Electronic Journal of Differential Equations, Vol. 2009(2009), No. 49, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND UNIQUENESS THEOREMS ON CERTAIN DIFFERENCE-DIFFERENTIAL EQUATIONS

BABURAO G. PACHPATTE

Abstract. In the present paper, existence and uniqueness theorems for the solutions of certain nonlinear difference-differential equations are established. The main tools employed in the analysis are based on the applications of the Leray-Schauder alternative and the well known Bihari's integral inequality.

1. Introduction

Let $\mathbb{R}^n$ denote the real n-dimensional Euclidean space with appropriate norm denoted by $|\cdot|$ and $J = [0, T]$ $(T > 0)$, $\mathbb{R}_+ = [0, \infty)$ be the given subsets of $\mathbb{R}$, the set of real numbers. In this paper, we consider the difference-differential equation of the form

$$x'(t) = f(t, x(t), x(t-1)), \tag{1.1}$$

for $t \in J$ under the initial conditions

$$x(t-1) = \phi(t) \ (0 \le t < 1), \quad x(0) = x_0, \tag{1.2}$$

where $f \in C(J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\phi(t)$ is a continuous function for $0 \le t < 1$, $\lim_{t\to 1-0} \phi(t)$ exists, for which we denote by $\phi(1-0) = c_0$. If we consider the solutions of (1.1) for $t \in J$, we obtain a function $x(t-1)$ which is unable to define as a solution for $0 \le t < 1$. Hence, we have to impose some condition, for example the first condition in (1.2). We note that, if T is less than 1, the problem is reduced to ordinary differential equation

$$x'(t) = f(t, x(t), \phi(t)), \tag{1.3}$$

for $0 \le t < 1$, with the second condition in (1.2). Here, it is essential to obtain the solutions of equation (1.1) for $0 \le t \le T$, so that, we suppose, in the sequel, T is not less than 1. It is easy to observe that the integral equations which are equivalent to (1.1)-(1.2) are

$$x(t) = x_0 + \int_0^t f(s, x(s), \phi(s))ds, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 34K5, 34K10, 34K40.
Key words and phrases. Existence and uniqueness; difference-differential equations; Leray-Schauder alternative; Bihari's integral inequality; Wintner type; Osgood type.

Submitted June 9, 2008. Published April 7, 2009.

for $0 \le t < 1$ and

$$x(t) = x_0 + \int_0^1 f(s, x(s), \phi(s))ds + \int_1^t f(s, x(s), x(s-1))ds, \tag{1.5}$$

for $1 \le t \le T$.

Problems of existence and uniqueness of solutions of equations of the form (1.1) and its more general versions, under various initial conditions have been studied by many authors by using different techniques. The fundamental tools used in the existence proofs, are essentially, the method of successive approximation, Schauder-Tychonoff's fixed point theorem, Banach contraction mapping principle and comparison method, see [2,6,9,12,14,15] and the references cited therein. Our main objective here is to investigate the global existence of solution to (1.1)-(1.2) by using simple and classical application of the topological transversality theorem of Granas [5, p. 61], also known as Leray-Schauder alternative. Osgood type uniqueness result for the solutions of (1.1)-(1.2) is established by using the well known Bihari's integral inequality. Existence and uniqueness theorems for certain perturbed difference-differential equation are also given.

2. Main Results

In proving existence of solution of (1.1)-(1.2), we use the following topological transversality theorem given by Granas [5, p.61].

Lemma 2.1. *Let B be a convex subset of a normed linear space E and assume $0 \in B$. Let $F : B \to B$ be a completely continuous operator and let $U(F) = \{x : x = \lambda F x\}$ for some $0 < \lambda < 1$. Then either $U(F)$ is unbounded or F has a fixed point.*

We also need the following integral inequality, often referred to as Bihari's inequality [8, p. 107].

Lemma 2.2. *Let $u(t), p(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. Let $w(u)$ be a continuous, nondecreasing function defined on $\mathbb{R}_+$, $w(u) > 0$ for $u > 0$ and $w(0) = 0$. If*

$$u(t) \le c + \int_0^t p(s) w(u(s))ds,$$

for $t \in \mathbb{R}_+$, where $c \ge 0$ is a constant, then for $0 \le t \le t_1$,

$$u(t) \le W^{-1}\Big[W(c) + \int_0^t p(s)ds\Big],$$

where

$$W(r) = \int_{r_0}^r \frac{ds}{w(s)}, \quad r > 0,\ r_0 > 0,$$

W^{-1} is the inverse function of W and $t_1 \in \mathbb{R}_+$ be chosen so that

$$W(c) + \int_0^t p(s)ds \in \mathrm{Dom}(W^{-1}),$$

for all $t \in \mathbb{R}_+$ lying in the interval $0 \le t \le t_1$.

The following theorem deals with the Wintner type global existence result for the solution of (1.1)-(1.2).

Theorem 2.3. *Suppose that the function f in (1.1) satisfies the condition*

$$|f(t,x,y)| \leq h(t)[g(|x|) + g(|y|)], \tag{2.1}$$

where $h \in C(J,R)$ and $g : \mathbb{R}_+ \to (0,\infty)$ is continuous and nondecreasing function. Then (1.1)-(1.2) has a solution $x(t)$ defined on J provided T satisfies

$$\int_0^T [h(s) + h(s+1)]ds < \int_\alpha^\infty \frac{ds}{g(s)}, \tag{2.2}$$

where

$$\alpha = |x_0| + \int_0^1 h(s)g(|\phi(s)|)ds. \tag{2.3}$$

Proof. The proof will be given in three steps.
Step I. To use Lemma 1, we establish the priori bounds on the solutions of the problem

$$x'(t) = \lambda f(t, x(t), x(t-1)), \tag{2.4}$$

under the initial conditions (1.2) for $\lambda \in (0,1)$. Let $x(t)$ be a solution of (2.4)-(1.2), then we consider the following two cases.

Case 1: $0 \leq t < 1$. From the hypotheses, we have

$$\begin{aligned} |x(t)| &= \big|x_0 + \int_0^t \lambda f(s, x(s), \phi(s))ds\big| \\ &\leq |x_0| + \int_0^t h(s)[g(|x(s)|) + g(|\phi(s)|)]ds \\ &= \alpha + \int_0^t h(s)g(|x(s)|)ds. \end{aligned} \tag{2.5}$$

Let $u(t)$ be defined by the right hand side of (2.5), then $u(0) = \alpha$, $|x(t)| \leq u(t)$ and

$$u'(t) = h(t)g(|x(t)|) \leq h(t)g(u(t));$$

that is,

$$\frac{u'(t)}{g(u(t))} \leq h(t). \tag{2.6}$$

Integration of (2.6) from 0 to t ($0 \leq t < 1$), the change of variable, and the condition (2.2) gives

$$\int_\alpha^{u(t)} \frac{ds}{g(s)} \leq \int_0^t h(s)ds \leq \int_0^1 h(s)ds < \int_\alpha^\infty \frac{ds}{g(s)}. \tag{2.7}$$

From this inequality, we conclude that, there is a constant Q_1 independent of $\lambda \in (0,1)$ such that $u(t) \leq Q_1$ for $0 \leq t < 1$ and hence $|x(t)| \leq Q_1$.

Case 2: $1 \le t \le T$. From the hypotheses, we have

$$
\begin{aligned}
|x(t)| &= |x_0 + \int_0^1 \lambda f(s, x(s), \phi(s))ds + \int_1^t \lambda f(s, x(s), x(s-1))ds| \\
&\le |x_0| + \int_0^1 h(s)[g(|x(s)|) + g(|\phi(s)|)]ds \\
&\quad + \int_1^t h(s)[g(|x(s)|) + g(|x(s-1)|)]ds \\
&= |x_0| + \int_0^1 h(s)g(|\phi(s)|)ds + \int_0^1 h(s)g(|x(s)|)ds \\
&\quad + \int_1^t h(s)g(|x(s)|)ds + \int_1^t h(s)g(|x(s-1)|)ds \\
&= \alpha + \int_0^t h(s)g(|x(s)|)ds + I_1
\end{aligned}
\tag{2.8}
$$

where

$$
I_1 = \int_1^t h(s)g(|x(s-1)|)ds. \tag{2.9}
$$

By the change of variable, from (2.9), we observe that

$$
I_1 = \int_0^{t-1} h(\sigma+1)g(|x(\sigma)|)d\sigma \le \int_0^t h(\sigma+1)g(|x(\sigma)|)d\sigma. \tag{2.10}
$$

Using this inequality in (2.8), we obtain

$$
|x(t)| \le \alpha + \int_0^t [h(s) + h(s+1)]g(|x(s)|)ds. \tag{2.11}
$$

Let $v(t)$ be defined by the right hand side of (2.11), then $v(0) = \alpha$, $|x(t)| \le v(t)$ and

$$
v'(t) = [h(t) + h(t+1)]g(|x(t)|) \le [h(t) + h(t+1)]g(v(t));
$$

that is,

$$
\frac{v'(t)}{g(v(t))} \le [h(t) + h(t+1)]. \tag{2.12}
$$

Integration of (2.12) from 0 to t, $1 \le t \le T$, the change of variable, and the condition (2.2) give

$$
\int_\alpha^{v(t)} \frac{ds}{g(s)} \le \int_0^t [h(s) + h(s+1)]ds \le \int_0^T [h(s) + h(s+1)]ds < \int_\alpha^\infty \frac{ds}{g(s)}. \tag{2.13}
$$

From (2.13) we conclude that there is a constant Q_2 independent of $\lambda \in (0,1)$ such that $v(t) \le Q_2$ and hence $|x(t)| \le Q_2$ for $1 \le t \le T$. Let $Q = \max\{Q_1, Q_2\}$. Obviously, $|x(t)| \le Q$ for $t \in J$ and consequently, $\|x\| = \sup\{|x(t)| : t \in J\} \le Q$.

Step II. We define $B = C(J, \mathbb{R}^n)$ to be the Banach space of all continuous functions from J into $\mathbb{R}^n$ endowed with sup norm defined above. We rewrite the problem (1.1)-(1.2) as follows. If $y \in B$ and $x(t) = y(t) + x_0$, it is easy to see that

$$
y(t) = \int_0^t f(s, y(s) + x_0, \phi(s))ds,
$$

for $0 \le t < 1$ and

$$y(t) = \int_0^1 f(s, y(s) + x_0, \phi(s))ds + \int_1^t f(s, y(s) + x_0, y(s-1) + x_0)ds,$$

for $1 \le t \le T$, $y(0) = y_0 = 0$ if and only if $x(t)$ satisfies (1.1)-(1.2). Let $B_0 = \{y \in B : y_0 = 0\}$ and define $F : B_0 \to B_0$ by

$$Fy(t) = \int_0^t f(s, y(s) + x_0, \phi(s))ds, \tag{2.14}$$

for $0 \le t < 1$ and

$$Fy(t) = \int_0^1 f(s, y(s) + x_0, \phi(s))ds + \int_1^t f(s, y(s) + x_0, y(s-1) + x_0)ds, \tag{2.15}$$

for $1 \le t \le T$. Then F is clearly continuous. Now, we shall prove that F is uniformly bounded. Let $\{b_m\}$ be a bounded sequence in B_0, that is, $\|b_m\| \le b$ for all m, where $b > 0$ is a constant. Let $N = \sup\{h(t) : t \in J\}$. We have to consider the two cases.

Case 1: $0 \le t < 1$. From (2.14) and hypotheses, we have

$$\begin{aligned} |Fb_m(t)| &\le \int_0^t |f(s, b_m(s) + x_0, \phi(s))|ds \\ &\le \int_0^t h(s)[g(|b_m(s)| + |x_0|) + g(|\phi(s)|)]ds \\ &\le \int_0^1 h(s)g(|\phi(s)|)ds + \int_0^t h(s)g(b + |x_0|)ds \\ &\le \gamma + \int_0^1 Ng(b + |x_0|)ds \\ &= \gamma + Ng(b + |x_0|). \end{aligned} \tag{2.16}$$

where

$$\gamma = \int_0^1 h(s)g(|\phi(s)|)ds. \tag{2.17}$$

Case 2: $1 \le t \le T$. From (2.15) and the hypotheses, we have

$$\begin{aligned} |Fb_m(t)| &\le \int_0^1 |f(s, b_m(s) + x_0, \phi(s))|ds + \int_1^t |f(s, b_m(s) + x_0, b_m(s-1) + x_0)|ds \\ &\le \int_0^1 h(s)[g(|b_m(s)| + |x_0|) + g(|\phi(s)|)]ds \\ &\quad + \int_1^t h(s)[g(|b_m(s)| + |x_0|) + g(|b_m(s-1)| + |x_0|)]ds \\ &= \int_0^1 h(s)g(|\phi(s)|)ds + \int_0^1 h(s)g(|b_m(s)| + |x_0|)ds \\ &\quad + \int_1^t h(s)g(|b_m(s)| + |x_0|)ds + \int_1^t h(s)g(|b_m(s-1)| + |x_0|)ds \\ &= \gamma + \int_0^t h(s)g(|b_m(s)| + |x_0|)ds + I_2, \end{aligned} \tag{2.18}$$

where γ is given by (2.17), and

$$I_2 = \int_1^t h(s)g(|b_m(s-1)| + |x_0|)ds. \tag{2.19}$$

By the change of variable, we have

$$I_2 = \int_0^{t-1} h(\sigma+1)g(|b_m(\sigma)| + |x_0|)d\sigma \le \int_0^t h(\sigma+1)g(|b_m(\sigma)| + |x_0|)d\sigma. \tag{2.20}$$

Using (2.20) in (2.18), we have

$$\begin{aligned}
|Fb_m(t)| &\le \gamma + \int_0^t [h(s) + h(s+1)]g(|b_m(s)| + |x_0|)ds \\
&\le \gamma + \int_0^T 2Ng(b + |x_0|)ds \\
&= \gamma + 2NTg(b + |x_0|).
\end{aligned} \tag{2.21}$$

From (2.16) and (2.21), it follows that $\{Fb_m\}$ is uniformly bounded.
Step III. We shall show that the sequence $\{Fb_m\}$ is equicontinuous. Let $\{b_m\}$ and N be as in Step II. We must consider three cases.

Case 1: t and t' are contained in $0 \le t < 1$. From (2.14), it follows that

$$\begin{aligned}
Fb_m(t) - Fb_m(t') &= \int_0^t f(s, b_m(s) + x_0, \phi(s))ds - \int_0^{t'} f(s, b_m(s) + x_0, \phi(s))ds \\
&= \int_{t'}^t f(s, b_m(s) + x_0, \phi(s))ds.
\end{aligned} \tag{2.22}$$

From the above equality and hypotheses, we have

$$\begin{aligned}
|Fb_m(t) - Fb_m(t')| &\le \Big|\int_{t'}^t |f(s, b_m(s) + x_0, \phi(s))|ds\Big| \\
&\le \Big|\int_{t'}^t h(s)[g(|b_m(s)| + |x_0|) + g(|\phi(s)|)]ds\Big| \\
&\le \Big|\int_{t'}^t N[g(b + |x_0|) + g(|c_0|)]ds\Big| \\
&= N[g(b + |x_0|) + g(|c_0|)]|t - t'|.
\end{aligned} \tag{2.23}$$

Case 2: t and t' are contained in $1 \le t \le T$. From (2.15), it follows that

$$\begin{aligned}
&Fb_m(t) - Fb_m(t') \\
&= \int_0^1 f(s, b_m(s) + x_0, \phi(s))ds + \int_1^t f(s, b_m(s) + x_0, b_m(s-1) + x_0)ds \\
&\quad - \int_0^1 f(s, b_m(s) + x_0, \phi(s))ds - \int_1^{t'} f(s, b_m(s) + x_0, b_m(s-1) + x_0)ds \\
&= \int_{t'}^t f(s, b_m(s) + x_0, b_m(s-1) + x_0)ds.
\end{aligned} \tag{2.24}$$

From this equality and the hypotheses, we have

$$
\begin{aligned}
|Fb_m(t)-Fb_m(t')| &\le \Big|\int_{t'}^{t} |f(s,b_m(s)+x_0,b_m(s-1)+x_0)|ds\Big| \\
&\le \Big|\int_{t'}^{t} h(s)[g(|b_m(s)|+|x_0|)+g(|b_m(s-1)|+|x_0|)]ds\Big| \\
&\le \Big|\int_{t'}^{t} Ng(|b_m(s)|+|x_0|)ds+I_3\Big|
\end{aligned}
\tag{2.25}
$$

where

$$I_3=\int_{t'}^{t} Ng(|b_m(s-1)|+|x_0|)ds. \tag{2.26}$$

By the change of variable, we have

$$I_3=\int_{t'-1}^{t-1} Ng(|b_m(\sigma)|+|x_0|)d\sigma \le Ng(b+|x_0|)(t-t'). \tag{2.27}$$

Using the above inequality in (2.25), we obtain

$$|Fb_m(t)-Fb_m(t')| \le 2Ng(b+|x_0|)|t-t'|. \tag{2.28}$$

Case 3: t and t' are respectively contained in $[0,1)$ and $[1,T]$. From (2.14) and (2.15), it follows that

$$
\begin{aligned}
&Fb_m(t)-Fb_m(t') \\
&=\int_0^t f(s,b_m(s)+x_0,\phi(s))ds \\
&\quad -\int_0^1 f(s,b_m(s)+x_0,\phi(s))ds-\int_1^{t'} f(s,b_m(s)+x_0,b_m(s-1)+x_0)ds \\
&=-\int_t^0 f(s,b_m(s)+x_0,\phi(s))ds-\int_0^1 f(s,b_m(s)+x_0,\phi(s))ds \\
&\quad -\int_1^{t'} f(s,b_m(s)+x_0,b_m(s-1)+x_0)ds.
\end{aligned}
\tag{2.29}
$$

From (2.29) and using the hypotheses, we have

$$
\begin{aligned}
&|Fb_m(t)-Fb_m(t')| \\
&\le \int_t^1 |f(s,b_m(s)+x_0,\phi(s))|ds \\
&\quad +\int_1^{t'} |f(s,b_m(s)+x_0,b_m(s-1)+x_0)|ds \\
&\le \int_t^1 h(s)[g(|b_m(s)|+|x_0|)+g(|\phi(s)|)]ds \\
&\quad +\int_1^{t'} h(s)[g(|b_m(s)|+|x_0|)+g(|b_m(s-1)|+|x_0|)]ds \\
&\le \int_t^1 N[g(b+|x_0|)+g(|c_0|)]ds+\int_1^{t'} N[g(b+|x_0|)+g(b+|x_0|)]ds \\
&\le M(t'-t),
\end{aligned}
\tag{2.30}
$$

where
$$M = \max\{N[g(b + |x_0|) + g(|c_0|)], 2Ng(b + |x_0|)\}.$$
From (2.23), (2.28), (2.30), we conclude that $\{Fb_m\}$ is equicontinuous. By Arzela-Ascoli theorem (see [4,7]), the operator F is completely continuous.

Moreover, the set $U(F) = \{y \in B_0 : y = \lambda Fy, \lambda \in (0,1)\}$ is bounded, since for every y in $U(F)$ the function $x(t) = y(t) + x_0$ is a solution of (2.4)-(1.2), for which we have proved $\|x\| \le Q$ and hence $\|y\| \le Q + |x_0|$. Now, an application of Lemma 1, the operator F has a fixed point in B_0. This means that (1.1)-(1.2) has a solution. The proof is complete. □

Remark. We note that the advantage of our approach here is that, it yields simutaneously the existence of solution of (1.1)-(1.2) and maximal interval of existence. In the special case, if we take $h(t) = 1$ in (2.2) and the integral on the right hand side in (2.2) is assumed to diverge, then the solution of (1.1)-(1.2) exists for every $T < \infty$; that is, on the entire interval $\mathbb{R}_+$. Our result in Theorem 1 yields existence of solution of (1.1)-(1.2) on $\mathbb{R}_+$, if the integral on the right hand side in (2.2) is divergent i.e., $\int_\alpha^\infty \frac{ds}{g(s)} = \infty$. Thus Theorem 1 can be considered as a further extension of the well known theorem on global existence of solution of ordinary differential equation due to Wintner given in [16].

The next theorem deals with the Osgood type uniqueness result for the solutions of (1.1)-(1.2).

Theorem 2.4. *Consider* (1.1) *with* $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$*, under the initial conditions in* (1.2)*. Suppose that:*

(i) *the function f satisfies*
$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le h(t)[g(|x-\bar{x}|) + g(|y-\bar{y}|)], \tag{2.31}$$
where $h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g(u)$ is a continuous, nondecreasing function for $u \ge 0, g(0) = 0$;

(ii) *let*
$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad (0 < r_0 \le r),$$
with G^{-1} being the inverse function of G and assume that $\lim_{r_0 \to +0} G(r) = \infty$, for any fixed r.

Then (1.1)-(1.2) *has at most one solution on* $\mathbb{R}_+$.

Proof. Let $x(t), y(t)$ be two solutions of equation (1.1), under the initial conditions
$$x(t-1) = y(t-1) = \phi(t), \quad (0 \le t < 1), \quad x(0) = y(0) = x_0, \tag{2.32}$$
and let $u(t) = |x(t) - y(t)|$, $t \in \mathbb{R}_+$. We consider the following two cases.

Case 1: $0 \le t < 1$. From the hypotheses, we have
$$\begin{aligned} u(t) &\le \int_0^t |f(s,x(s),\phi(s)) - f(s,y(s),\phi(s))|ds \\ &\le \int_0^t h(s)g(|x(s)-y(s)|)ds \\ &\le \varepsilon_1 + \int_0^t h(s)g(u(s))ds, \end{aligned} \tag{2.33}$$

where $\varepsilon_1 > 0$ is sufficiently small constant. Now, an application of Lemma 2 to (2.33) yields

$$|x(t) - y(t)| \le G^{-1}\big[G(\varepsilon_1) + \int_0^t h(s)ds\big]. \tag{2.34}$$

Case 2: $1 \le t < \infty$. From the hypotheses, we have

$$\begin{aligned} u(t) &\le \int_0^1 |f(s, x(s), \phi(s)) - f(s, y(s), \phi(s))|ds \\ &\quad + \int_1^t |f(s, x(s), x(s-1)) - f(s, y(s), y(s-1))|ds \\ &\le \int_0^1 h(s)g(u(s))ds + \int_1^t h(s)[g(u(s)) + g(u(s-1))]ds \\ &= \int_0^1 h(s)g(u(s))ds + \int_1^t h(s)g(u(s))ds + \int_1^t h(s)g(u(s-1))ds \\ &= \int_0^t h(s)g(u(s))ds + I_4, \end{aligned} \tag{2.35}$$

where

$$I_4 = \int_1^t h(s)g(u(s-1))ds. \tag{2.36}$$

By the change of variable, we observe that

$$I_4 \le \int_0^t h(s+1)g(u(s))ds. \tag{2.37}$$

Using (2.37) in (2.35), we obtain

$$u(t) \le \int_0^t [h(s) + h(s+1)]g(u(s))ds \le \varepsilon_2 + \int_0^t [h(s) + h(s+1)]g(u(s))ds, \tag{2.38}$$

where $\varepsilon_2 > 0$ is sufficiently small constant. Now, an application of Lemma 2 to (2.38) yields

$$|x(t) - y(t)| \le G^{-1}[G(\varepsilon_2) + \int_0^t [h(s) + h(s+1)]ds]. \tag{2.39}$$

To apply the estimations in (2.34), (2.39) to the uniqueness problem, we use the notation $G(r, r_0)$ instead of $G(r)$ and impose the assumption $\lim_{r_0 \to +0} G(r, r_0) = +\infty$, for fixed r, then we obtain $\lim_{r_0 \to +0} G^{-1}(r, r_0) = 0$, see [15, p. 77]. From (2.34), (2.39), it follows that $|x(t) - y(t)| \le 0$ for $t \in \mathbb{R}_+$ and hence $x(t) = y(t)$ on $\mathbb{R}_+$. Thus, there is at most one solution to (1.1)-(1.2) on $\mathbb{R}_+$. $\square$

Remark. We note that the condition (2.31) corresponds to the Osgood type condition concerning the uniqueness of solutions in the theory of differential equations (see [4, p. 35]).

3. Perturbed equations

In this section, we consider the difference-differential equation of the form

$$x'(t) = A(t)x(t) + B(t)x(t-1) + f(t, x(t), x(t-1)), \tag{3.1}$$

for $t \in J$ with the initial conditions (1.2). The equation (3.1) is treated as a perturbation of the linear system

$$x'(t) = A(t)x(t) + B(t)x(t-1), \tag{3.2}$$

for $t \in J$ with the initial conditions (1.2), where $A(t), B(t)$ are continuous functions on J and f, ϕ are the functions as in (1.1), (1.2). Following Sugiyama [14], let $K(t,s)$ be a matrix solution of equations:

$$\begin{gathered}
\frac{\partial}{\partial t}K(t,s) = A(t)K(t,s) + B(t)K(t-1,s) \quad (0 \le s < t-1), \\
\frac{\partial}{\partial t}K(t,s) = A(t)K(t,s) \quad (0 < t-1 < s < t, 0 \le s < t < 1), \\
K(t,t) = 1, \\
K(t,s) = 0 \quad (-1 \le t < 0).
\end{gathered}$$

The function $K(t,s)$ satisfying the above properties is called the kernel function for equation (3.2). For more details concerning the kernel function and its use in the study of various properties of solutions of difference-differential equations with perturbed terms, see Bellman and Cooke [1,2]. By means of the kernel function $K(t,s)$, it follows that the solution of (3.1) with (1.2) considered as a perturbation of (3.2) with (1.2) is represented by (see [13, p.457])

$$x(t) = x_0(t) + \int_0^t K(t,s)f(s, x(s), x(s-1))ds, \tag{3.3}$$

where $x_0(t)$ is a unique solution of (3.2) with (1.2). It is easy to observe that (see [13]) the integral equations which are equivalent to (3.3) are

$$x(t) = x_0(t) + \int_0^t K(t,s)f(s, x(s), \phi(s))ds, \tag{3.4}$$

for $0 \le t < 1$ and

$$x(t) = x_0(t) + \int_0^1 K(t,s)f(s, x(s), \phi(s))ds + \int_1^t K(t,s)f(s, x(s), x(s-1))ds, \tag{3.5}$$

for $1 \le t \le T$.

The following theorems concerning the existence and uniqueness of solutions of (3.1)-(1.2) hold.

Theorem 3.1. *Suppose that:*

(i) *the function f in (3.1) satisfies (2.1),*

(ii) *the unique solution $x_0(t)$ of (3.2)-(1.2) is bounded, that is*

$$|x_0(t)| \le c, \tag{3.6}$$

for $t \in J$, where c is a positive constant,

(iii) *the kernel function $K(t,s)$ for (3.2) is bounded; that is,*

$$|K(t,s)| \le L, \tag{3.7}$$

for $0 \le s \le t \le T$, where $L \ge 0$ is a constant, and for each $t' \in J$,

$$\lim_{t \to t'} \int_0^T |K(t,s) - K(t',s)|ds = 0, \tag{3.8}$$

is satisfied for $t \in J$.

Then (3.1)-(1.2) *has a solution* $x(t)$ *defined on* J *provided* T *satisfies*

$$\int_0^T L[h(s)+h(s+1)]ds < \int_\beta^\infty \frac{ds}{g(s)}, \tag{3.9}$$

where

$$\beta = c + \int_0^1 Lh(s)g(|\phi(s)|)ds. \tag{3.10}$$

Theorem 3.2. *Consider* (3.1) *with* $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ *and the conditions in* (1.2), *as a perturbation of* (3.2) *for* $t \in \mathbb{R}_+$ *with* (1.2). *Suppose that:*

(i) *the condition (i) of Theorem 3 holds and the kernel function* $K(t,s)$ *satisfies the condition* (3.7),

(ii) *the conditions (i)-(ii) of Theorem 2 hold.*

Then (3.1)-(1.2) *has at most one solution on* $\mathbb{R}_+$.

The proofs of Theorems 3 and 4 can be completed by following the proofs of Theorems 1 and 2 given above, with suitable modifications and closely looking at the proofs of existence results given in [9,10]. We omit the details here.
Remark. We note that our approach to the existence study of (1.1)-(1.2) and (3.1)-(1.2) is different from those used in [11-15] and we believe that the results given here are of independent interest. For the study of numerical solution of general Volterra integral equation with delay arguments of the form (3.3), see [3].

Acknowledgements. The author wishes to express his sincere thanks to the anonymous referee and Professor J. G. Dix for helpful comments and suggestions.

References

[1] R. Bellman and K. L. Cooke; *Stability theory and adjoint operators for linear differential-difference equations*, Trans.Amer.Math.Soc. 92(1959), 470-500.

[2] R. Bellman and K. L. Cooke; *Differential-Difference Equations*, Academic Press, New York, 1963.

[3] B. Cahlon, L. J. Nachman and D. Schmidt; *Numerical solution of Volterra integral equations with delay arguments*, J. Integral Equations 7(1984), 191-208.

[4] C. Corduneanu; *Principles of Differential and Integral Equations*, Chelsea, New York, 1971.

[5] J. Dugundji and A. Granas; *Fixed point Theory, Vol. I, Monografie Mathematyczne*, PWN, Warsaw, 1982.

[6] J. K. Hale; *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[7] M. A. Krasnoselskii; *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1964.

[8] B. G. Pachpatte; *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.

[9] B. G. Pachpatte; *On higher order differential equation with retarded argument*, In: Differential Equations and Applications, Vol. 4, 2007, Nova Science Publishers, Inc., New York, 2007, Editors: Yeol Je Cho et.al., pp. 93-103.

[10] B. G. Pachpatte; *On a perturbed system of Volterra integral equations*, Numer. Funct. Anal. Optim. 29(2008), 197-212.

[11] B. G. Pachpatte; *Some basic theorems on difference-differential equations*, Elect. J. Diff. Eqs. Vol. 2008(2008), No. 75, pp. 1-11.

[12] S. Sugiyama; *On the existence and uniqueness theorems of difference-differential equations*, Kodi Math.Sem.Rep. 12(1960), 179-190.

[13] S. Sugiyama; *On the boundedness of solutions of difference-differential equations*, Proc. Japan Acad. 36(1960), 456-460.

[14] S. Sugiyama; *Existence theorems on difference-differential equations*, Proc. Japan. Acad. 38(1962), 145-149.

[15] S. Sugiyama; *Dependence properties of solutions of the retardation and initial values in the theory of difference-differential equations*, Kodai Math. Sem. Rep. 15(1963), 67-78.

[16] A. Wintner; *The nonlocal existence problem for ordinary differential equations*, Amer. J. Math. 67(1945), 277-284.

Baburao G. Pachpatte
57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad 431 001 (Maharashtra), India
E-mail address: bgpachpatte@gmail.com