing and non-radially symmetric solutions to the following Emden-Fowler equation:

$$-\Delta u = |u|^{p-1}u, \quad x \in \Omega, \tag{1.1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{1.2}$$

$$u(gx) = u(x), \quad x \in \Omega,\ g \in G, \tag{1.3}$$

where Ω is a unit ball $\Omega := \{x \in \mathbb{R}^N : |x| < 1\}$ or an annulus $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$, $0 < a < b$, $N \geq 3$, $1 < p < (N+2)/(N-2)$ and G is a closed subgroup of the orthogonal group $O(N)$ of degree N. Here gx is the product of the column vector x and the matrix g and a solution of (1.1)-(1.3) will be called a G-invariant solution.

It is known that (1.1)-(1.2) has infinitely many sign-changing radially symmetric solutions when $1 < p < (N+2)/(N-2)$ (cf. [1, 2, 3, 4]) and each one of them has finitely many zero points. The existence of sign-changing solutions of (1.1)-(1.2) with further information on the nodal domains is considered in [5] but no conclusions on the non-radial symmetry are derived.

Clearly, a radially symmetric solution is a G-invariant solution, for any subgroup G of $O(N)$. The converse problem was considered in [6] where the author proved that there exist solutions which are G-invariant and not radially symmetric if G is not transitive on $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. In the sequel, we say that G is transitive if for any two points $x, y \in S^{N-1}$ there exists a $g \in G$ such that $y = gx$. Under this assumption, in [6, Theorem 1] it is proved that the problem (1.1)-(1.3) admits an unbounded sequence of G-invariant and non-radially symmetric

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solutions. According to a celebrated theorem by Gidas, Ni and Nirenberg [8], non-radially symmetric solutions must change their sign if the domain is a ball. In this note we derive the conclusion that these solutions must indeed change sign in Ω, even if the domain is an annulus. Precisely, we prove the following.

Theorem 1.1. *If G is not transitive on S^{N-1}, then there exists a sequence $\{w_k\}$ of solutions of (1.1)-(1.3) such that each w_k is G-invariant, sign-changing and non-radially symmetric. Moreover, $\|w_k\| \to \infty$ as $k \to \infty$.*

We denote by $\|\cdot\|$ the usual norm in $H_0^1(\Omega)$. We mention that, by construction, the solutions w_k have a well-determined Morse index (cf. [7, 9]), so that it is likely that further conclusions on their nodal domains can be derived, in the line of the work in [10].

We recall from [6, Corollaries 1 and 2] that Theorem 1.1 applies in case G is finite or has dimension not greater than $N-2$. A typical example is $G = \{Id, -Id\}$, where Id is the unit matrix. It follows that (1.1)-(1.3) has infinitely many *sign-changing* non-radially symmetric and even solutions. Another example is

$$G = \Big\{ \begin{pmatrix} e & 0 \\ 0 & w \end{pmatrix} : e \in O(m),\ \ w \in O(N-m) \Big\}, \quad 1 \le m < N.$$

Then by Theorem 1.1, (1.1)-(1.3) has a sequence of solutions $\{u_k\}$ such that each u_k is sign-changing and $u_k(x) = u_k(|x'|, |x''|)$ for all $x' \in \mathbb{R}^m, x'' \in \mathbb{R}^{N-m}$ with $x = (x', x'') \in \Omega$, but $u_k(x) \neq u_k(|x|)$.

The proof of Theorem 1.1 is given in the next section. We combine the approach in [6] (namely the crucial estimates in Lemmas 2.1 and 2.2) with the method introduced in [9] for finding sign-changing solutions to superlinear elliptic equations such as the one in (1.1), which is essentially contained in the strict inequality (2.6) below.

2. Proof of Theorem 1.1

Let

$$H_0^1(\Omega, G) = \{u \in H_0^1(\Omega) : u(gx) = u(x), x \in \Omega, g \in G\}$$

equipped with the inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v dx$ and the corresponding norm $\|u\| = \langle u, u\rangle^{1/2}$. We also denote by $\|u\|_{p+1}$ the $L^{p+1}(\Omega)$ norm of u. Solutions of (1.1)-(1.3) are critical points of the functional I defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \quad u \in H_0^1(\Omega, G).$$

We denote by $\{\lambda_k(\Omega, G)\}_{k\in\mathbb{N}}$ the increasing sequence of eigenvalues of the problem

$$\begin{gathered} -\Delta u = \lambda u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \\ u(gx) = u(x), \quad x \in \Omega,\ g \in G. \end{gathered} \tag{2.1}$$

Lemma 2.1 (cf. [6, 1, 2, 3])**.** *The set of radially symmetric critical points of I consists of a sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ and the zero solution. Moreover,*

$$0 < \beta_1 < \beta_2 < \dots < \beta_k < \dots \to \infty, \quad \text{where } \beta_k = I(\pm u_k),$$

and there exists $A_0 > 0$ independent of k such that

$$A_0 k^{\frac{2(p+1)}{p-1}} \le \beta_k, \quad k \in \mathbb{N}.$$

Let
$$G(x) = \{gx : g \in G\}, \quad x \in S^{N-1}.$$
Then $G(x)$ is a closed submanifold of S^{N-1} and we denote by $\dim G(x)$ its dimension, so that $0 \le \dim G(x) \le N-1$. Let
$$m := m(G) := \max\{\dim G(x) : x \in S^{N-1}\}.$$

Lemma 2.2 (cf. [6]). *Assume that G is not transitive on S^{N-1}. Then $0 \le m \le N-2$ and there exists a positive constant C_1 independent of k such that*
$$\lambda_k(\Omega, G) \le C_1 k^{\frac{2}{N-m}}, \quad k \in \mathbb{N}.$$

Now, let E_k be the eigenspace associated to the eigenvalues $\lambda_i(\Omega, G)$ with $i = 1, \dots, k$ and $S_k := \{u \in E_{k-1}^{\perp} : \|u\|_{p+1} = 1\}$. As observed in [6], it follows from Lemma 2.2 that
$$\sup_{E_k} I \le B_0 k^{\frac{2(p+1)}{(N-m)(p-1)}}, \tag{2.2}$$
while a simple computation shows that
$$\inf_{S_k} I \ge B_1 \lambda_k(\Omega, G)^{\alpha} - B_2, \tag{2.3}$$
for some positive constants B_0, B_1, B_2 independent of k, where α is given by $\alpha = \frac{(2+N)-p(N-2)}{2(p+1)} > 0$.

By observing that $I(u) > 0$ if u is a nontrivial critical point of I, we define
$$\begin{aligned} N_1 := \sup\big\{& c \in \mathbb{R} : c > 0 \text{ is a critical value of } I \text{ corresponding to } G\text{-invariant} \\ & \text{sign-changing and non-radially symmetric critical points}\big\} \end{aligned}$$
(We set $N_1 = 0$ in case this set is empty). To prove Theorem 1.1 we must show that the above set is nonempty and that $N_1 = \infty$. In the sequel we argue by contradiction by assuming that $N_1 < \infty$.

According to (2.3), we can fix $k_0 > 0$ such that
$$\inf_{S_k} I > N_1 \quad \text{for all } k \ge k_0. \tag{2.4}$$
Let
$$N_2 := \max\{k \in \mathbb{N} : A_0(k - k_0 + 1)^{\frac{2(p+1)}{p-1}} \le B_0 k^{\frac{2(p+1)}{(N-m)(p-1)}}\}.$$
Thanks to Lemma 2.2, N_2 is finite. We choose k^* large enough such that $k^* > \max\{k_0, N_2\}$. From now on we only consider the integers k lying in the interval $[k_0, k^*]$. Let
$$C^* = \sup_{E_{k^*}} I < \infty. \tag{2.5}$$
We also fix $R_k > 0$ in such a way that
$$\|u\|_{p+1} > 1, \quad I(u) < 0 \quad \text{for all } u \in E_k \text{ with } \|u\| \ge R_k.$$
We may assume that R_k increases with k.

Let P denote the positive cone of $H_0^1(\Omega, G)$, that is $P := \{u \in H_0^1(\Omega, G) : u(x) \ge 0, x \in \Omega\}$. It follows from [9, Lemma 2.4] that
$$\operatorname{dist}\Big(\big(\cup_{k=k_0}^{k^*} S_k\big) \cap I^{C^*}, \pm P\Big) > 0, \tag{2.6}$$
where $I^{C^*} := \{u : I(u) \le C^*\}$. Let $D := \{u \in H_0^1(\Omega, G) : \operatorname{dist}(u, P) < \varepsilon_0\}$, $D^* := -D \cup D$, $\mathcal{U} := E \backslash D^*$. Then, for ε_0 small enough, we have
$$\big(\cup_{k=k_0}^{k^*} S_k\big) \cap I^{C^*} \subset \mathcal{U}. \tag{2.7}$$

Moreover, as shown in [11], $D^* \cap \mathcal{K} \subset (-P \cup P)$, where $\mathcal{K} := \{u \in H_0^1(\Omega, G) : I'(u) = 0\}$. For $k \in [k_0, k^*]$, we set

$$T_k := \{h : h \in C(\Theta_k, E), h \text{ is odd }, h(u) = u \text{ on } \partial\Theta_k\},$$
$$\Theta_k := \{u \in E_k : \|u\| < R_k\}, \quad \partial\Theta_k := \{u \in E_k : \|u\| = R_k\}.$$

Define

$$\begin{gathered} Z_k := \big\{h(\overline{\Theta_i \backslash A}) : h \in T_i,\ i \in [k, k^*],\ A \in \mathcal{E}, \\ \gamma(A) \leq i - k,\ I(h(\overline{\Theta_i \backslash A})) \leq C^*\big\}, \end{gathered} \tag{2.8}$$

where $\mathcal{E}$ is the family of closed subsets A of $H_0^1(\Omega, G)$ such that $0 \notin A$ and $-u \in A$ whenever $u \in A$; $\gamma(A)$ denotes the genus of A. Clearly, $Z_k \neq \emptyset$ since $Id \in T_k$; also, $Z_{k+1} \subset Z_k$.

Lemma 2.3. *$B \cap \mathcal{U} \cap S_k \neq \emptyset$ for any $B \in Z_k$.*

Proof. Thanks to (2.7) it is sufficient to prove that $B \cap S_k \neq \emptyset$. This, in turn, can be derived in a standard way. For completeness, we sketch the argument as in [12, Proposition 9.23]. We write $B = h(\overline{\Theta_i \backslash A})$ with $h \in T_i, k^* \geq i \geq k$ and $\gamma(A) \leq i-k$. Let $W_1 := \{u \in \Theta_i : \|h(u)\|_{p+1} < 1\}$ and $W_2 := \{u \in \Theta_i : \|h(u)\|_{p+1} = 1\}$. Then W_1 is a symmetric bounded neighborhood of 0 in Θ_i and hence $\gamma(\partial W_1) = i$, while $\partial W_1 \subset W_2$ by our choice of R_k. Thus $\gamma(W_2) \geq i$ and so $\gamma(h(\overline{W_2 \backslash A})) \geq \gamma(\overline{W_2 \backslash A}) \geq k > k-1$. Hence $h(\overline{W_2 \backslash A}) \cap E_{k-1}^{\perp} \neq \emptyset$ and this proves the claim. □

Now, for $k_0 \leq k \leq k^*$ we define

$$c_k = \inf_{B \in Z_k} \max_{u \in B \cap \mathcal{U}} I(u).$$

Thanks to (2.4) and Lemma 2.3, c_k is well defined and $c_k \geq \inf_{S_k} I > N_1$. Clearly, $c_{k_0} \leq c_{k_0+1} \leq \dots \leq c_{k^*}$.

Lemma 2.4. *If $c_k = c_{k+1} = \dots = c_{k+\ell} =: c$, then $\gamma(\mathcal{K}_c \cap \mathcal{U}) \geq \ell + 1$, where $\mathcal{K}_c := \{u \in H_0^1(\Omega, G) : I(u) = c, I'(u) = 0\}$.*

Proof. In view of a contradiction, assume that $\gamma(\mathcal{K}_c \cap \mathcal{U}) \leq \ell$. Since $\mathcal{K}_c^s := \mathcal{K}_c \cap \mathcal{U}$ is compact and $0 \notin \mathcal{K}_c^s$, there exists a closed neighborhood U of $\mathcal{K}_c^s$ such that $\gamma(U) \leq \ell$. Let V be an open neighborhood of $\mathcal{K}_c \cap (-P \cup P) := \mathcal{K}_c^{pn}$ such that $V \subset D^*$. The well-known deformation lemma implies that for $\varepsilon > 0$ small enough we can find a flow $\eta \in C([0,1] \times E, E)$ such that $\eta(1, u)$ is odd in u, $\eta(1, I^{c+\varepsilon} \backslash (\overset{\circ}{U} \cup V)) \subset I^{c-\varepsilon}$ and $\eta(1, \cdot) = Id$ on $\partial\Theta_i$ for $i \in [k, k^*]$ (here we use the fact that $I < 0$ on $\partial\Theta_i$ and $c > N_1 \geq 0$). Moreover, the flow η keeps $\pm D$ invariant, that is $\eta(t, \pm D) \subset \pm D$ for every t (see for example [13, 11, 9, 14]). Hence, $\eta(1, I^{c+\varepsilon} \backslash \overset{\circ}{U}) \subset I^{c-\varepsilon} \cup D^*$. Choose $B \in Z_{k+\ell}$ such that $\max_{B \cap \mathcal{U}} I \leq c + \varepsilon$, $B = h(\overline{\Theta_i \backslash A})$ with $h \in T_i, i \in [k+\ell, k^*], \gamma(A) \leq i - (k+\ell), \sup_B I \leq C^*$. Similarly to [12, Proposition 9.18] we find that $\overline{B \backslash U} \in Z_k$. Since η is a descending flow, also $\eta(1, \overline{B \backslash U}) \in Z_k$. But $\eta(1, \overline{B \backslash U}) \cap \mathcal{U} = \eta(1, \mathcal{U} \cap \overline{B \backslash U}) \cap \mathcal{U} \subset \eta(1, I^{c+\varepsilon} \backslash \overset{\circ}{U}) \cap \mathcal{U} \subset (I^{c-\varepsilon} \cup D^*) \cap \mathcal{U} \subset I^{c-\varepsilon}$. This contradicts the definition of c and proves the lemma. □

Proof of Theorem 1.1. Thanks to Lemma 2.4, we can conclude similarly to [6], and so we only sketch the argument. Since $c_k > N_1$ for all $k \in [k_0, k^*]$, by Lemma 2.1, we see that $\{c_{k_0}, c_{k_0+1}, \dots, c_{k^*}\} \subset \{\beta_1, \beta_2, \dots\}$. Assume $c_k = c_{k+1}$ for some $k \in [k_0, k^* - 1]$. Then, by Lemma 2.4, $\gamma(\mathcal{K}_{c_k} \cap \mathcal{U}) \geq 2$. But $c_k = \beta_i$ for some i, and so $\mathcal{K}_{c_k} \cap \mathcal{U} = \{u_i, -u_i\}$. This is a contradiction and it follows that $\{c_k\}_{k=k_0}^{k^*}$ is strictly increasing. Therefore, $c_{k^*} = \beta_j$ for some $j \geq k^* - k_0 + 1$. Hence, by Lemma 2.1 and (2.2),

$$A_0(k^* - k_0 + 1)^{\frac{2(p+1)}{p-1}} \leq \beta_j = c_{k^*} \leq B_0(k^*)^{\frac{2(p+1)}{(N-m)(p-1)}}.$$

The very definition of N_2 implies $k^* \leq N_2$. This contradicts our choice of k^* and proves our claim that $N_1 = \infty$. □

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